Generalized Kelvin theorem for incompressible fluids

Theodore D. Drivas

Let Ω be a smooth bounded domain and let u be a *strong* solution of the incompressible Navier-Stokes equation with no-slip boundary conditions

$$\partial_t u + \pounds_u^T u = -\nabla q + \nu \Delta u, \tag{1}$$
$$\nabla \cdot u = 0, \tag{2}$$
$$u|_{\partial\Omega} = 0, \tag{3}$$
$$u|_{t=0} = u_0, \tag{4}$$

where q is related to the usual pressure by $q := p + \frac{1}{2}|u|^2$ and where the Lie derivative of a 1-form is

$$\pounds_u^T f = u \cdot \nabla f + (\nabla u)^t \cdot f. \tag{5}$$

Suppose now that h solves the following linear backward PDE

$$\partial_t h + \pounds_u h = -\nu \Delta h, \tag{6}$$
$$\nabla \cdot h = 0, \tag{7}$$
$$h|_{\partial\Omega} = 0, \tag{8}$$
$$h|_{t=T} = h_T, \tag{9}$$

where the Lie derivative of a vector field is

$$\pounds_u f = u \cdot \nabla f - f \cdot \nabla u. \tag{10}$$

The following is a *generalized Kelvin theorem*, which holds in the presence of boundaries and viscosity.

PROPOSITION 1. *Let u be a strong solution of* (1)–(4) *and h be* ***any*** *solution of* (6)–(9). *Then*

$$\int_\Omega u(t) \cdot h(t)\, \mathrm{d}x = \int_\Omega u(t') \cdot h(t')\, \mathrm{d}x, \qquad t, t' \in [0, T]. \tag{11}$$

PROOF. We proceed by direct computation starting with

$$\begin{aligned} \frac{d}{dt} \int_\Omega u \cdot h &= \int_\Omega (\partial_t u \cdot h + u \cdot \partial_t h) \\ &= -\int_\Omega (\pounds_u^T u \cdot h + u \cdot \pounds_u h) - \nu \int_\Omega u \cdot \Delta h + \nu \int_\Omega \Delta u \cdot h, \end{aligned} \tag{12}$$

where we used the fact that h is divergence-free and $h|_{\partial\Omega} = 0$ to eliminate the contribution from the pressure. Now, using the facts that u is divergence-free and $u|_{\partial\Omega} = 0$, we have for any regular f, g that

$$\int_\Omega \pounds_u^T f \cdot g := \int_\Omega (u \cdot \nabla f + \nabla u \cdot f) \cdot g = -\int_\Omega (u \cdot \nabla g - g \cdot \nabla u) \cdot f := -\int_\Omega \pounds_u g \cdot f.$$

Moreover, since $u|_{\partial\Omega} = 0$ and $h|_{\partial\Omega} = 0$, we have

$$-\nu \int_\Omega u \cdot \Delta h + \nu \int_\Omega \Delta u \cdot h = \nu \int_\Omega \nabla u : \nabla h - \nu \int_\Omega \nabla u : \nabla h = 0.$$

□

REMARK 1 (Relation to Kelvin's theorem for loops). Proposition 1 tells that there is an infinite dimensional family of non-local invariants for the Euler and Navier-Stokes equations. A special, singular, case within this family correspond to taking h to be "singular loops". Specifically, we can solve (6)–(9) with distributional final data, interpreting h has a solution in the weak sense. Treating u as a test function in the weak form of (6), the L^2 pairing of h with u is time invariant on $[0,T]$. In the case of Euler $\nu = 0$, then

$$h(x,t) = \oint_{X_{T,t}(C)} \delta^d(x-y)\mathrm{d}y = \int_0^1 \delta^d(x - X_{T,t}(C(s)))\frac{d}{ds}X_{T,t}(C(s))\mathrm{d}s \tag{13}$$

is distributionally divergence free and is a weak solution of (6) with $\nu = 0$ and final-time data

$$h(x,T) = \oint_C \delta^d(x-y)\mathrm{d}y. \tag{14}$$

Whence, the usual Kelvin theorem [**1**] is recovered from Proposition 1

$$\oint_C u(T)\cdot \mathrm{d}\ell = \oint_{X_{T,t}(C)} u(t)\cdot \mathrm{d}\ell, \qquad t\in[0,T]. \tag{15}$$

In the case of Navier-Stokes on the torus, we note that

$$h(x,t) = \mathbb{E}\left[\oint_{\tilde{X}_{T,t}(C)} \delta^d(x-y)\mathrm{d}y\right] \tag{16}$$

is once again distributionally divergence free and a weak solution of the backward equation (6) (smooth for any $t > 0$). Then, the Constantin-Iyer Kelvin theorem [**2**] is recovered from Proposition 1

$$\oint_C u(T)\cdot \mathrm{d}\ell = \mathbb{E}\left[\oint_{\tilde{X}_{T,t}(C)} u(t)\cdot \mathrm{d}\ell\right], \qquad t\in[0,T]. \tag{17}$$

References

[1] Kelvin, T.: On vortex motion. Trans. Roy. Soc. Edinb. 25 (1869): 217-260
[2] Constantin, P., and Iyer, G.: A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations. Commun. Pure Appl. Math 61.3 (2008): 330-345.

DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NY, 11794
Email address: `tdrivas@math.stonybrook.edu`