PRINCIPLE OF LEAST ACTION AND PERFECT FLUIDS

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ABSTRACT. In this note, we describe Arnold's variational formulation of the Euler equations as geodesic motion on the group of volume preserving diffeomorphisms.

Let $M \subset \mathbb{R}^d$ be a bounded simply connected open domain, possibly with boundary ∂M having exterior unit normal $\hat{n}$. The Euler equations governing the velocity $u(t,x) : [0,T] \times M \to \mathbb{R}^d$ of a fluid which is perfect and confined to M read

$$\partial_t u + u \cdot \nabla u = -\nabla p, \qquad \text{in} \quad M, \tag{0.1}$$

$$\nabla \cdot u = 0, \qquad \text{in} \quad M, \tag{0.2}$$

$$u|_{t=0} = u_0, \qquad \text{in} \quad M, \tag{0.3}$$

$$u \cdot \hat{n} = 0, \qquad \text{on} \quad \partial M \tag{0.4}$$

where $p(t,x) : [0,T] \times M \to \mathbb{R}$ is the hydrodynamic pressure which enforces incompressibility. We shall sometimes refer to M as the *fluid vessel*.

The Euler equations have a beautiful geometric interpretation. Let $\mathscr{D}_\mu(M)$ denote the group of smooth volume-preserving diffeomorphisms of M which leave the boundary invariant. This acts as the configuration space of the fluid, labelling particle positions. Perfect fluid motion is governed by the ODE for $t \mapsto \Phi_t$ in the space $\mathscr{D}_\mu(M)$:

$$
\begin{aligned}
\ddot{\Phi}_t(x) &= -\nabla p\,(t, \Phi_t(x)) & (t,x) &\in [0,T] \times M, \\
\Phi_0(x) &= x & x &\in M, \\
\Phi_t(\cdot) &\in \mathscr{D}_\mu(M) & t &\in [0,T],
\end{aligned}
\tag{0.5}
$$

In these equations, the acceleration (the pressure gradient) acts in keeping with its role as a constraint to enforce incompressibility. The system (0.5) can be thought of as arising from d'Alembert's principle of constrained motion, namely $\Phi_t \in \mathscr{D}_\mu$, $\dot{\Phi}_t \in T_{\Phi_t}\mathscr{D}_\mu$ (the tangent space to $\mathscr{D}_\mu$ at Φ_t consisting of divergence-free vector fields tangent to ∂M) and $\ddot{\Phi}_t \in (T_{\Phi_t}\mathscr{D}_\mu)^\perp$ (the orthogonal complement to the tangent space consisting of gradient vector fields). Said another way, the acceleration (the pressure gradient) is orthogonal to the constraint (that Φ_t remain in $\mathscr{D}_\mu$). The system (0.5) can be considered as the definition of perfect fluid motion.

Arnold interpreted the ODE (0.5) for $t \mapsto \Phi_t$ as a *geodesic* equation on $\mathscr{D}_\mu(M)$. To understand this view, fix $\gamma_1, \gamma_2 \in \mathscr{D}_\mu(M)$. Then, for any path $\gamma_\cdot : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ satisfying $\gamma_{t_1} = \gamma_1$ and $\gamma_{t_2} = \gamma_2$, define the action functional

$$\mathscr{A}[\gamma]_{t_1}^{t_2} := \int_{t_1}^{t_2} \int_M \frac{1}{2} |\dot{\gamma}_t(x)|^2 \mathrm{d}x \mathrm{d}t. \tag{0.6}$$

We take variations of $\mathscr{A}$ in path space as follows. Consider a smooth one-parameter family of paths $\gamma^\varepsilon_\cdot : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ for $\varepsilon \in (-1,1)$ with fixed endpoints $\gamma^\varepsilon_{t_1} = \gamma_1$ and $\gamma^\varepsilon_{t_2} = \gamma_2$ and where $\gamma^0 = \gamma$. Then we define the variation by

$$\delta\mathscr{A}[\gamma]_{t_1}^{t_2} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathscr{A}[\gamma^\varepsilon]_{t_1}^{t_2} \Big|_{\varepsilon=0}. \tag{0.7}$$

In order to compute this object we need the variation of the path, defined by

$$\delta\gamma_t(x) := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon(x)\right|_{\varepsilon=0}. \tag{0.8}$$

Fixing $x \in M$, the variation $\delta\gamma_\cdot(x) : [t_1, t_2] \mapsto T_{\gamma_\cdot(x)}M$ defines an element of the tangent space of the manifold at $\gamma_\cdot(x)$ (formally $\delta\gamma_\cdot : [t_1, t_2] \mapsto T_{\gamma_\cdot}\mathscr{D}_\mu(M)$ defines an element of the tangent space of $\mathscr{D}_\mu(M)$ along the path γ). Composing with γ^{-1}, $\delta\gamma \circ \gamma^{-1} : [t_1, t_2] \mapsto T_{\mathrm{id}}\mathscr{D}_\mu(M)$ gives an element of the tangent space to the volume preserving diffeomorphism group at the identity. Let $\mathfrak{X}_\mu(M)$ be the space of smooth divergence-free vector fields over M which are tangent to the boundary. The tangent space $T_{\mathrm{id}}\mathscr{D}_\mu(M)$ can be identified with $\mathfrak{X}_\mu(M)$. For our discussion, we require only that for any variation defined by pinned paths as above, it holds

$$\delta\gamma_t(\gamma_t^{-1}(x)) = v(t, x), \tag{0.9}$$

for some $v : [t_1, t_2] \mapsto \mathfrak{X}_\mu(M)$ with $v(t_1) = v(t_2) = 0$, and vice versa. The proof is elementary as everything is taken to be smooth:

Lemma 0.1. *Fix $\gamma_\cdot : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$. The following two statements hold*

(1) *Fix $v : [t_1, t_2] \mapsto \mathfrak{X}_\mu(M)$ with $v(t_1) = v(t_2) = 0$. There is a family $\gamma_\cdot^\varepsilon : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ for $\varepsilon \in (-1, 1)$ with $\gamma_{t_1}^\varepsilon = \gamma_1$, $\gamma_{t_2}^\varepsilon = \gamma_2$ and $\gamma^0 = \gamma$ such that (0.9) holds.*

(2) *Let $\gamma_\cdot^\varepsilon : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ for $\varepsilon \in (-1, 1)$ be paths with $\gamma_{t_1}^\varepsilon = \gamma_1$, $\gamma_{t_2}^\varepsilon = \gamma_2$ and $\gamma^0 = \gamma$. There exists $v : [t_1, t_2] \mapsto \mathfrak{X}_\mu(M)$ with $v(t_1) = v(t_2) = 0$ such that (0.9) holds.*

Proof. To establish the first direction, define the family $\gamma_\cdot^\varepsilon : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon(x) = v(t, \gamma_t^\varepsilon(x)), \qquad \gamma_t^0(x) = \gamma_t(x). \tag{0.10}$$

Since $v \in \mathfrak{X}_\mu(M)$, by Liouville's theorem it follows that $\det(\nabla\gamma_t^\varepsilon(x)) = 1$ and thus $\gamma^\varepsilon \in \mathscr{D}_\mu(M)$ for all ε. Note that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon\big|_{t_1} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon\big|_{t_2} = 0$ since v vanishes at those times so that $\gamma_{t_1}^\varepsilon = \gamma_1$, $\gamma_{t_2}^\varepsilon = \gamma_2$. According to definition (0.8), it follows (0.9) holds.

In the other direction, define

$$v^\varepsilon(t, x) := \left(\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon\right)((\gamma_t^\varepsilon)^{-1}(x)), \tag{0.11}$$

so that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_t^\varepsilon(x) = v^\varepsilon(t, \gamma_t^\varepsilon(x))$. Since γ_t^ε preserves volume, again by Liouville's theorem we have that $v^\varepsilon : [t_1, t_2] \mapsto \mathfrak{X}_\mu(M)$ for all $\varepsilon \in [0, 1)$. Moreover $v^\varepsilon(t_1) = v^\varepsilon(t_2) = 0$ since $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_{t_1}^\varepsilon = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_{t_2}^\varepsilon = 0$. With $v := v^0$, the claim follows. $\square$

With this in hand, we arrive at the formal variational principle:

Theorem 0.2 (Action principle for perfect fluid motion). *Let $\Phi_{t_1}, \Phi_{t_2} \in \mathscr{D}_\mu(M)$ be configurations of the fluid at times t_1 and t_2 with $t_1 < t_2$. A trajectory $\Phi_t : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ is an perfect fluid flow, i.e. $u := \dot\Phi_t \circ \Phi_t^{-1}$ satisfies equations (0.1)–(0.4), if and only if Φ is a critical point of the action $\mathscr{A}$, i.e.*

$$\delta\mathscr{A}[\Phi]_{t_1}^{t_2} = 0, \qquad with \qquad \delta\Phi_{t_1} = 0, \ \ \delta\Phi_{t_2} = 0. \tag{0.12}$$

Proof. Compute the first variation

$$\delta\mathscr{A}[\Phi]_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_M \dot\Phi_t(x)\cdot\delta\dot\Phi_t(x)\mathrm{d}x\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\int_M \ddot\Phi_t(x)\cdot\delta\Phi_t(x)\mathrm{d}x\mathrm{d}t \qquad (\text{using that } \delta\Phi_{t_1}=\delta\Phi_{t_2}=0).$$

According to Lemma 0.1, $\delta\Phi_t(\Phi_t^{-1}(x)) = v(x,t)$ for some divergence-free velocity field v which satisfies $v\cdot\hat n|_{\partial M} = 0$ and $v(t_1) = v(t_2) = 0$. This in hand, using that Φ_t^{-1} preserves volume and $\Phi_t(M) = M$ we write the variation of the action as

$$\delta\mathscr{A}[\Phi]_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_M \ddot\Phi_t(\Phi_t^{-1}(x))\cdot v(x,t)\mathrm{d}x\mathrm{d}t. \tag{0.13}$$

Assume first that $\delta\mathscr{A}[\Phi]_{t_1}^{t_2} = 0$, namely the action is stationary on Φ under any variation. By Lemma 0.1(1), the object (0.13) must vanish in particular for vector fields of the form $v(x,t) = f(t)\psi(x)$ for $f\in C_0^\infty(t_1,t_2)$ and $\psi\in\mathfrak{X}_\mu(M)$. Thus

$$0 = \int_{t_1}^{t_2} f(t)g(t)\mathrm{d}t, \qquad g(t) := \int_M \ddot\Phi_t(\Phi_t^{-1}(x))\cdot\psi(x)\mathrm{d}x \tag{0.14}$$

for all $\psi\in\mathfrak{X}_\mu(M), f\in C_0^\infty(t_1,t_2)$. Since g is continuous in time, we may take f to approximate g on $[t_1,t_2]$ to conclude that $g(t) = 0$ for each $t\in[t_1,t_2]$ (the fundamental lemma of calculus of variations). We deduce for each $t\in[t_1,t_2]$ that

$$0 = \int_M \ddot\Phi_t(\Phi_t^{-1}(x))\cdot\psi(x)\mathrm{d}x\mathrm{d}t, \qquad \forall\psi\in\mathfrak{X}_\mu(M). \tag{0.15}$$

The arbitrariness of $\psi\in\mathfrak{X}_\mu(M)$ together with the Hodge decomposition allows us to conclude the existence of $p(t,x) : [t_1,t_2]\times M\to\mathbb{R}$ such that

$$\ddot\Phi_t(x) = -\nabla p(t,\Phi_t(x)), \qquad \forall t\in[t_1,t_2]. \tag{0.16}$$

Since $\ddot\Phi_t(x) = (\partial_t u + u\cdot\nabla u)(\Phi_t(x),t)$ where $u = \dot\Phi_t\circ\Phi_t^{-1}$, we see that (0.16) implies that u solves Euler. Contrariwise, if u solves the Euler equations then (0.16) holds with p as the pressure field so that $\delta\mathscr{A}[\Phi]_{t_1}^{t_2} = 0$ by (0.13) and $v\in\mathfrak{X}_\mu(M)$. $\qquad\square$

Next we show that for short times the Euler flow minimizes the action. This fact was pointed out by Ebin and Marsden [8, Section 9]. The following version is due to Y. Brenier in [4, Section 5] or [6, Proposition 3.2].

Theorem 0.3 (Perfect fluid flow minimizes the action for short times). *Let $u\in C^1([0,T]\times\overline M)$, $p\in C([0,T];C_x^2(\overline M))$. Suppose that $T > 0$ is such that*

$$T^2 \leqslant \pi^2/K \tag{0.17}$$

with $K := \sup_{t\in[0,T]}\sup_{x\in\overline M}\sup_{|y|=1} y\cdot\nabla^2 p(x,t)\cdot y$. If (u,p) is a solution of the Euler (0.1)–(0.4) and $\dot\Phi_t = u(\Phi_t,t)$ with $\Phi_0 = \mathrm{id}$, then

$$\mathscr{A}[\Phi]_0^T \leqslant \mathscr{A}[\gamma]_0^T \tag{0.18}$$

among all $\gamma. : [0,T]\mapsto\mathscr{D}_\mu(M)$ with $\gamma_0 = \mathrm{id}$ and $\gamma_T = \Phi_T$. If $T^2 < \pi^2/K$, equality holds if and only if $\gamma = \Phi$.

From a geometric standpoint, the pressure gradient is the second fundamental form – encoding how the submanifold of volume preserving diffeomorphisms $\mathscr{D}_\mu(M)$ sits inside the ambient group of all diffeomorphisms $\mathscr{D}(M)$ [13].

Proof. Since (u, p) is an Euler solution, we have $\ddot{\Phi}_t(x) = -\nabla p(\Phi_t(x))$. Note first

$$\mathscr{A}[\Phi]_0^T = \mathscr{A}[\gamma]_0^T + \int_0^T \int_M \dot{\Phi}_t(x) \cdot (\dot{\Phi}_t(x) - \dot{\gamma}_t(x)) \mathrm{d}x\mathrm{d}t - \mathscr{A}[\Phi - \gamma]_0^T. \qquad (0.19)$$

The last two terms are negative for short time. Indeed, by Poincaré's inequality,[†]

$$\mathscr{A}[\Phi - \gamma]_0^T = \frac{1}{2} \int_0^T \int_M |\dot{\Phi}_t(x) - \dot{\gamma}_t(x)|^2 \mathrm{d}x\mathrm{d}t \geqslant \frac{\pi^2}{2T^2} \int_0^T \int_M |\Phi_t(x) - \gamma_t(x)|^2 \mathrm{d}x\mathrm{d}t.$$

On the other hand, since $\Phi_0(x) = \gamma_0(x)$ and $\Phi_T(x) = \gamma_T(x)$, we have

$$\int_0^T \int_M \dot{\Phi}_t(x) \cdot (\dot{\Phi}_t(x) - \dot{\gamma}_t(x)) \mathrm{d}x\mathrm{d}t = \int_0^T \int_M \ddot{\Phi}_t(x) \cdot (\Phi_t(x) - \gamma_t(x)) \mathrm{d}x\mathrm{d}t$$

$$= -\int_0^T \int_M \nabla p(\Phi_t(x)) \cdot (\Phi_t(x) - \gamma_t(x)) \mathrm{d}x\mathrm{d}t.$$

Since the pressure is twice differentiable, by Taylor's theorem we have

$$p(\gamma_t(x)) = p(\Phi_t(x)) + \nabla p(\Phi_t(x)) \cdot (\Phi_t(x) - \gamma_t(x))$$
$$+ \frac{1}{2}(\Phi_t(x) - \gamma_t(x)) \cdot \nabla^2 p(Y_t(x)) \cdot (\Phi_t(x) - \gamma_t(x)),$$

where $Y_t(x)$ is on the chord connecting $\gamma_t(x)$ to $\Phi_t(x)$. Upon integrating, using the fact that γ_t and Φ_t preserve volume, we obtain

$$-\int_0^T \int_M \nabla p(\Phi_t(x)) \cdot (\Phi_t(x) - \gamma_t(x)) \mathrm{d}x\mathrm{d}t$$

$$= \frac{1}{2} \int_0^T \int_M (\Phi_t(x) - \gamma_t(x)) \cdot \nabla^2 p(Y_t(x)) \cdot (\Phi_t(x) - \gamma_t(x)) \mathrm{d}x\mathrm{d}t.$$

It follows that

$$\left| \int_0^T \int_M \dot{\Phi}_t(x) \cdot (\dot{\Phi}_t(x) - \dot{\gamma}_t(x)) \mathrm{d}x\mathrm{d}t \right| \leqslant \frac{K}{2} \int_0^T \int_M |\Phi_t(x) - \gamma_t(x)|^2 \mathrm{d}x\mathrm{d}t. \qquad (0.20)$$

Thus we obtain the upper bound

$$\mathscr{A}[\Phi]_0^T \leqslant \mathscr{A}[\gamma]_0^T + \frac{1}{2}\left(K - \frac{\pi^2}{T^2} \right) \int_0^T \int_M |\Phi_t(x) - \gamma_t(x)|^2 \mathrm{d}x\mathrm{d}t, \qquad (0.21)$$

whence for $T \leqslant \pi^2/K$ we have $\mathscr{A}[\Phi]_0^T \leqslant \mathscr{A}[\gamma]_0^T$ as claimed. If equality holds, then from (0.21) we deduce $\int_0^T \int_M |\Phi_t(x) - \gamma_t(x)|^2 \mathrm{d}x\mathrm{d}t = 0$ so that $\gamma = \Phi$ as claimed. $\qquad \square$

Remark 0.4 (Failure to be a minimizer). The condition (0.17) on the time is sharp in the following senses. Consider the two-dimensional example of solid body rotation, i.e. $u(x) = x^\perp$. This is an exact stationary solution of the Euler equations having pressure $p(x) = \frac{1}{2}|x|^2$ on a disc domain. The corresponding flowmap is $\Phi_t(x) = \mathsf{R}_{t \bmod 2\pi} x$ where R_θ denotes the (counterclockwise) rotation matrix by angle $\theta \in [0, 2\pi)$ about 0. Brenier

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[†]Indeed, consider any absolutely continuous curve $f : [0, T] \mapsto \mathbb{R}^d$ with $f(0) = f(T) = 0$ and with $\mathrm{d}f(t)/\mathrm{d}t \in L^2([0, T])$. Using the Fourier series representation of f together with Plancherel's theorem we find immediately that $\|f(\cdot)\|_{L^2([0,T])}^2 \leqslant \frac{T^2}{\pi^2} \|\frac{\mathrm{d}}{\mathrm{d}t} f(\cdot)\|_{L^2([0,T])}^2$.

[4] points out that at time $T = \pi$ (a half rotation of the disk), there fails to be a unique minimizer of the action. Indeed, the action does not depend on whether the rotation is clockwise or counter clockwise, both of which are geodesics connecting these two states. Note that since $\nabla^2 p = I$ we have $K = 1$. Thus, at exactly this moment, $T = \pi$, the condition (0.17) is violated illustrating its sharpness. For $T > \pi$, there exists a shorter path (just rotate clockwise) and thus after this moment, the original fluid flow is not the minimum of the action any longer. We remark that solid body rotation has a *cut point* at $T = \pi$ – see [7] for further discussion of this geometric notion. It is also an example of *isochronal* flow: one for which the Lagrangian flowmap is time periodic. Geometrically, this corresponds to Φ being a *closed geodesic* in $\mathscr{D}_\mu(M)$ [16].

Remark 0.5 (Euler as geodesic motion on $\mathscr{D}_\mu(M)$). We now describe V.I. Arnold's geometric picture in greater detail. Formally, one can view the space $\mathscr{D}_\mu(M)$ as an infinite-dimensional manifold with the metric inherited from the embedding in $L^2(M; \mathbb{R}^d)$, and with tangent space made by the divergence-free vector fields tangent to the boundary of M. We can define the length of a path $\gamma. : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ by the expression

$$\mathscr{L}[\gamma]_{t_1}^{t_2} := \int_{t_1}^{t_2} \|\dot{\gamma}_t(\cdot)\|_{L^2(M)} \mathrm{d}t. \tag{0.22}$$

We formally define the *geodesic distance* connecting two states $\gamma_1, \gamma_2 \in \mathscr{D}_\mu(M)$ by

$$\mathrm{dist}_{\mathscr{D}_\mu(M)}(\gamma_0, \gamma_1) = \inf_{\substack{\gamma. : [0,1] \mapsto \mathscr{D}_\mu(M) \\ \gamma(0) = \gamma_0, \ \gamma(1) = \gamma_1}} \mathscr{L}[\gamma]_0^1. \tag{0.23}$$

A geodesic curve $\Phi. : [t_1, t_2] \mapsto \mathscr{D}_\mu(M)$ is defined to be one so that for all $t_1 \in \mathbb{R}$ there exists a $\tau > 0$ such that if $t_1 < t_2 < t_1 + \tau$ then

$$\mathrm{dist}_{\mathscr{D}_\mu(M)}(\Phi(t_1), \Phi(t_2)) = \mathscr{L}[\Phi]_{t_1}^{t_2}. \tag{0.24}$$

If additionally the parametrization by t is chosen so that $\|\dot{\Phi}_t(\cdot)\|_{L^2(M)} = (\mathrm{const.})$, then Φ minimizes the action (0.6) among all smooth paths connecting $\Phi(t_1)$ and $\Phi(t_2)$. Indeed by Schwarz's inequality, we have $\mathscr{A}[\gamma]_{t_1}^{t_2} \geqslant (\mathscr{L}[\gamma]_{t_1}^{t_2})^2 / 2(t_2 - t_1)$ with equality if and only if $\|\dot{\gamma}_t(\cdot)\|_{L^2(M)} = (\mathrm{const.})$. In general, according to Theorem 0.2 perfect fluid motion is a critical point of both functionals $\mathscr{A}[\gamma]_{t_1}^{t_2}$ and $\mathscr{L}[\gamma]_{t_1}^{t_2}$. It is, in fact, a geodesic according to Theorem 0.3, although as we say in Remark 0.4 it need not be the curve of minimal length for long times.
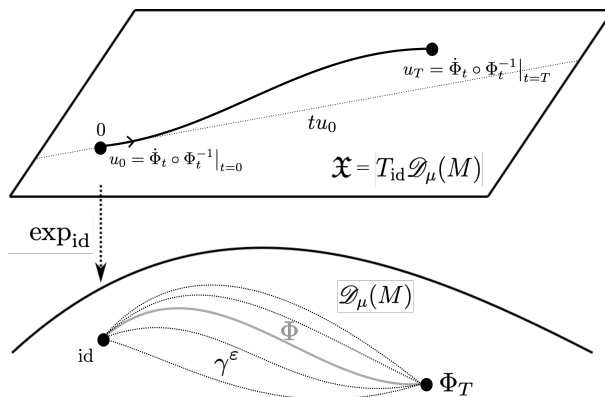

FIGURE 1. Depiction of the geometry of fluid motion.

The above discussion is somewhat formal in that it ignores issues of regularity required for precise definitions of the variations. To make things more precise, one may consider the group $\mathscr{D}_\mu^s(M)$ for $s > d/2 + 1$, which is a submanifold of all H^s diffeomorphisms of M, see [8]. The L^2 exponential map is defined as the solution map of the geodesic equations: it maps lines through the origin in the tangent space at a given diffeomorphism onto geodesics in the diffeomorphism group. More precisely, at the identity we set

$$\exp_{\mathrm{id}} : T_{\mathrm{id}}\mathscr{D}_\mu^s \to \mathscr{D}_\mu^s, \qquad \exp_{\mathrm{id}} tu_0 = \Phi_t, \quad t \in \mathbb{R} \tag{0.25}$$

where Φ_t is the unique L^2 geodesic (at least for short times) starting from id with velocity u_0. We note that in any spatial dimension, $\exp_{\mathrm{id}}$ is a local diffeomorphism near id and in two dimensions, it is defined on the whole tangent space. The study of analytical properties of this map is the subject of the classical work of Ebin and Marsden [8]. This framework opened up the possibility to ask purely geometric questions about fluid motion, such as those concerning the existence of conjugate and cut points. See [14, 19, 7]. We remark also that Arnold's geometric viewpoint has since been generalized to accommodate systems such as compressible and quantum fluids, by the inclusion of an appropriate potential related to material properties of the system. See the recent survey [11].

Remark 0.6 (Two-point fluid problem). The principle of least action suggests the following 'infinite dimensional Dirichlet principle': given two isotopic configurations γ_1 and $\gamma_2 \in \mathscr{D}_\mu$, construct a perfect fluid flow connecting them by identifying the shortest path between them in the diffeomorphism group (in the L^2 metric). From the above discussion, if such a path exists it is automatically a perfect fluid flow. This problem was first investigated by Shnirelman [18], where he proved that if $d \geqslant 3$, this variational problem does not have minimizers for all pairs of configurations γ_1 and γ_2. In $d = 2$, this is open even in the following relaxed form:

Open Problem 0.7. *Let $M \subset \mathbb{R}^2$ be a domain with smooth boundary. Does there exist perfect fluid flow connecting any two isotopic states $\gamma_1, \gamma_2 \in \mathscr{D}_\mu(M)$?*

See also [10, Section 4.B]. In particular, it is not known whether the image of the space of incompressible vector fields in $\mathscr{D}_\mu$ defined by the L^2 exponential map is the whole group $\mathscr{D}_\mu$, i.e. that any such diffeomorphism could be realized as a time-1 value of some solution of the Euler-Lagrange equation. It is a question of accessibility of the entire configuration space by perfect fluid flows. If Question 0.7 is answered in the affirmative, it would represent a hydrodynamical analogue of the Hopf-Rinow theorem for the group of diffeomorphisms. The difficulty is that this group is not locally compact. However, Misiolek, and Preston proved that the L^2 exponential map is a covering space map on an open connected component $\mathcal{U} \subset \mathscr{D}_\mu^s$ of the identity whose L^2 diameter is infinite, cf. [15]. This is a consequence of the fact that $\exp_{\mathrm{id}}$ is a nonlinear Fredholm map of index zero, see [9, 20]. An affirmative resolution of Open Question 0.7 (conjectured by Shnirelman in [18]) would result from showing this connected component is the whole group. We remark finally that a different but related question is that of finding minimizers (shortest paths) connecting the states γ_1 and γ_2. The existence of conjugate and cut points along geodesics generated by simple steady solutions on $M = \mathbb{T}^2$ having streamfunction $\psi(x_1, x_2) = \sin(nx_1)\sin(mx_2)$ (Kolmogorov flows) [14, 7] indicate that the minimum may fail to be achieved at late times (see the numerical study [12]), as was conjectured in [17].

In view of the fact that classical minimizers of the two-point problem need not always exist [18], Brenier introduced *generalized flows*, which are a wider class of (stochastic) objects over which the variation problem is always solvable [4]. Shnirelman used this idea to show that any sufficiently long geodesic in $\mathscr{D}_\mu^s(M)$ will contain a local cut point if $\dim M = 3$ (a point such that shorter paths can be chosen arbitrarily close to the given geodesic in the manifold topology).

References

[1] L.Ambrosio and A.Figalli, *Geodesics in the space of measure-preserving maps and plans*, Arch. Ration. Math. Anal., **194** (2009), no. 2, 421–462.

[2] V.Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble), **16** (1966), fasc. 1, 319–361.

[3] V.Arnold and B.A.Khesin, (2008). *Topological methods in hydrodynamics*, (Vol. 125). Springer Science & Business Media.

[4] Y.Brenier, *The least action principle and the related concept of generalized flows for incompressible perfect fluids,* J. Amer. Mat. Soc., **2** (1989), 225–255.

[5] Y.Brenier, *The dual least action problem for an ideal, incompressible fluid,* Arch. Rational Mech. Anal., **122** (1993), 323–351.

[6] Y.Brenier, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations,* Comm. Pure Appl. Math., **52** (1999), 411–452.

[7] T.D.Drivas, G.Misiołek, B.Shi, and T.Yoneda (2021). Conjugate and cut points in ideal fluid motion. arXiv preprint arXiv:2105.11869.

[8] D.G.Ebin and J.Marsden, *Groups of diffeomorphisms and the motion of an ideal incompressible fluid,* Annals of Math., **2** (1970), 102–163.

[9] D.G.Ebin, G.Misiołek and S.Preston, *Singularities of the exponential map on the volume-preserving diffeomorphism group*, Geom. funct. anal. **16** (2006), 850-868.

[10] B.Khesin. Topological fluid dynamics. Notices of the AMS, 52:1 (2005), 9-19;

[11] B.Khesin, G.Misiolek and K.Modin, Geometric hydrodynamics and infinite-dimensional Newton's equations. Bulletin of the Amer. Math. Soc., vol.58, no.3 (2021), 377-442.

[12] R.Lan, The Minimizer of the Dirichlet Integral, Diss. Concordia University, 2012.

[13] G.Misiolek, Stability of flows of ideal fluids and the geometry of groups of diffeomorphisms, Indiana Univ. Math. J. 42 (1993), 215-235.

[14] G.Misiołek, *Conjugate points in $\mathscr{D}_\mu(\mathbb{T}^2)$*, Proc. Amer. Math. Soc. **124** (1996), 977-982.

[15] G.Misiołek and S.Preston, *Fredholm properties of Riemannian exponential maps on diffeomorphism groups*, Invent. math. **179** (2010), 191-227.

[16] D.Serre. Sur le principe variationnel des équations de la mécanique des fluides parfaits. ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 27(6), 739-758 (1993).

[17] A.I.Shnirelman, *Attainable diffeomorphisms,* Geometric & Functional Analysis GAFA 3.3 (1993): 279-294.

[18] A.I.Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid (Russian),* Mat. Sb. (N.S.), **128 (170)** (1985), no. 1, 82–109.

[19] A.I.Shnirelman, *Generalized fluid flows, their approximation and applications,* Geom. Funct. Anal., **4** (1994), no. 5, 586–620.

[20] A.I.Shnirelman, *Microglobal analysis of the Euler equations.* Journal of mathematical fluid mechanics 7.3 (2005): S387-S396.

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