Self-regularization and intermittency in the Burgers equation

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ABSTRACT. We demonstrate that finite fluxes of infinitely many nonlinear ideal conserved quantities, analogous to the Kolmogorov $\frac{4}{5}$–law for the energy in incompressible turbulence, regularize vanishing viscosity solutions of one-dimensional conservation laws and impose strong constraints on spatial intermittency.

Consider the viscous Burgers equation [1] on $x \in \mathbb{T}$, $t \geq 0$

$$\partial_t u^\nu + u^\nu \partial_x u^\nu = \nu \partial_x^2 u^\nu + f, \tag{1}$$
$$u|_{t=0} = u_0.$$

The most basic estimates for the solution u^ν are derived from energy balance

$$\nu \int_0^T \int_{\mathbb{T}} |\partial_x u^\nu(x,t)|^2 \mathrm{d}x\mathrm{d}t = \tfrac{1}{2}\|u_0\|_{L^2}^2 - \tfrac{1}{2}\|u^\nu(\cdot,T)\|_{L^2}^2 + \int_0^T \int_{\mathbb{T}} u^\nu(x,t) f \mathrm{d}x\mathrm{d}t, \tag{2}$$

which confers $L^2(0,T;H^1(\mathbb{T}))$ regularity to the solution, but not uniformly in the viscosity. The strongest *uniform* estimate at first sight is $L^\infty(0,T;L^\infty(\mathbb{T}))$, following from the maximum principle

$$\sup_{t\in[0,T]} \|u^\nu(t)\|_{L^\infty(\mathbb{T})} \leq \|u_0\|_{L^\infty(\mathbb{T})} + T\|f\|_{L^\infty(\mathbb{T})}. \tag{3}$$

Our main result is a uniform fractional regularity of u^ν related to the pth–order structure functions

$$S_p^\nu(\ell,T) := \frac{1}{T}\int_0^T \int_{\mathbb{T}} |u^\nu(x+\ell,t) - u^\nu(x,t)|^p \mathrm{d}x\mathrm{d}t.$$

In turbulent flows, both incompressible and compressible, these objects develop scaling ranges

$$S_p^\nu(\ell,T) \sim |\ell|^{\zeta_p} \qquad \text{for} \quad \ell_\nu \ll \ell \ll 2\pi$$

while, mathematically, the aim is to establish uniform upper bounds $S_p^\nu(\ell,T) \leq C_p|\ell|^{\zeta_p}$ for all $|\ell| > 0$.

Theorem 1. *For each $T > 0$ and $p \geq 3$ and data $u_0 \in L^\infty(\mathbb{T})$, $f \in L^\infty(\mathbb{T})$, there is a constant $C_p := C_p(\|u_0\|_{L^\infty}, \|f\|_{L^\infty}, T)$ so that $\zeta_p \geq 1$ for all $p \geq 3$, that is for all $|\ell| > 0$ and $T > 0$, we have*

$$S_p^\nu(\ell,T) \leq C_p|\ell|. \tag{4}$$

If f is time-periodic, then C_p is independent of T. If $f = 0$, for $c_p := c_p(\|u_0\|_{L^\infty})$ we have uniform decay

$$S_p^\nu(\ell,T) \leq c_p \frac{|\ell|}{T}. \tag{5}$$

Condition (4) is equivalent to $⨍_0^T \|u^\nu(t)\|^p_{\dot{B}^{1/p}_{p,\infty}(\mathbb{T})} \mathrm{d}t \leq C_p$, where $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{T})} := \sup_{|\ell|<1} |\ell|^{-s}\|f(\cdot+\ell) - f\|_{L^p(\mathbb{T})}$ is the Besov semi-norm. As such, we have a self-regularization result for viscous solutions – from bounded data, they immediately enter into the space $X = \bigcap_{p\geq 3} L^\infty(0,T;B^{1/p}_{p,\infty}(\mathbb{T}))$ *uniformly* in the viscosity. Entropic shocks solutions saturate this regularity in that they live in X and no better space within the Besov scale.

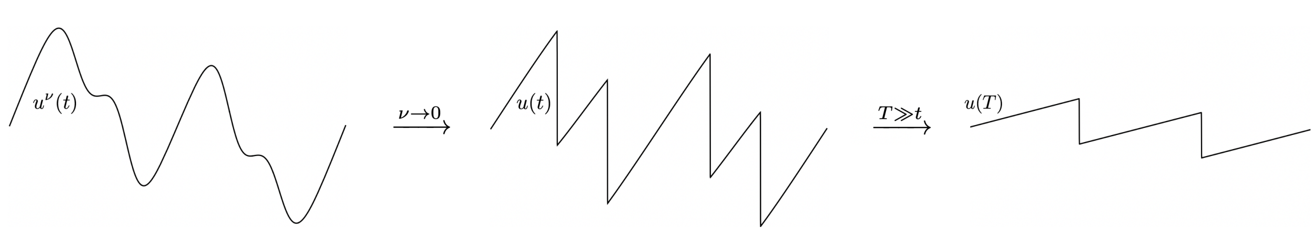

FIGURE 1. Burgers solution with $\nu > 0$, $f = 0$, first for small viscosity and subsequently for long times.

The above result itself is well known [5, 4, 8]; in fact there is a stronger uniform estimate for the bounded variation $L^\infty(0,T;(L^\infty\cap BV)(\mathbb{T}))\subset L^\infty(0,T;B^{1/p}_{p,\infty}(\mathbb{T}))$ for all $p\geq 1$. Our primary interest is in giving a short, intrinsically physical space, argument based on the following principle: nonlinear ideal conserved quantities that may be anomalously dissipated limit the degree to which the solution may suffer irregularities. A bound of this type was established by Goldman, Josien and Otto [3] using a modified energy flux. For Burgers, there are infinitely many such quantities (any convex function of the solution), suggesting there should be infinitely many such bounds (4), which we here show. These estimates show that pth-order absolute structure functions obey a uniform bound with $\zeta_p=1$ for $p\geq 3$. This fact underlies the rigidity of the so-called multifractal spectrum of anomalous exponents in Burgulence. For Navier-Stokes, the only known inviscid invariant which is dissipated is the kinetic energy. Its flux, related to the dissipation by the Kolmogorov $\frac{4}{5}$–law [6], is not coercive unlike the present setting. With an additional alignment hypothesis, the law does confer limited regularity [2]. Whether this is a generic feature of turbulence is open.

We remark that much of the analysis here could carry over to general one-dimensional conservation laws, even those with non-degenerate, nonlinear viscosity, taking the form

$$\partial_t u^\varepsilon+\partial_x h(u^\varepsilon)=\varepsilon\partial_x(\nu(u^\varepsilon)\partial_x u^\varepsilon),$$

provided, at least, that the Hamiltonian function $h(u)$ is sufficiently close to quadratic. For simplicity, we choose to focus here on the viscous Burgers equation. We require the identity (similar to that appearing in [7])

Lemma 1. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ have Lipschitz first derivative. Let Φ be the primitive of φ and $\tilde{\Phi}$ of $x\varphi'(x)$. Then*

$$\partial_t\varphi(\delta_\ell u)+\partial_x J_\ell[u]+\partial_\ell\Pi[\delta_\ell u]=-\nu\varphi''(\delta_\ell u)|\partial_x\delta_\ell u|^2+\varphi'(\delta_\ell u)\delta_\ell f \tag{6}$$

where we defined

$$\begin{aligned}J_\ell[u]&:=u'\varphi(\delta_\ell u)-\tilde{\Phi}(\delta_\ell u)-\nu\partial_x\varphi(\delta_\ell u),\\ \Pi[\delta_\ell u]&:=\tilde{\Phi}(\delta_\ell u)-\Phi(\delta_\ell u).\end{aligned}$$

PROOF OF LEMMA 1. To see this, let $u'=u(\cdot+\ell)$, $u=u(\cdot)$ and $\delta_\ell u=u'-u$. Then

$$\partial_t\delta_\ell u+u'\partial_x\delta_\ell u+\delta_\ell u\partial_\ell\delta_\ell u-\delta_\ell u\partial_x\delta_\ell u=\nu\Delta\delta_\ell u+\delta_\ell f,$$

since $u'\partial_x u'-u\partial_x u=\delta_\ell u\partial_x u'+u\partial_x\delta_\ell u=\delta_\ell u\partial_\ell\delta_\ell u+u\partial_x\delta_\ell u=\delta_\ell u\partial_\ell\delta_\ell u+u'\partial_x\delta_\ell u-\delta_\ell u\partial_x\delta_\ell u$. Multiplying the above by $\varphi'(\delta_\ell u)$, we obtain the evolution, which is equivalent to (6)

$$\partial_t\varphi(\delta_\ell u)+\partial_x(u'\varphi(\delta_\ell u))-\varphi'(\delta_\ell u)\delta_\ell u\partial_x\delta_\ell u+\Big(\varphi'(\delta_\ell u)\delta_\ell u-\varphi(\delta_\ell u)\Big)\partial_\ell\delta_\ell u=\nu\varphi'(\delta_\ell u)\Delta\delta_\ell u+\varphi'(\delta_\ell u)\delta_\ell f.$$

□

PROOF OF THEOREM 1. For $p\geq 3$ and $\alpha\in\mathbb{R}$, let $\varphi(x)=\alpha x|x|^{p-2}$. We compute $\varphi'(x)=\alpha(p-1)|x|^{p-2}$ and $\varphi''(x)=\alpha(p-1)(p-2)x|x|^{p-4}\in L^\infty$. The primitive of φ is $\Phi=\frac{\alpha}{p}|x|^p$ and for $x\varphi'(x)=(p-1)\varphi(x)$, it is $\tilde{\Phi}=(p-1)\Phi$. Thus $\Pi[x]=(p-2)\Phi(x)=\alpha\frac{p-2}{p}|x|^p$. Letting $\alpha=\frac{p}{p-2}$, we obtain the balance

$$\begin{aligned}\frac{1}{\ell}\int_0^T\!\!\int_{\mathbb{T}}|\delta_\ell u|^p\mathrm{d}x\mathrm{d}t=\frac{1}{T}\fint_0^\ell\!\!\int_{\mathbb{T}}\varphi(\delta_\ell u_0)\mathrm{d}x-\frac{1}{T}\fint_0^\ell\!\!\int_{\mathbb{T}}\varphi(\delta_\ell u_T)\mathrm{d}x+\fint_0^\ell\fint_0^T\!\!\int_{\mathbb{T}}\varphi'(\delta_{\ell'}u)\delta_{\ell'}f\mathrm{d}x\mathrm{d}t\mathrm{d}\ell'\\ -\nu\fint_0^\ell\fint_0^T\!\!\int_{\mathbb{T}}\varphi''(\delta_{\ell'}u)|\partial_x\delta_{\ell'}u|^2\mathrm{d}x\mathrm{d}t\mathrm{d}\ell'.\end{aligned}$$

Using energy balance (2) and the maximum principle (3) to bound each term on the right-hand-side, we obtain a uniform-in-viscosity bound. This yields the claimed estimates. If f is time-periodic, the uniform-in-time estimate follows from [5] which established $\|u(t)\|_{L^\infty}\leq C_0$ for a t independent constant C_0, improving the bound (3). When $f=0$, by (2) we have the decaying estimate

$$\left|\nu\fint_0^\ell\fint_0^T\!\!\int_{\mathbb{T}}\varphi''(\delta_{\ell'}u)|\partial_x\delta_{\ell'}u|^2\mathrm{d}x\mathrm{d}t\mathrm{d}\ell'\right|\leq\frac{2}{T}\|\varphi''\|_{L^\infty(0,\|u_0\|_{L^\infty})}\|u_0\|^2_{L^2},$$

which gives (5). This completes the proof. □

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