Mathematical Aspects of Turbulence

Outline: Lecture 3 : Phenomenology and Mathematics of fully developed turbulence

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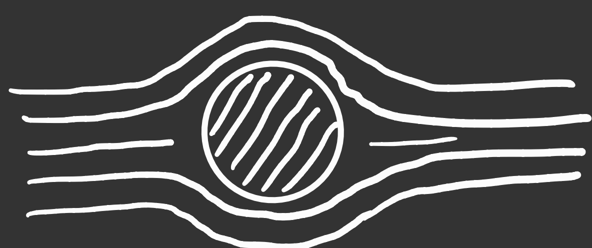
D'Alembert's Paradox: in an irrotational Euler flow, the drag force on a body moving with constant velocity relative to the fluid is zero!

"planes cannot fly in an irrotational Euler flow"



Eulerian picture

Reality

Solution: One must take into account friction forces between adjacent molecules, i.e. viscosity

C.L. Navier (1822) and G. Stokes (1845) derived a model for this under the assumption that the shear stress is proportional to the symmetric part of the gradient:

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f$$
$$\nabla \cdot u = 0$$

- the parameter $\nu > 0$ is the kinematic viscosity of the fluid
- the function $f(x,t)$ is an external body force.
- widely accepted model of Newtonian fluid flow, arising in the joint limit of small Knudsen number and Mach number

Kinetic energy is dissipated:
$$\frac{1}{2}\|u^\nu(t)\|_{L^2}^2 - \frac{1}{2}\|u^\nu(0)\|_{L^2}^2 = -\nu \int_0^t \int |\nabla u^\nu|^2 \, dx \, dt =: \varepsilon^\nu$$

Non-dimensionalization (physical laws hold independent of units)

- U: characteristic velocity of the flow, e.g. rms $\left(\int |u|^2\right)^{1/2}$.
- L: characteristic length in the flow, e.g. domain size or period

Note, all the terms in NS have units of acceleration $\frac{L^3}{U^2}$.

Non-dimentionalizing $u \mapsto u/U$, $x \to x/L$, $t \to t/_{L/U}$

$$\partial_t u + u\cdot\nabla u = -\nabla p + \frac{1}{Re}\Delta u$$

$$\nabla\cdot u = 0$$

The non-dimentional number

$$Re = \frac{UL}{\nu} \sim \frac{u\cdot\nabla u}{\nu\Delta u}$$

is the Reynolds number. It measures relative strength of inertial forces (nonlinearity) and viscous forces:

- bacteria $\quad Re \approx 10^{-5}$
- blood flow $\quad Re \approx 10^{2}$
- MLB pitch $\quad Re \approx 10^{5}$
- wake of blue whale $\quad Re \approx 10^{8}$
- wake of Boeing 747 $\quad Re \approx 10^{12}$

$\vdots$

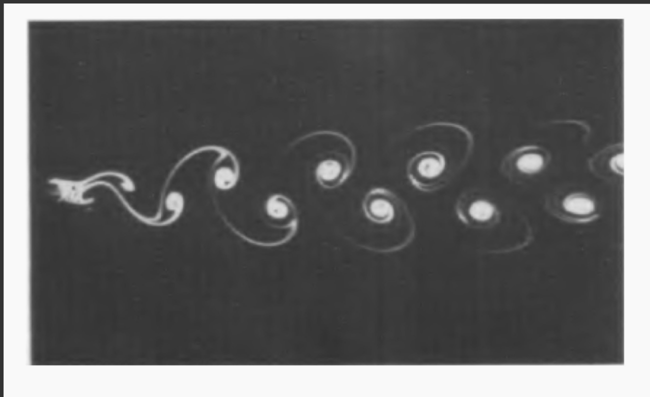
Note: in experiments, one often uses the Taylor-scale Re

$$Re_\lambda = \frac{U\lambda}{\nu}, \quad \lambda^2 = \frac{\int |u|^2}{\int |\nabla u|^2}$$

Typically $Re_\lambda \gg Re$.

Van Kármán vortex street behind cylinder.

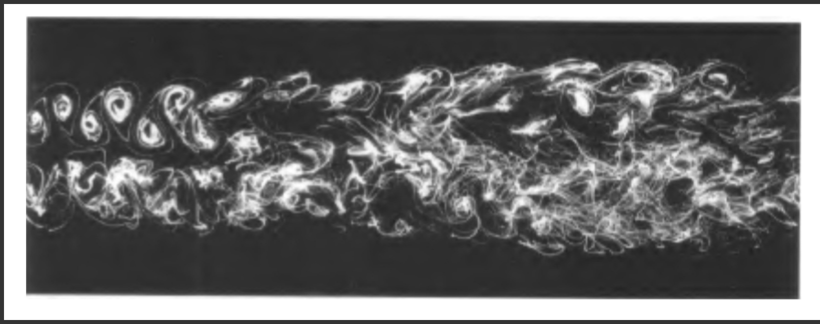
Van Dyke (1982)
Frisch (1995)



Re = 105

Wake behind two cylinders

Frisch (1995)



Re = 240

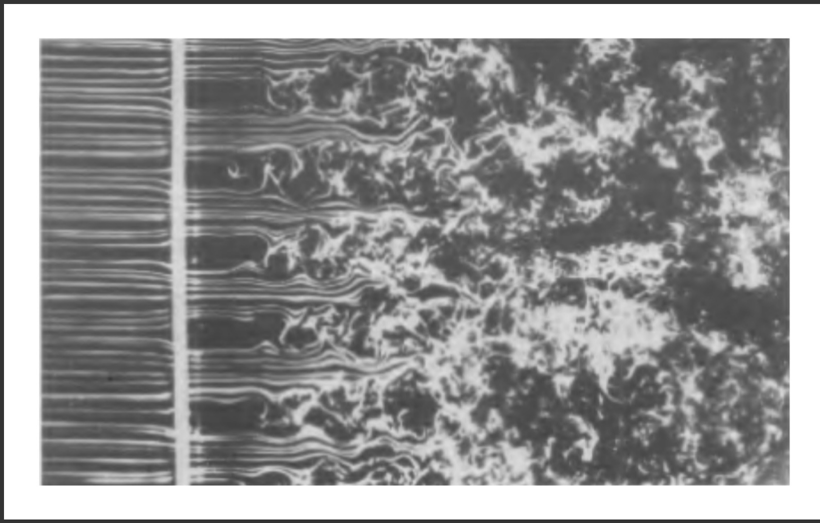
Wake behind two cylinders

Frisch (1995)

Re = 1800

Homogeneous turbulence behind grid.

Frisch (1995)



Re = 2300

What happens as $Re \to \infty$? (without boundary)

- If a classical solution of Euler exists, u_*, then $u^\nu \to u_*$ strongly. Moreover $\varepsilon^\nu = \mathcal{O}(\nu)$.

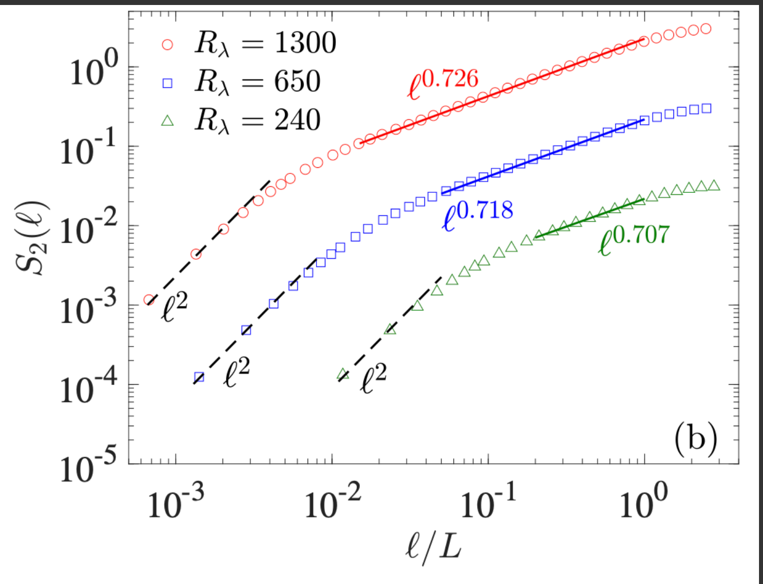
- If classical solution does not exist because either
  - finite time singularity
  - $u_0^\nu \to u_0$ = rough data
    $f^\nu \to f$ = rough forcing

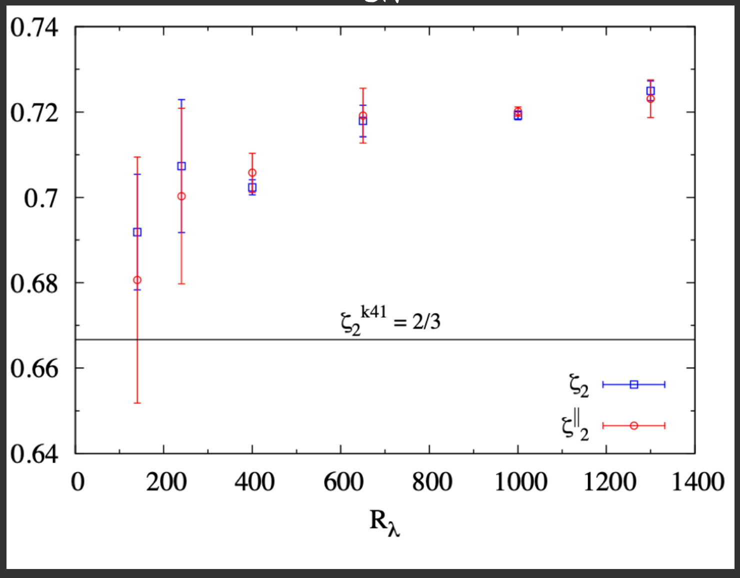
Then assuming $\{u^\nu\}$ cmp in L^2, then $u^\nu \xrightarrow{L^2} u$, a weak solution of Euler

$$(u, \partial_t \phi)_{L^2_{x,t}} + (u \otimes u ; \nabla \phi)_{L^2_{t,x}} = 0 \qquad \forall \phi \in C_0^\infty, \ \operatorname{div} \phi = 0$$

$$(u ; \nabla \phi)_{L^2_{x,t}} = 0$$

Evidence: $S_2(\ell) = \sup_{|r| \le \ell} \| u(\cdot + \ell) - u(\cdot) \|^2_{L^2_{t,x}} \lesssim \ell^{\zeta_2}$





Being a weak solution, momentum is conserved (take $\phi_i \to 1$ on the domain). But energy need not

In fact, assuming $\{u^\nu\}$ cmp in L^3, from

$$\partial_t\left(\frac{1}{2}|u^\nu|^2\right) + \operatorname{div}\left(\left(\frac{|u^\nu|^2}{2} + p^\nu\right)u^\nu - \nu\nabla\frac{|u^\nu|^2}{2}\right) = -\nu|\nabla u^\nu|^2 =: -\varepsilon^\nu$$

we deduce LHS $\xrightarrow{\nu\to 0}$ to

Calderon–Zygmund $p^\nu \to p$ in $L^{3/2}$

$$\partial_t\left(\frac{1}{2}|u|^2\right) + \operatorname{div}\left(\left(\frac{|u|^2}{2} + p\right)u\right)$$

Thus, in the sense of distributions, we have

$$\varepsilon^\nu \to \varepsilon[u] := \partial_t\left(\frac{1}{2}|u|^2\right) + \operatorname{div}\left(\left(\frac{|u|^2}{2} + p\right)u\right)$$

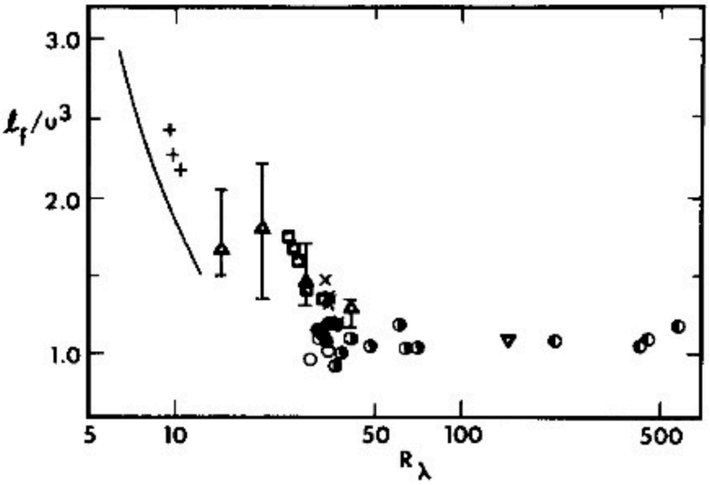
If $(\varepsilon, \varphi) > 0$, we say anomalous dissipation occurs.

Thus, $u^\nu \to u$ could be a dissipative Euler solution

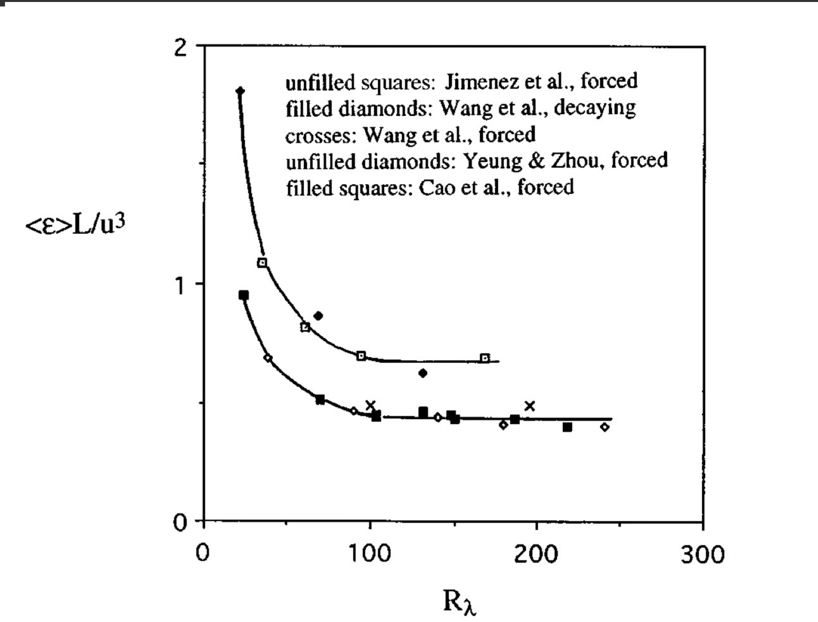
$$\frac{1}{2}\int |u(t)|^2 - \frac{1}{2}\int |u(0)|^2 = -\int_0^T\!\!\int \varepsilon[u] < 0.$$

But there is another way to compute energy balance of the weak solution...

Sreenivasan, 1998



IG. 1. The quantity $\epsilon L_f/u^3$ for biplane square-mesh grids. All data except
– are for the initial period of delay, and are explained in Table I. + indi-
ate typical data[13] in the final period of decay. — corresponds to Eq. (1).



Pearson, Krogstad, de Water, (2001)

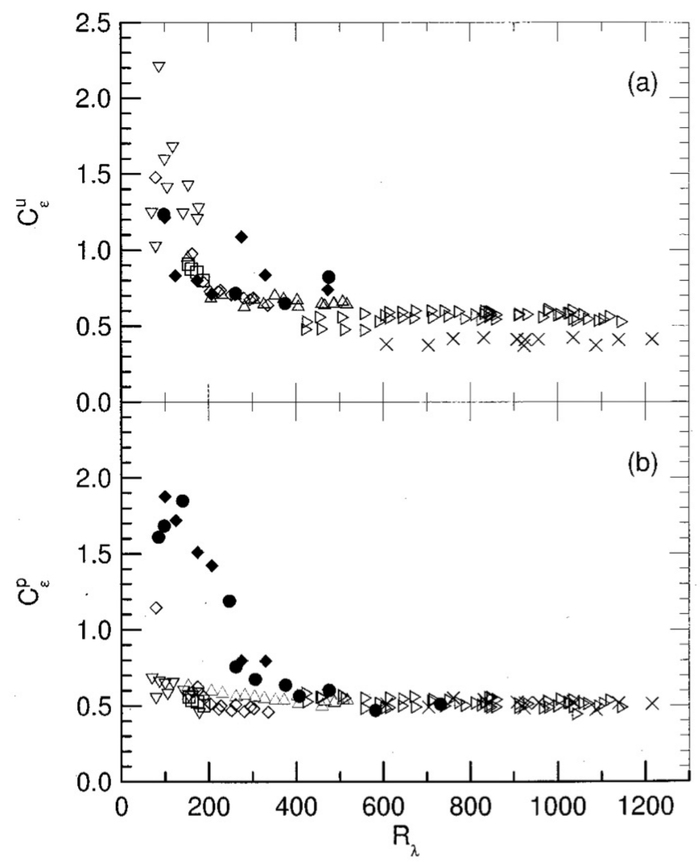
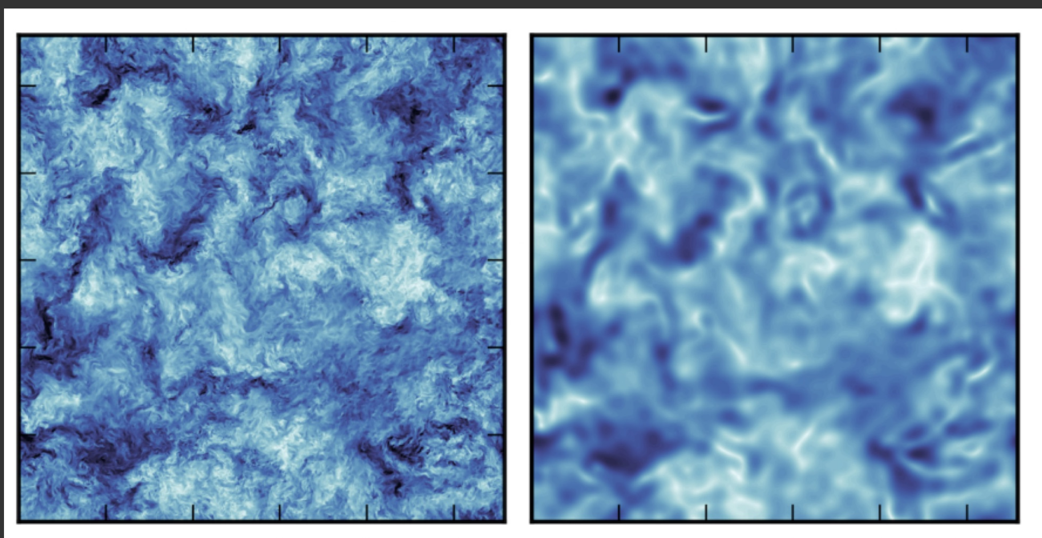


FIG. 1. Normalized dissipation rate for a number of shear flows. Details as found in this work and Refs. 14–16. (a) C^u_ϵ [Eq. (3)]; (b) C^p_ϵ [Eq. (4)]. □, circular disk, $154 \lesssim R_\lambda \lesssim 188$; ▽, pipe, $70 \lesssim R_\lambda \lesssim 178$; ◇, normal plate, $79 \lesssim R_\lambda \lesssim 335$; △, NORMAN grid, $174 \lesssim R_\lambda \lesssim 516$; × NORMAN grid (slight mean shear, $dU/dy \approx dU/dy|_{max}/2$), $607 \lesssim R_\lambda \lesssim 1217$; ▷, NORMAN grid (zero mean shear), $425 \lesssim R_\lambda \lesssim 1120$; ●, "active" grid Refs. 14, 15, $100 \lesssim R_\lambda \lesssim 731$; ◆, "active" grid, with L_u estimated by Ref. 16. For Ref. 14 data, we estimate $L_p \approx 0.1$ m and for Ref. 15 data we estimate $L_p \approx 0.225$ m.

Energy transfer through scale

Coarse-graining $\bar{u}_\ell(x) = \int G_\ell(r)\, u(x+r)\, dr$,



$G_\ell(r) = \ell^{-d}\, G\left(\frac{r}{\ell}\right)$

$G \geq 0 \quad \int G = 1 \quad G(x) = G(-x)$

$\operatorname{supp} G \subseteq B_1(0)$

$$\partial_t \bar{u}_\ell + \nabla \cdot (\bar{u}_\ell \otimes \bar{u}_\ell) = -\nabla \bar{p}_\ell - \nabla \cdot \tau_\ell(u,u)$$

$$\nabla \cdot \tau_\ell(u,u) := \overline{(u \otimes u)}_\ell - \bar{u}_\ell \otimes \bar{u}_\ell$$

Reynolds stress

Energy balance in scales $\geq \ell$:

$$\partial_t\left(\tfrac{1}{2}|\bar{u}_\ell|^2\right) + \operatorname{div}\left(\left(\tfrac{1}{2}|\bar{u}_\ell|^2 + \bar{p}_\ell\right)\bar{u}_\ell + u_\ell \tau_\ell\right) = -\Pi_\ell[u]$$

$$\Pi_\ell[u] = -\nabla \bar{u}_\ell : \tau_\ell(u,u)$$

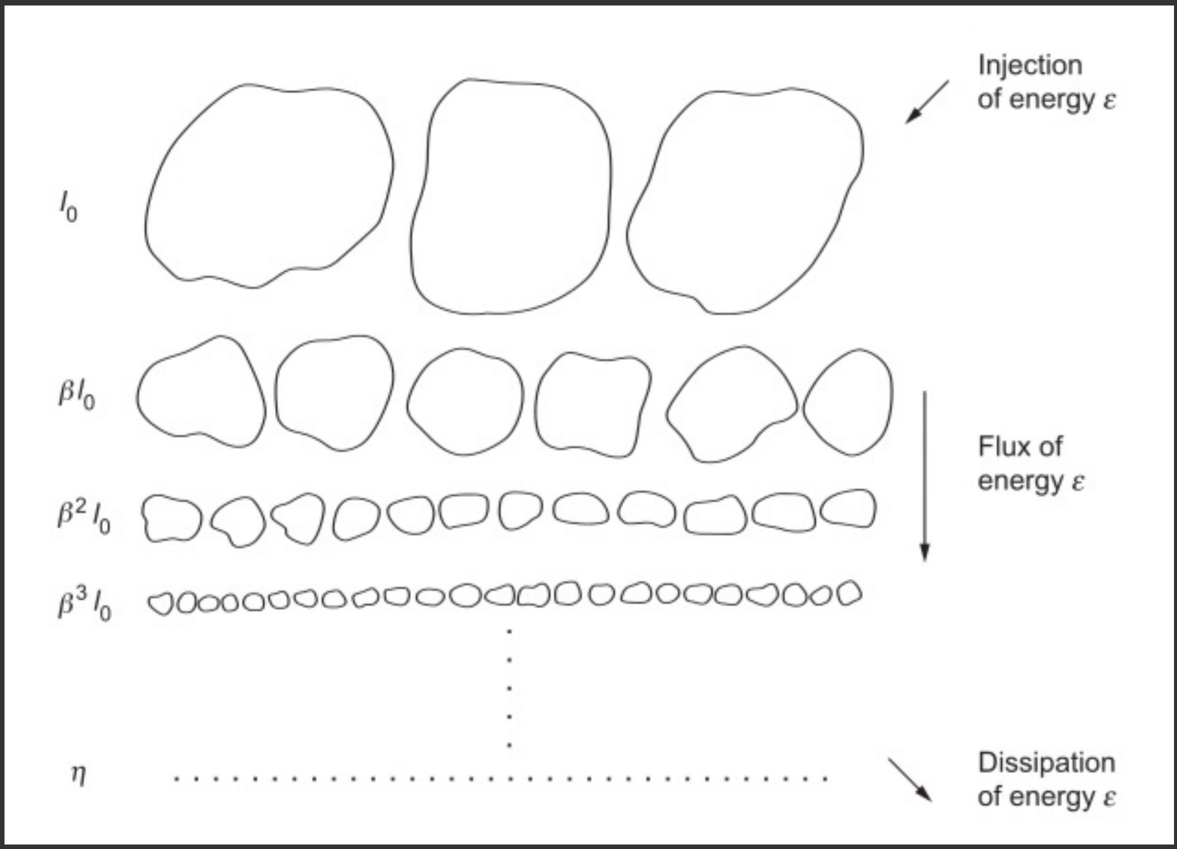
energy flux through scale

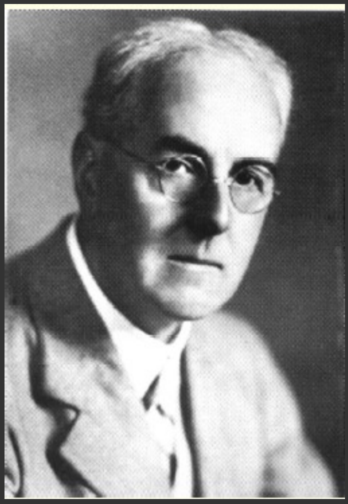
- <u>Locality of cascade:</u> one can show that

energy flux $\Pi_\ell[u]$

has contributions primarily from a band of scales $[\ell-\delta, \ell+\Delta]$ using Littlewood-Paley

Eyink, 2005
Constantin-Cheskidov-Friedlander-Shvydkoy, 2008





Big whirls have little whirls,That feed on their velocity;And little whirls have lesser whirls,And so on to viscosity

We wish to study the limit $\ell \to 0$.

Note that, since $u \in L^3$, $\bar{u}_\ell \xrightarrow{L^3} u$, $\bar{P}_\ell \xrightarrow{L^{3/2}} P$ strongly since

$$\bar{u}_\ell(x) - u(x) = \int G_\ell(r)\left(u(x+r) - u(x)\right) dr$$

$$\|\bar{u}_\ell - u\|_{L^3} \leq \sup_{|r| \leq \ell} \|\delta_r u\|_{L^3} \xrightarrow{\ell \to 0} 0.$$

Moreover

$$\tau_\ell(u,u) = \overline{(u \otimes u)}_\ell - \bar{u}_\ell \otimes \bar{u}_\ell$$

$$= \left\langle (u - \bar{u}_\ell(x)) \otimes (u - \bar{u}_\ell(x)) \right\rangle_G$$

where $\langle f \rangle_G = \int f(r) G(r) dr$. Thus

$$\|\bar{u}_\ell \, \tau_\ell(u,u)\|_{L^1} \leq \|\bar{u}_\ell\|_{L^3} \|\tau_\ell\|_{L^{3/2}} \leq \|u\|_{L^3} \sup_{r \leq \ell} \|\delta_r u\|_{L^3}^2 \to 0$$

Thus,

$$\partial_t\left(\tfrac{1}{2}|\bar{u}_\ell|^2\right) + \operatorname{div}\left(\left(\tfrac{1}{2}|\bar{u}_\ell|^2 + \bar{P}_\ell\right)\bar{u}_\ell + u_\ell \tau_\ell\right)$$

$$\to \partial_t\left(\tfrac{1}{2}|u|^2\right) + \operatorname{div}\left(\left(\frac{|u|^2}{2} + P\right)u\right) = -\lim_{\ell \to 0} \Pi_\ell[u]$$

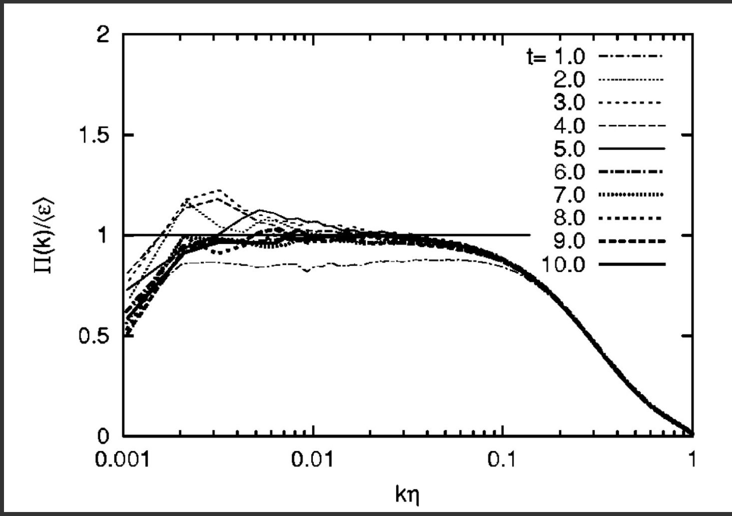
So $$\lim_{\ell \to 0} \Pi_\ell[u] = D[u]$$ ← inertial dissipation

Duchon–Robert '00

Thus we obtain an inviscid formula for $\mathcal{E}$

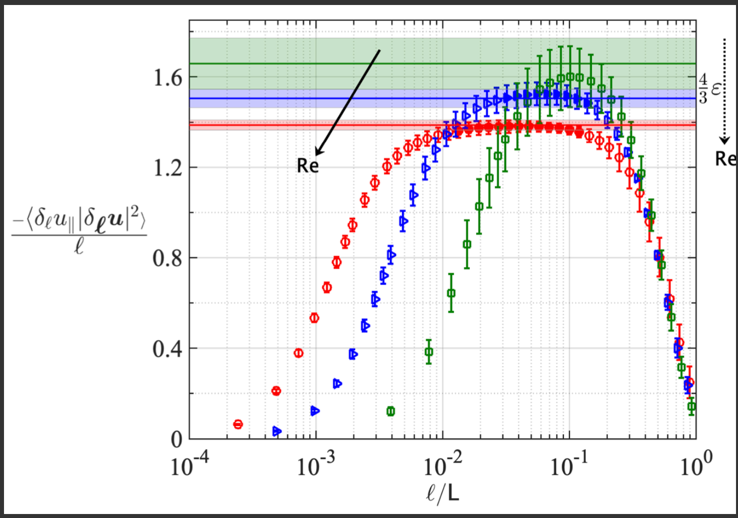
$$\boxed{\lim_{\nu \to 0} \nu |\nabla u|^2 = \mathcal{E}[u] = D[u] = \lim_{\ell \to 0} \Pi_\ell[u]}$$

Evidence of balance



$$\boxed{\varepsilon[u] = D[u]}$$

Letting $\varphi = \bar{u}_\ell$ be the test function, we obtain

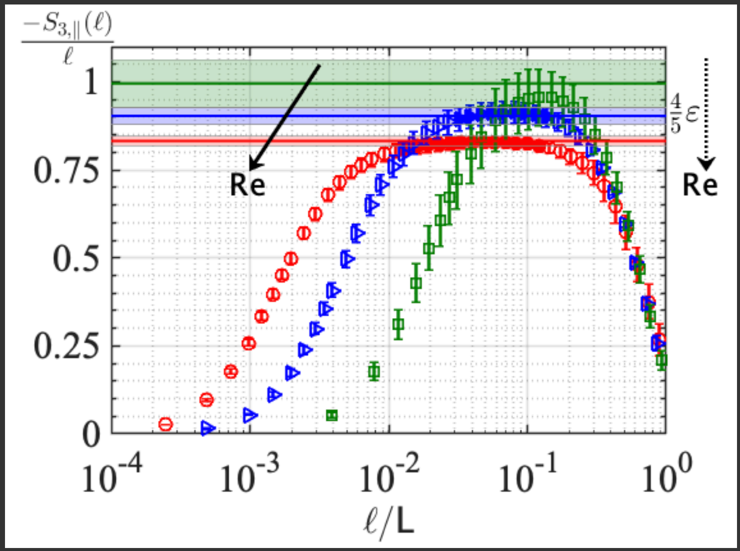


$$D[u] = -\lim_{\ell\to 0} \frac{1}{4}\int \nabla G(r)\cdot \frac{\delta_{\ell z} u}{\ell} |\delta_{\ell z} u|^2 \, dz$$

letting $G(z) \to \frac{\mathbb{1}_{B_1(0)}}{|B_1(0)|}$,

$$D[u] = -\lim_{\ell\to 0}\frac{d}{4}\int_{S^{d-1}} \frac{z\cdot\delta_{\ell z}u(x)}{\ell}|\delta_{\ell z}u(x)|^2 \, d\sigma(z)$$

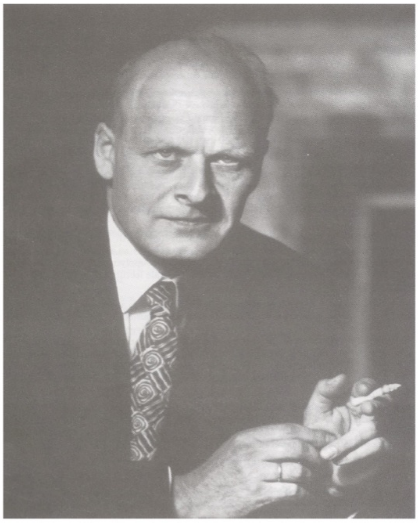
Kolmogorov-$\frac{4}{3}$ law: $\langle \delta_\ell u\cdot \ell \, |\delta_\ell u|^2\rangle \sim -\frac{4}{3}\varepsilon$



Further manipulation shows (Eyink, Novack)

$$D[u] = -\lim_{\ell\to 0}\frac{d(d+2)}{12}\int_{S^{d-1}}\frac{(\delta_{\ell z}u(x)\cdot z)^3}{\ell} \, d\sigma(z)$$

Kolmogorov-$\frac{4}{5}$ law: $\langle(\delta_\ell u\cdot \ell)^3\rangle \sim -\frac{4}{5}\varepsilon$



Lars Onsager (1903-1976)

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

(26) $|\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')| < (\text{const.})r^n$

for any order n greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description...

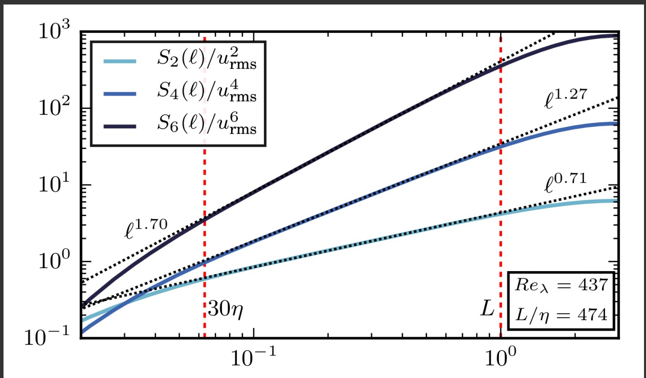
"Statistical Hydrodynamics" (1949)

$$\boxed{\varepsilon[u] = -\lim_{\ell\to 0} \frac{d(d+2)}{12} \int_{S^{d-1}} \frac{(\delta_{\ell z} u(r)\cdot z)^3}{\ell}\, d\sigma(z)}$$

$u \in B^{\sigma}_{p,\infty}$ for $p \geq 1$

iff $\begin{cases} u \in L^p \\ \|\delta_\ell u\|_{L^p} \leq |\ell|^\sigma \quad \forall\, |\ell| < 1 \end{cases}$

Note : $S_p(\ell) = \|\delta_\ell u\|^p_{L^p}$.



THEOREM: (Eyink 1992, Constantin-E-Titi 1994)

Let $u \in L^3(0,T; B_3^{\sigma,\infty}(\mathbb{T}^d)) \cap C(0,T; L^2(\mathbb{T}^d))$ with $\sigma > 1/3$, then

$$\frac{1}{2}\int_{\mathbb{T}^d} |u(x,t)|^2\, dx = \frac{1}{2}\int_{\mathbb{T}^d} |u_0(x)|^2\, dx \qquad \forall\, t \in [0,T].$$

Flexible Side of Onsager's Conjecture

THEOREM (Isett, 2018, Buckmaster - De Lellis - Szekelyhidi - Vicol 19)
Let $e:[0,T]\to\mathbb{R}$ be a strictly positive smooth function. For any $\alpha\in(0,1/3)$, there exists a weak solution $u\in C^\alpha([0,T]\times\mathbb{T}^3)$ of the Euler equations with

$$\int \frac{1}{2}|u(x,t)|^2\,dx = e(t) \qquad \forall\ t\in[0,T]$$

Long history: Scheffer 1993, Shnirelman 1997, 2000, De Lellis - Szekelyhidi 2009-2011, 2012, Buckmaster - De Lellis - Szekelyhidi 2013, 2014

Built off ideas of Nash-Kuiper Theorem and Gromov's h-principle. (Buckmaster - Vicol 2020, De Lellis - Szekelyhidi, 2011 great reviews!)

<u>Ideas</u>: Inverse Renormalization group (Frisch)

- Stages $S_0, S_1, \ldots, S_q, \ldots$ adding ever smaller motions ($2\pi\to 2\pi/2\to\ldots\to 2\pi/2^q\to\ldots$)

$$\partial_t\bar{u}_{\ell_q}+\nabla\cdot(\bar{u}_{\ell_q}\otimes\bar{u}_{\ell_q}) = -\nabla\bar{p}_{\ell_q}+\nabla\cdot\tau_{\ell_q}$$

$$\tau_\ell = \overline{(u\otimes u)}_\ell-\bar{u}_\ell\otimes\bar{u}_\ell \geq 0$$

where $\bar{u}_\ell = \sum_{|k|<\ell^{-1}} e^{ik\cdot x}\,\hat{u}(t)$ and $\ell := \lambda^{-q}$

Must show $\tau_{\ell_q}\to 0$ as $q\to\infty$.

Remarks:

subsolution

↓

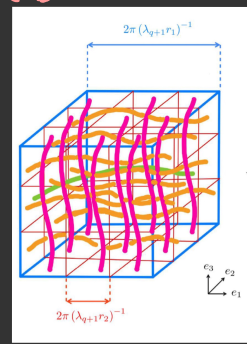
weak solution

↑

high freq oscillations introduce stresses in passing to weak limits

- Because the equation for subsolutions is highly underconstrained, it is easy to construct them.
- An iteration process reintroduces high-wavenumber oscillations, perturbations designed to cancel low frequencies of old stress.
- Difficulty lies in controlling error terms. Accomplished by judicious choice of building blocks.
- oscillations introduced in highly non-unique way $\Rightarrow$ infinitely many solutions.

(B.M.N.V 21')



- Solutions are "monofractal" in that the velocity has just one exponent h, which can be $1/3^-$. They have "Kolmogorov-like" spectra (not quite).

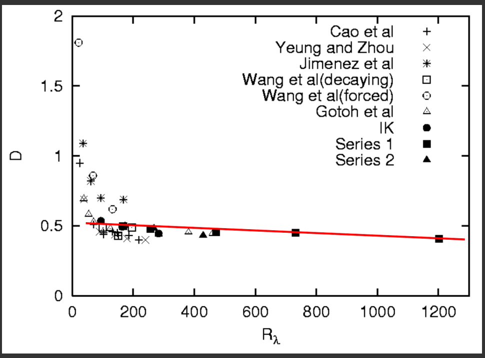
- Notable recent exception: Buckmaster-Masmoudi-Novak-Vicol, 21
  Solutions with $>1/3$ derivative in L^2 but $<1/3$ derivative in L^3 constructed. Towards more realistic flows!

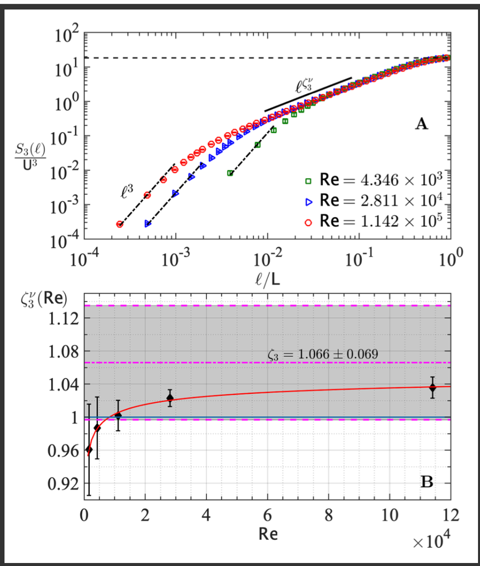
- High degree of non-uniqueness! Dissipative Euler solutions do not provide a predictive theory alone. Must consider viscosity!

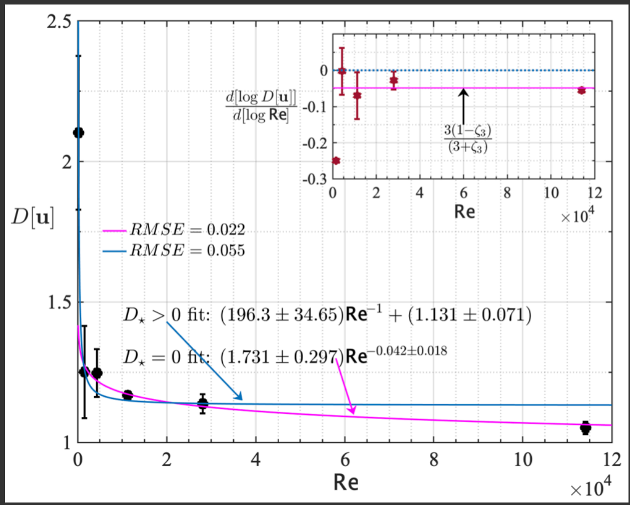
Wrinkle: no anomaly without boundary in steady state?

THEOREM: (Drivas – Eyink 2018) if $\{u^\nu\}_{\nu>0}$ and $\sigma\in(0,1]$,

$$\sup_{\nu>0}\|u^\nu\|_{L^3_t B^{\sigma,\infty}_3}<\infty \implies \int_0^T\int_{\mathbb{T}^d} \nu|\nabla u^\nu|^2\,dx\,dt \lesssim \nu^{\frac{3\sigma-1}{\sigma+1}}.$$







MATHEMATICAL GOAL

dissipative anomaly (1) There is data $u_0\in L^2$, $T>0$ and $\varepsilon>0$ s.t.

$$\int_0^T\int_{\mathbb{T}^3}\nu|\nabla u|^2\,dx\,dt > \varepsilon$$

weak compactness (2) The family $\{\bar u^\nu\}_{\nu>0}$ is compact and along subsequences converge to weak Euler

Onsager Conjecture (3) These Euler solutions exhibit constant mean flux $\langle\Pi(\bar u)\rangle=\varepsilon$ and live in $L^3_t B^{1/3,\infty}_3$

Moreover, this behavior should be generic (statistically stationary)

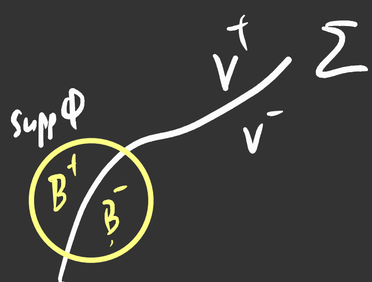
Note that the assumption $u \in B^{1/3+}_{3,\infty}$ is overkill, as only a piece of the increment appears, and the flux is not coersive

$$\boxed{\mathcal{E}[u] = -\lim_{\ell \to 0} \frac{d(d+2)}{12} \int_{S^{d-1}} \frac{(\delta_{\ell z} u(x) \cdot z)^3}{\ell} \, d\sigma(z)}$$

Thus, if the singularity set of a weak solution $u \notin B^{1/3+}_{3,\infty}$ is structured, dissipation may be impossible!

Example: Regular Vortex sheets

Consider v.f. $v : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ which is smooth away from a codimension-1 surface S. Let Σ be an arbitrary surface



If $\nabla_{x,t} \cdot v \overset{L^\infty_{t,x}}{=} \mu \geq 0$ (measure), then

$$\mu(\Sigma) = \int_\Sigma (v^+ - v^-) \cdot \hat{n} \, d\mathcal{H}^{d-1}$$

Proof: $\int \varphi \, d\mu = \int_B \varphi \, \mathrm{div}\, v = -\int_B v \cdot \nabla \varphi = -\int_{B^+} v \cdot \nabla \varphi - \int_{B^-} v \cdot \nabla \varphi$

$= \int_\Sigma v^+ \cdot n \varphi \, d\mathcal{H}^{d-1} - \int_\Sigma v^- \cdot n \, \varphi \, d\mathcal{H}^{d-1} + \int_{B^+ \cup B^-} \varphi \, d\mu$. $\square$

① $\nabla_{x,t} \cdot (1, u) = 0$ (1 mass)

② $\nabla_{x,t} \cdot (u, u \otimes u + pI)$ (d momenta)

③ $\nabla_{x,t} \cdot \left(\frac{1}{2}|u|^2, \left(\frac{1}{2}|u|^2 + p\right) u\right) = -D[u]$ (1 energy)

From ① & ②, we deduce (with $n = (n_t, n_x)$)

①' $[u \cdot n_x] = 0$ so: $u^+ \cdot n_x = u^- \cdot n_x =: u_n$

②' $[u\, n_t] + [u\, u_n + p\, n_x] = 0$

From ②', dotting with n_x gives

②'' $[p] = 0$ (contact discontinuity)

whence ②' becomes

$$[u]\,(u_n + n_t) = 0$$

Either $u_n + n_t = 0$ or not. If $u_n + n_t \neq 0$ then $[u] = 0$, so $D(\Sigma) = 0$. If $u_n + n_t = 0$, energy gives

$$-D(\Sigma) = \left[\tfrac{1}{2}|u|^2\right] n_t + \left[\tfrac{1}{2}|u|^2 u_n\right] = \left[\tfrac{1}{2}|u|^2\right](u_n + n_t) = 0$$

Thus $D(\Sigma) = 0$ for every surface Σ. $\Rightarrow$ $D[u] = 0$.

This result is due to Shvydkoy, and crucially uses incompressibility. Note that such a vortex sheet lives in $B^{1/3}_{3,\infty}$ exactly!

Indeed, $BV \cap L^\infty \hookrightarrow B^{1/p}_{p,\infty}$ sharply since

$$\| u(\cdot + \ell) - u(\cdot) \|_{L^1_x} \lesssim |\ell| \quad \& \quad \|u\|_{L^\infty} \leq C$$

then, by interpolation,

$$\| u(\cdot + \ell) - u(\cdot) \|_{L^p_x} \lesssim |\ell|^{1/p} \qquad \text{for all } p \geq 1.$$

Thus, these solutions are "critical", yet conservative.

$$|D_\ell[u]|_{L^1} = \left| \frac{1}{4} \int \nabla G(z) \cdot \frac{\delta_{\ell z} u}{\ell} |\delta_{\ell z} u|^2 \, dz \right|_{L^1} \lesssim \sup_{r \leq \ell} \frac{\|\delta_r u\|}{r} \|u\|^2_{L^\infty} \leq C$$

De Rosa & Inversi made a beautiful observation:

> **Theorem:** If $u \in BV \cap L^\infty$ is a weak solution, then it conserves energy

To do so, they exploited freedom in G, and Alberti's Lemma.

Kolmogorov 1941 Theory

For $\ell_\nu \ll \ell \ll L$, if $\mathcal{E}_\ell = \int_{B_\ell(x,t)} d\mathcal{E}[u]$

$$\boxed{-\frac{4}{5}\,\mathcal{E}_\ell[u] \approx \int_{S^{d-1}} \frac{(\delta_{\ell z} u(r)\cdot z)^3}{\ell}\, d\sigma(z)}$$

- $\varepsilon = \liminf_{Re\to\infty} \varepsilon^{Re} > 0$ zeroth law

$$C_p\, \mathcal{E}_\ell[u]^{p/3} \approx \int_{S^{d-1}} \left(\frac{\delta_{\ell z} u(r)\cdot z}{\ell^{1/3}}\right)^p d\sigma(z)$$ homogeneity and isotropy

Integrating in space-time

$$\int_0^T\int_{\mathbb{T}^d}\int_{S^{d-1}} \left(\frac{\delta_{\ell z} u(r)\cdot z}{\ell^{1/3}}\right)^p d\sigma(z)\, dx\, dt = C_p \int_0^T\int_{\mathbb{T}^d} \mathcal{E}_\ell[u]^{p/3}\, dx\, dt$$

"no intermittency" $\approx C_p \left(\int_0^T\int_{\mathbb{T}^d} \mathcal{E}_\ell[u]\, dx\, dt\right)^{p/3}$

$= C_p \langle \varepsilon \rangle^{p/3}$

Thus, prediction:

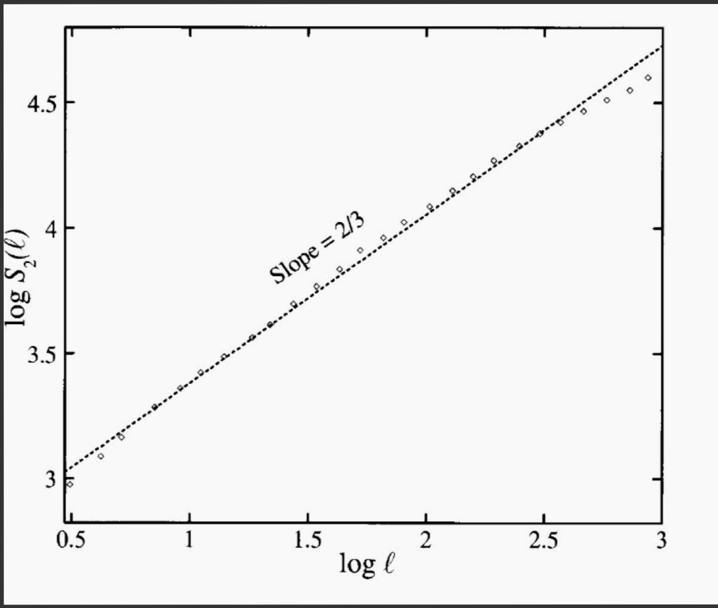
$$S_p^{\parallel}(\ell) = \left\langle \left(\delta_{\ell z} u(x,t)\cdot z\right)^p \right\rangle_{x,\,ang} \equiv C_p \left(\ell \langle \varepsilon\rangle\right)^{p/3}$$

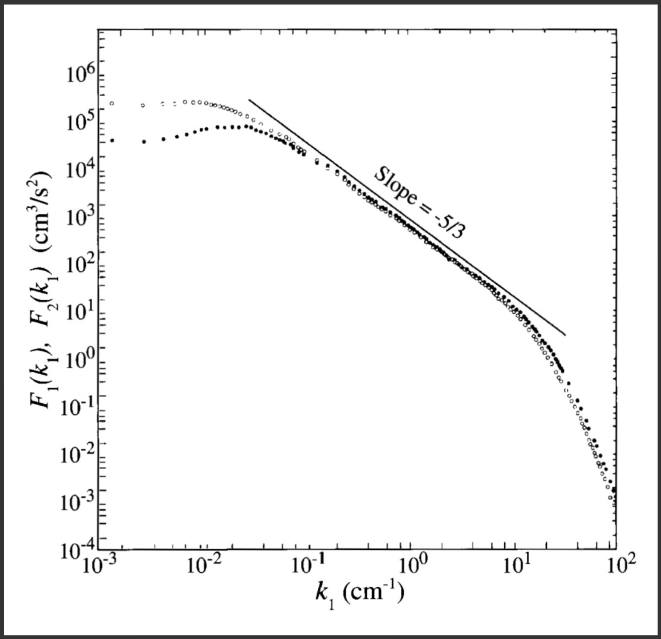
$$= C_p\, \ell^{\zeta_p^{K41}}, \quad \zeta_p^{K41} = \frac{p}{3}.$$

$$S_2^{\parallel}(\ell) = \left\langle \left(\delta_{\ell z} u(x,t)\cdot z\right)^2 \right\rangle_{x,\,ang} \equiv C_p \left(\ell \langle \varepsilon \rangle\right)^{2/3},$$

Connection with Kolmogorov Spectra

"K41" $\zeta_2 = \frac{2}{3}$ 2/3rds law $\Longleftrightarrow$ $-5/3$ law





$$E(k) = \sum_{p\in\mathbb{Z}^d} \delta(k-|p|)\,|\hat{u}(k)|^2$$

By Wiener-Khinchin theorem

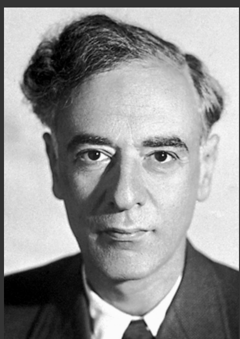
$$S_2(\ell) \sim \ell^{2s} \Longleftrightarrow E(k) \sim k^{-(2s+1)}$$

$s = 1/3$ $\quad E(k) \sim \varepsilon^{2/3} k^{-5/3} \Longleftrightarrow S_2(\ell) \sim (\varepsilon \ell)^{2/3}$

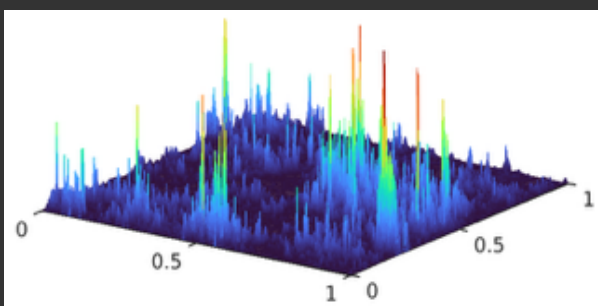
Recall $1/3$ is the Onsager exponent, which is maximal regularity consistent with anomalous dissipation

Landau's Remark and Intermittency

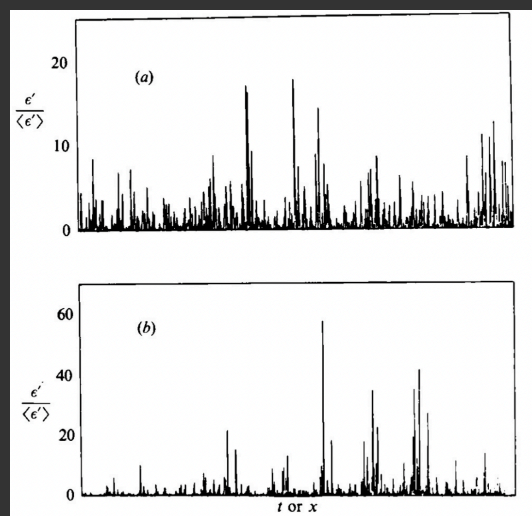
Landau: 1942

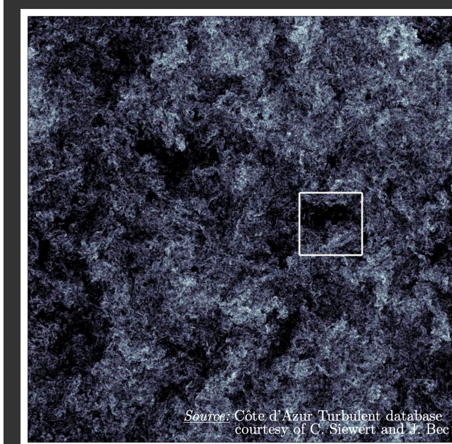


The rate of energy dissipation is intermittent. I.e., it is spatially / temporally inhomogeneous.



Meneveau & Sreenivasan 1991





Namely, $\varepsilon[u]$ can be a wild measure, giving supported on lower dimensional sets... We should expect

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon_\ell[u]^p \, dx\,dt \sim \langle \varepsilon \rangle^p \left(\frac{\ell}{L}\right)^{-\alpha_p} \qquad p \geq 1, \quad \alpha_p > 0.$$

Thus ζ_p should not be a constant multiple of p:

$$S_p^{\parallel}(\ell) = \left\langle \left(\delta_{\ell z} u(x,t) \cdot z\right)^p \right\rangle_{x,\,ang} \approx C_p \left(\ell \langle \varepsilon \rangle\right)^{p/3} \left(\frac{\ell}{L}\right)^{\alpha_p}$$

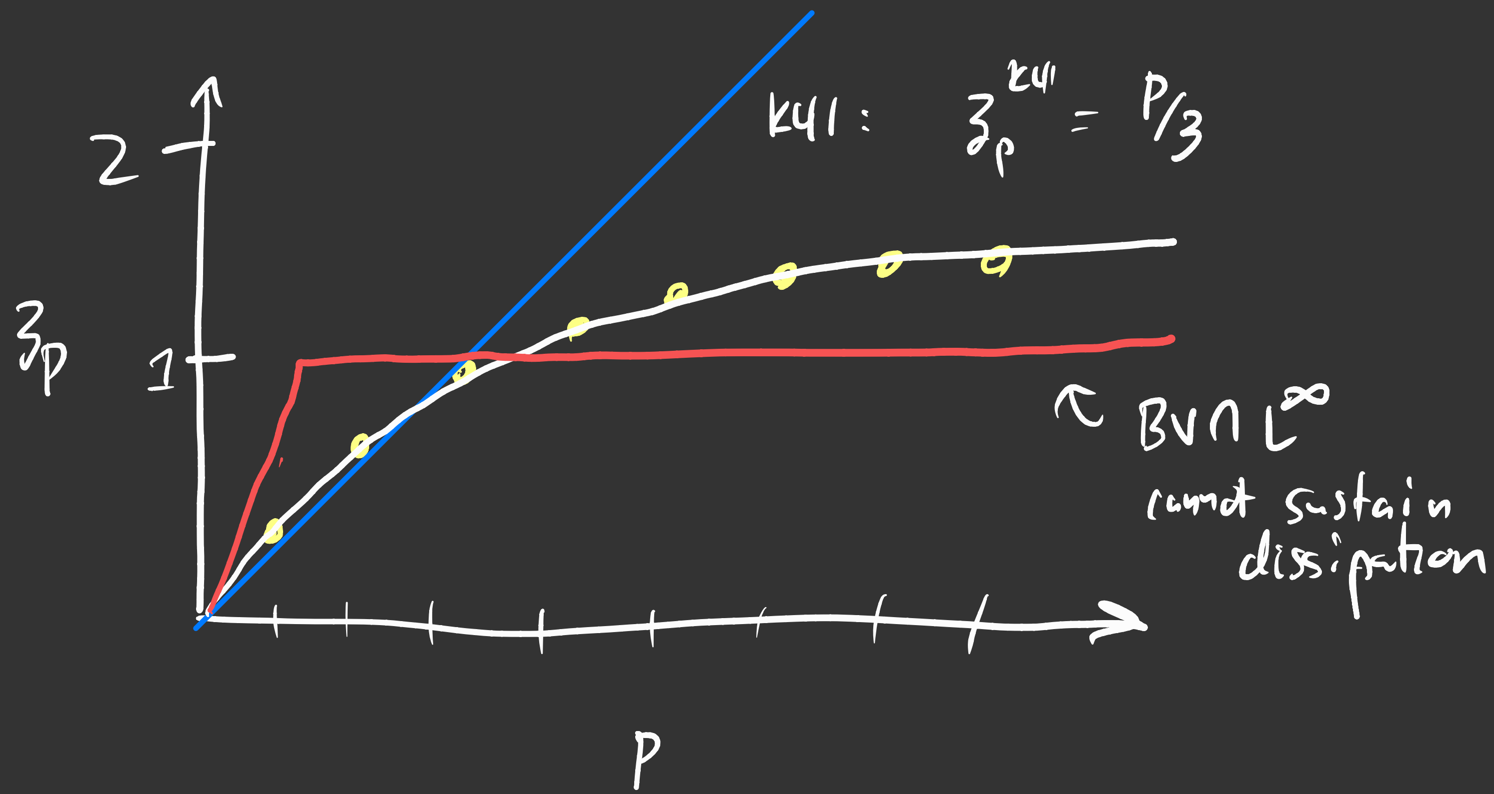
so that $S_p^{\parallel}(\ell) \sim \ell^{\zeta_p}$ where $\zeta_p = \frac{p}{3} - \delta\zeta_p$.

dimensional analysis

K41: unique scaling so $S_p^{\parallel}(\ell)$ independent of largescale L.

Conflating $S_p''(\ell)$ with $S_p(\ell)$

$$S_p(\ell) = \langle |\delta_\ell u|^p \rangle_{x,t} \sim \ell^{\zeta_p}$$



This phenomenon of regularity depending on moment is a manifestation of <u>intermittency</u>.

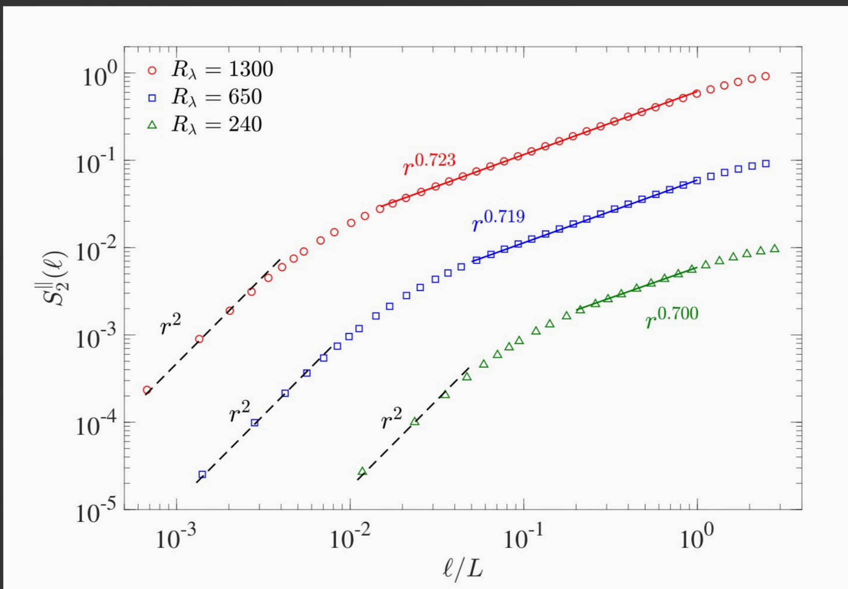
Physicist's Goal: predict ζ_p

Mathematician's Goal: constrain ζ_p, constructions

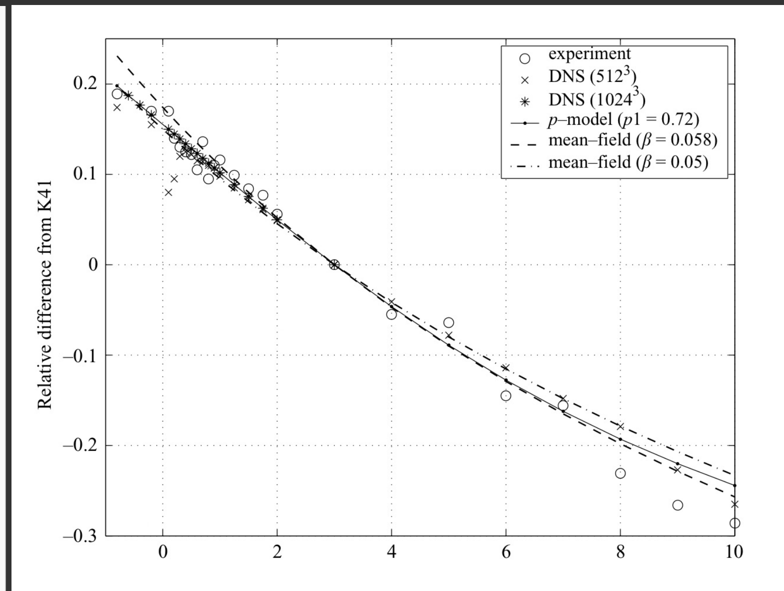
$\langle \varepsilon \rangle > 0 \Rightarrow \zeta_p \leq p/3 \quad \forall p \geq 3.$

what about intermittent ε?

Numerical & Experimental Evidence

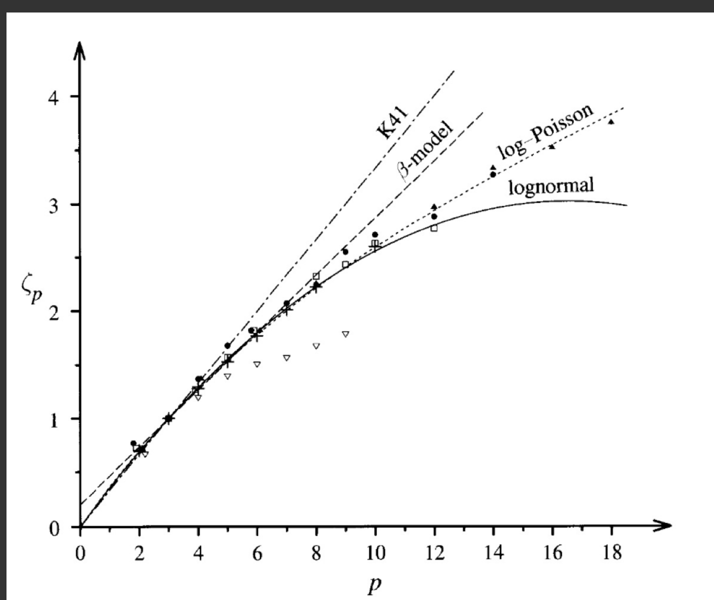


Iyer et al. 2020



Chen et al 2005
$(\zeta_{|p|} - p/3)\frac{3}{p}$ v.s. p

Frisch 1995



Models:

- log-normal: $\zeta_p = \frac{p}{3} - \frac{\mu}{18} p (p-3)$, $\mu = 0.25$ Kolmogorov 1962
- β-model: $\zeta_p = \frac{p}{3} + (3-D)\left(1-\frac{p}{3}\right)$, $D = 2.8$ Frisch et al 1978
- log-Poisson: $\zeta_p = \frac{p}{9} + 2\left(1-\left(\frac{2}{3}\right)^{\frac{p}{3}}\right)$ She-Leveque 1994
- mean-field: $\zeta_p = \frac{ap}{b-cp}$, $a = 0.185$, $b = 0.475$, $c = 0.0275$ Yakhot 2001

NEED: mathematical framework to impose constraints.

AIM: Provide lower bounds on the dimension of the support set for anomalous dissipation

All take the form $\nabla \cdot V = -D$ ← positive Radon measure

$\nabla = (\partial_t, \partial_{x_1}, \dots, \partial_{x_d})$

- Incompressible Euler from NS $u^\nu \to u$

$$V = \left(\tfrac{1}{2}|u|^2, \left(\tfrac{|u|^2}{2} + p\right)u\right), \quad D = \lim_{\nu \to 0} \nu |\nabla u^\nu|^2$$

- Transported Scalars $\quad |\theta^\kappa|_{L^\infty} \leq C.$

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

$$V = \left(\frac{\theta^2}{2}, u\frac{\theta^2}{2}\right), \quad D = \lim_{\kappa \to 0} \kappa |\nabla \theta^\kappa|^2$$

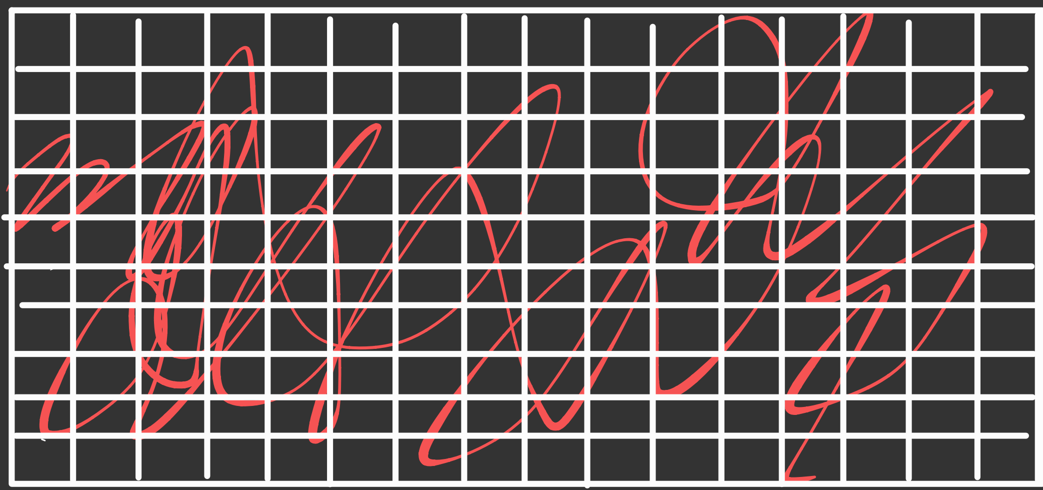
- Compressible Euler

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$
$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u + pI) = 0$$
$$\partial_t E + \nabla \cdot ((p+E)u) = 0$$

$$V = (s, us), \quad D = \lim_{\nu, \kappa, 2 \to 0} \text{Dissipation}$$

Fractal Dimensions

$\delta \{$



S

$N_\delta(S)$ = # of cubes of size δ which cover S

$$\dim_B(S) = \lim_{\delta \to 0} \frac{\log N_\delta(S)}{\log(1/\delta)}$$

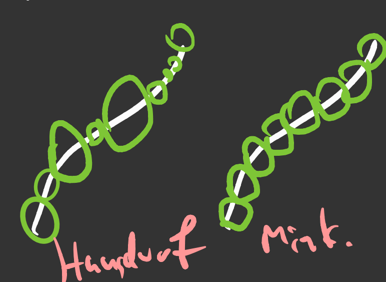
e.g. $\dim_B(S) = d$, $N_\delta(S) \sim \delta^{-d}$

(upper and lower w/ limsup, liminf). Equiv.

$$\mathcal{H}^d(S_\delta) \lesssim \delta^{d-\gamma'}, \quad \forall \gamma' < \gamma = \dim_B(S)$$

Hausdorff dimension $d_H(S)$ is a more measure theoretic notion. Basic difference is one can cover with balls of radius $\leq \delta$, not just $= \delta$. Thus

$$\dim_H(S) \leq \underline{\dim}_B(S) \leq \overline{\dim}_B(S)$$



(rational #s have $\dim_H = 0$, $\dim_B = 1$)

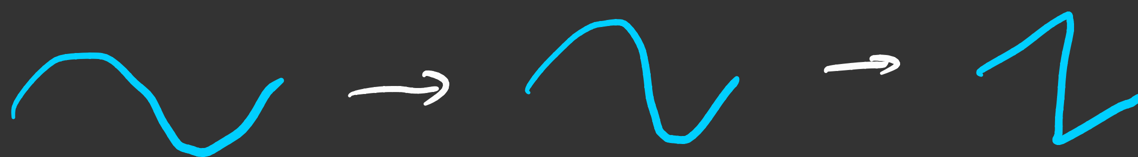
In following statements, you can just think Box counting.

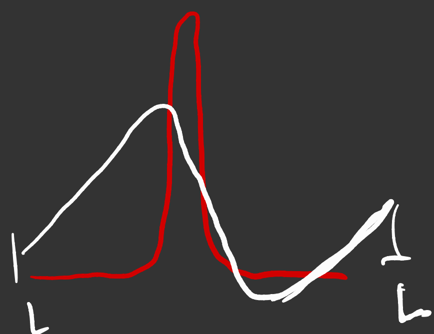
Examples

Burgers equation: $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u + u\,\partial_x u = \nu\,\partial_x^2 u$$

Remarks:
- for $\nu > 0$, model is globally wellposed
- for $\nu = 0$, model shocks in finite time.



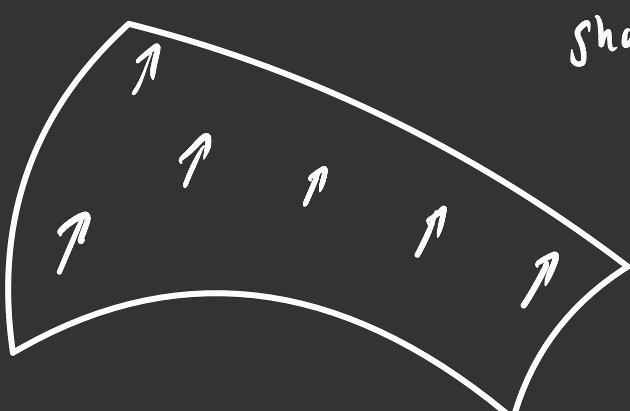


$$\varepsilon^\nu(x,t) = \nu\,|\partial_x u^\nu|^2 \xrightarrow{\nu \to 0} \frac{(\Delta u)^3}{12}\,\delta(x)$$

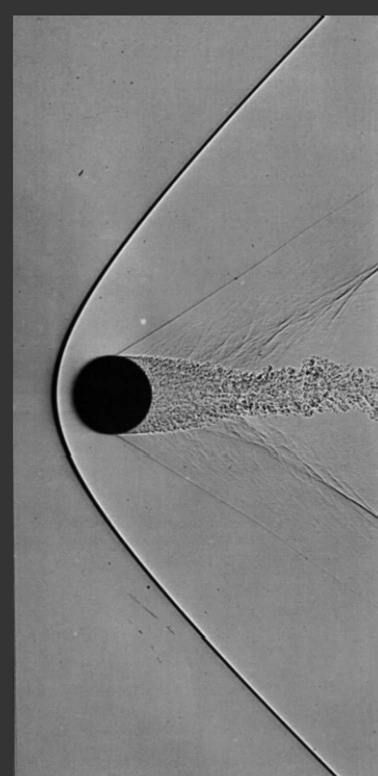
Dissipation happening at points moving ctn in time,

$\dim_H(\text{diss}) = 1.$ (codimension 1)

Shocks in compressible Euler



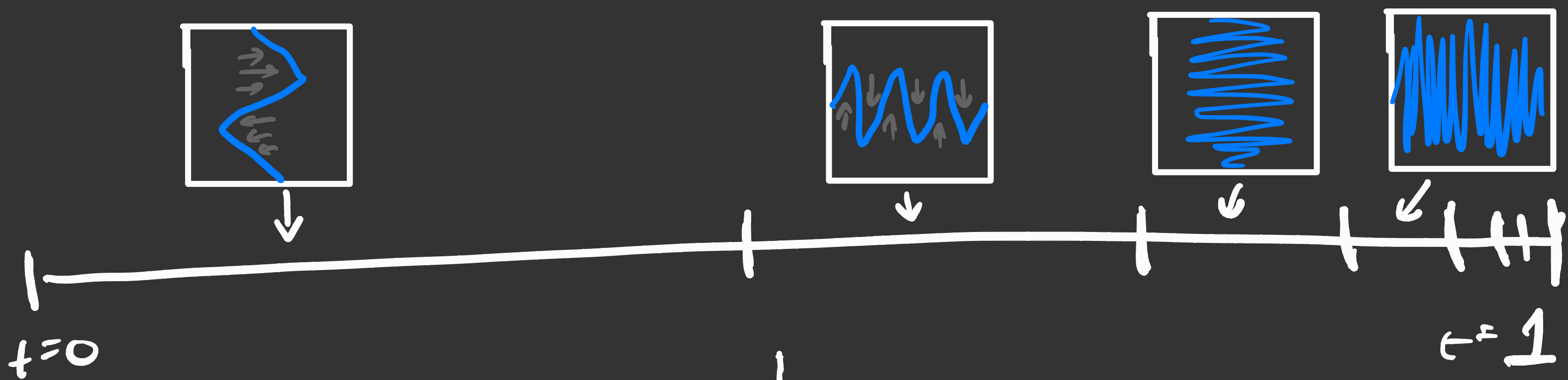
shock surface: codimension 1 at each time. smoothly evolving



$\dim_H(\text{entropy prod}) = d.$

Example:

Constructions: (D. Elgindi - Iyer - Jeong)



$$\partial_t \theta + u\cdot\nabla\theta = \kappa\Delta\theta \qquad \int_0^1 \kappa \|\nabla\theta\|_{L^2}^2 \, ds \geq c_0 > 0$$

All the dissipation is happening at one time.

Similar constructions for Navier-Stokes, (2+½ dim)
(Bruè - De Lellis, Bruè - Colombo - Crippa - De Lellis - Sorella)

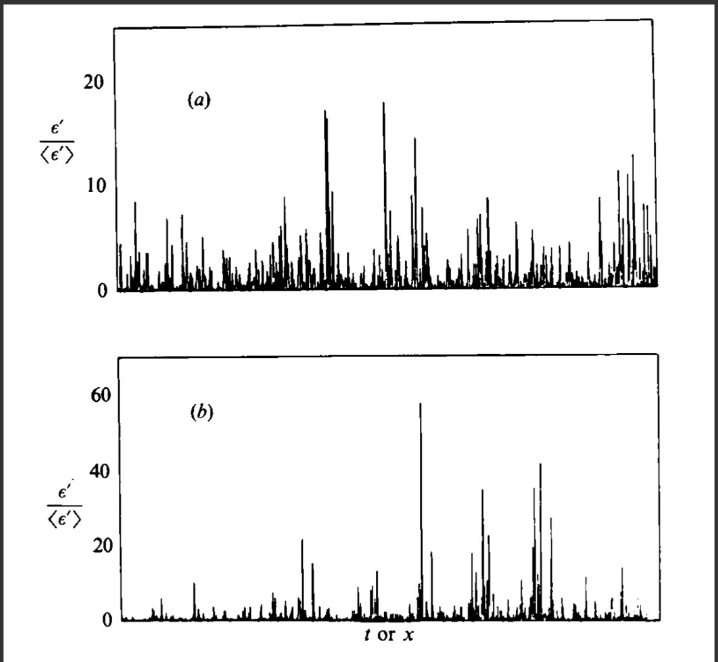
In all cases; our result shows that at time=1, dissipation is full measure in space:

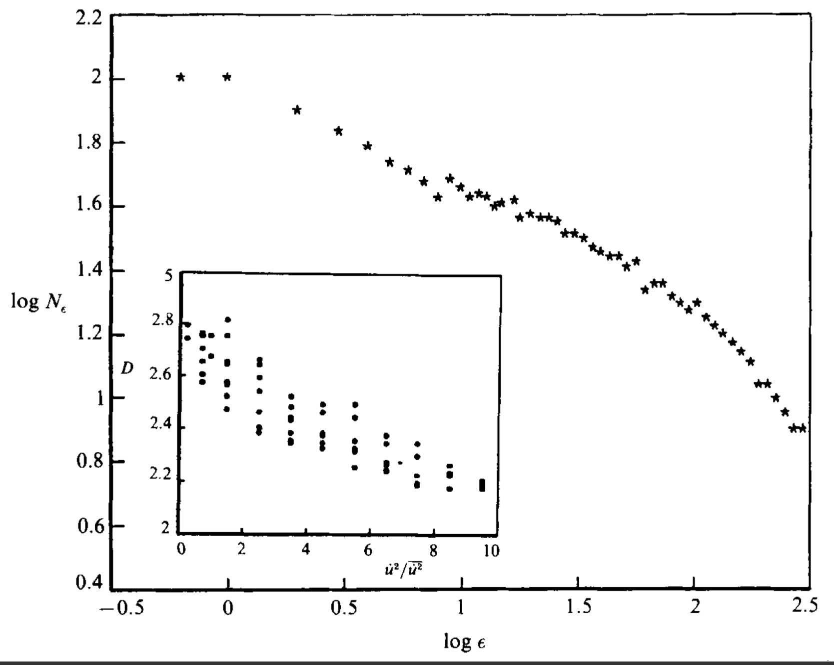
$$\dim_H(\text{diss}) = d$$

Recent construction of (Armstrong - Vicol 2023)

$$\dim_H(\text{diss}) = d+1$$

Incompressible Turbulence: Meneveau & Sreenivasan 1991
(Experiments)





They estimate that

$$\dim_B(\text{dissipation}) \approx 3.87$$

Occurring on a fractal set!

$\boxed{\text{Any restrictions?}}$

Implications?

Theorem: Bounded weak solutions of any of those equations, on domains $\Omega \subseteq \mathbb{R}^d$ which produce entropy anomalously, have

$$\dim_H (\operatorname{Spt} D) \geq d.$$

space-time

Remark: various previous examples show this theorem is sharp.

↳ Burgers, Scalar constructions, comp shocks

Remark. Theorem holds assuming $u \in L^p$. then

$$\dim_H (\operatorname{Spt} D) \geq d+1 - \frac{p}{p-1}$$

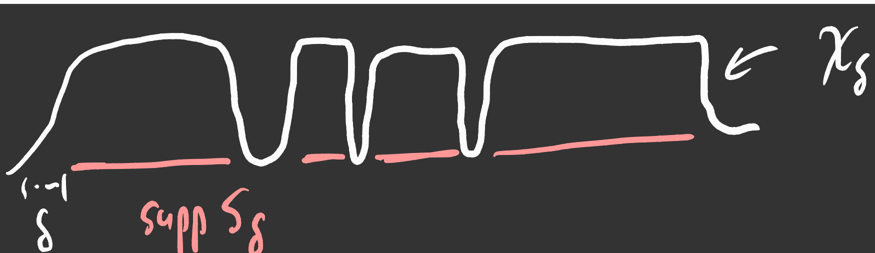
Also, if $u \in L^p_t L^q_x$, there are results which can optimize with non isotropic Hausdorff measure.

Theorem: (Frostman): Let μ be a pos Borel measure on $\mathbb{R}^d \times \mathbb{R}$. Let $\delta_0 > 0$ and assume $\forall\ (x,t) \in \operatorname{Spt}\mu$, and any $\delta \in (0, \delta_0)$, $\exists\ w(\delta)$ s.t.

$$\mu\left(B_\delta(x,t)\right) \le w(\delta)\, \delta^s$$

Then μ is abs-cont w.r.t. $\mathcal{H}^s$. If $w(\delta)$ is a modulus, $\mu(A) = 0$ for any Borel set A st $\mathcal{H}^s(A) < \infty$. E.g. since μ nontrivial $\mu(\operatorname{Spt}\mu) > 0$

$$\dim_{\mathcal{H}}(\operatorname{Spt}\mu) \ge s.$$



Fix $x \in \mathbb{R}^d$ and let χ_δ be cutoff localizing to $B_\delta(x)$ s.t. $\chi_\delta \equiv 1$ on $B_\delta(x)$ and $|\nabla \chi_\delta| \le \delta^{-1}$.

$$\mu\left(B_\delta(x)\right) \underset{\text{pos measure}}{\le} \int_{\mathbb{R}^d} \chi_\delta(x)\, d\mu = -\int V \cdot \nabla \chi_\delta(x)\, dx$$

$$\le \|V\|_{L^p(B_{2\delta})} \left(\int_{B_{2\delta}} |\nabla \chi|^q \right)^{1/q} \qquad q = \frac{p}{p-1}$$

$$\lesssim w_V(2\delta)\, \delta^{d\frac{p-1}{p} - 1}$$

$$\Rightarrow \dim_{\mathcal{H}}(\operatorname{Spt}\mu) \ge d\,\frac{p-1}{p} - 1$$

Now, we wish to bound ζ_p assuming knowledge of γ:

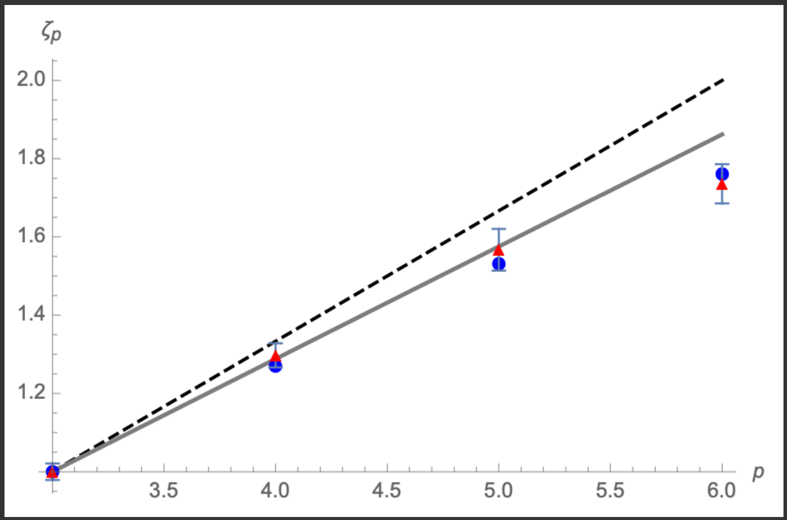
$$\boxed{\begin{array}{l}\textbf{Theorem: } \text{Let } u \in L^3_{x,t} \text{ be a weak solution of Euler,} \\ \text{whose dissipation measure is supported on a} \\ \text{spacetime set } S \text{ with } \dim_{\mathcal{H}}(S) = \gamma. \text{ For all } p \in [3,\infty), \\ \text{s.t. } u \in L^p_t B^{\sigma_p}_{p,\infty} \text{ it holds} \\ \qquad \dfrac{2\sigma_p}{1-\sigma_p} \leq 1 - \dfrac{p-3}{p}(d+1-\gamma)\end{array}}$$

Note: if $\gamma < d+1$, then $\sigma_p < 1/3$. $\forall\, p > 3$. Intermittency!

In terms of ζ_p, $\quad \zeta_p \leq \dfrac{p}{3} - \dfrac{2k(p-3)p}{9p - 2k(p-3)}$ $\quad$ codim $k = d+1-\gamma$

$$\left.\frac{d\zeta_p}{dp}\right|_{p=3} = \frac{3-2k}{9} = \frac{2\gamma-5}{9}$$

data: $\left.\dfrac{d\zeta_p}{dp}\right|_{p=3} \simeq 0.303$



$\implies$ bound with $\gamma \simeq 3.75$.

Agrees with Meneveau & Sreenivasan 1991 !

To prove this result, we use a new representation of $D[u]$

> **Lemma:** Let $u \in L^3_{x,t}$ be a weak Euler solution. Then
>
> $$-D[u] = \bar{D}_t E^\ell + \operatorname{div} Q^\ell + C^\ell$$
>
> where
>
> $$\bar{D}_t = \partial_t + \bar{u}_\ell \cdot \nabla$$
>
> $$E^\ell = \frac{|u - \bar{u}_\ell|^2}{2}$$
>
> $$Q^\ell = \left(\frac{|u - \bar{u}_\ell|^2}{2} + p - \bar{p}_\ell \right)(u - \bar{u}_\ell)$$
>
> $$C^\ell = -(u - \bar{u}_\ell) \operatorname{div} \tau_\ell + (u - \bar{u}_\ell) \otimes (u - \bar{u}_\ell) : \nabla \bar{u}_\ell$$

Another consequence:

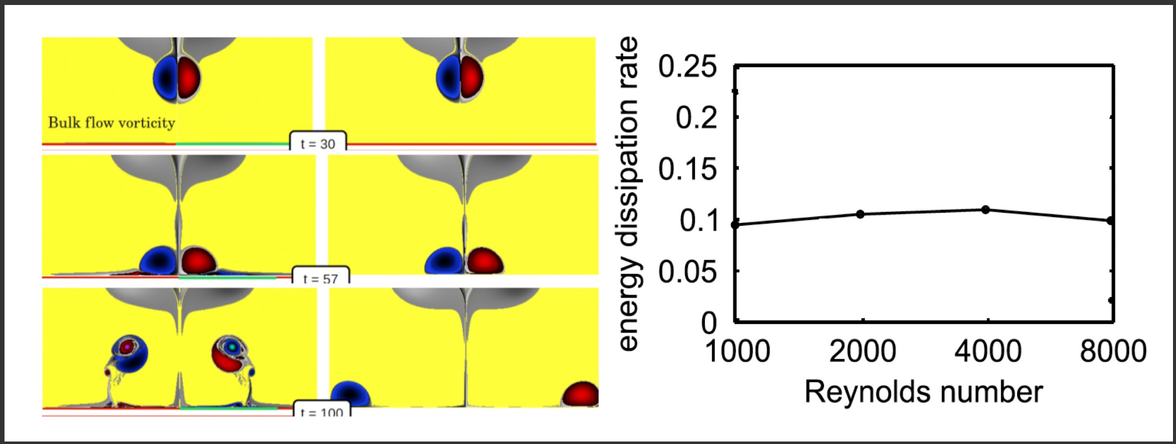
Theorem: The anomalous dissipation measure satisfies

$$\varepsilon_r = \varepsilon(B_r(x,t)) \leq r^{\frac{2\sigma}{1-\sigma} - 1 + \frac{p-3}{p}(d+1)}$$

In particular, if $\int |\varepsilon_\ell|^p \, dx\, dt \geq \left(\frac{\ell}{L}\right)^{-\alpha_p}$ for $\alpha_p > 0$

then $\alpha_p \leq 1 - \frac{p-3}{p}(d+1) - \frac{2\sigma_p}{1-\sigma_p}$

Effects of walls and additives



Nguyen, Farge, Schneider, 2011

- bounded domains: Bardos-Titi (2018), Drivas-Nguyen (2018)

$\sigma \in [1/3, 1]$ $\quad u \in B_p^{\sigma,\infty}$ (interior)

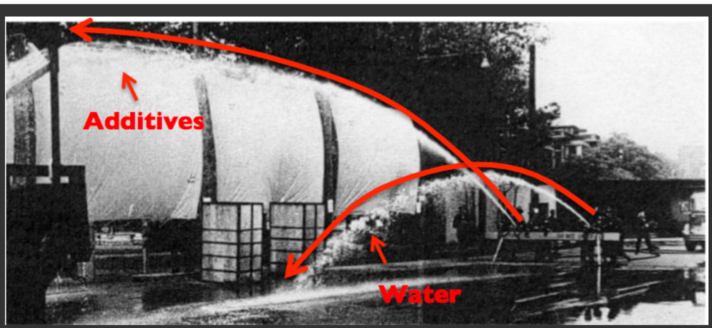
$\leftarrow \mathrm{dist}(\partial\Omega) \sim \nu^{\frac{1}{2(1-\sigma)}}$

$\leftarrow$ velocity equicontinuous

kolmogorov length

$\sigma = 1/3 \Rightarrow \nu^{3/4}$ B.L. Theory makes contact Blasius.

QUESTION: Identify physical mechanism that suppress this behavior. Drag reduction! Engineering



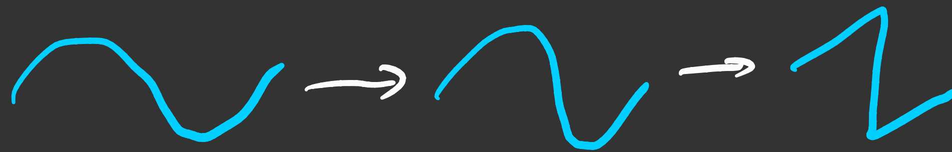
Polymer: Drivas - La, 2019 $\leftarrow$ polymer included only at walls

Rough walls: Mikelić, Jäger $\leftarrow$ only near very special flows.

1D Burgers: a case study

$$\begin{cases} \partial_t u + \partial_x\left(\tfrac{1}{2}u^2\right) = 0 \\ u\big|_{t=0} = u_0 \end{cases}$$

$x \in \mathbb{T},\ t \in \mathbb{R}$



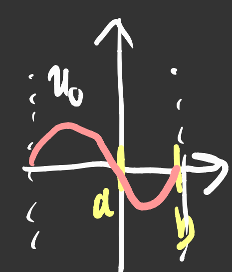
Theorem (Singularity): there does not exist a global-in-time continuous weak solution.

Alinhac

Proof:

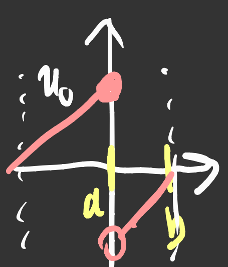
- First, if x_* is a zero, $u(x_*)=0$, and a point of continuity, then it remains a zero.

  Indeed, $\dot{X}_t = u(X_t, t) = u_0 \implies X_t(a) = a + t u_0(a)$ for points of continuity (see Dafermos 2006)



2 zeros

- all continuous functions of zero mean (w.l.o.g) have at least two zeros, $a, b \in \mathbb{T}$ $u_0(a) = u_0(b) = 0$
- Let $\phi \in C_0^\infty(\mathbb{T})$ s.t. $[a,b] \subseteq \operatorname{supp}\phi$, $\phi_x\big|_{[a,b]} > 0$.
- Introduce $A(t) = \int_a^b u(t)\,\phi\,dx$. Then



1 zero!

$$\frac{d}{dt}A(t) = \int_a^b \tfrac{1}{2}u^2\phi_x\,dx \geq \frac{1}{2K}A^2.$$

$$A^2 = \left(\int_a^b u\phi\right)^2 \leq \int_a^b u^2\phi_x \underbrace{\int_a^b \frac{\phi^2}{\phi_x}dx}_{K}$$

↖ Blows up!

$A_0 > 0 \Rightarrow A(t) \nearrow \infty$ at some time.

Indeed, the Khokhlov sawtooth solution is

$$u(x,t) = \begin{cases} \frac{x+L}{t} & -L \le x \le 0 \\ \frac{x-L}{t} & 0 \le x \le L \end{cases}$$

which has a jump discontinuity, and dissipates:

$$\frac{1}{2}\|u(t)\|^2_{L^2} = \frac{1}{2L}\int_0^L \left(\frac{x-L}{t}\right)^2 dx = \frac{1}{6}\left(\frac{L}{t}\right)^2.$$

Anomalous dissipation! If $\Delta u := u^- - u^+ = \frac{2L}{t}$,

$$\langle \varepsilon \rangle = -\frac{d}{dt}\frac{\|u\|^2_{L^2}}{2} = \frac{1}{3}\frac{L^2}{t^3} = \frac{(\Delta u)^2}{12t} = \frac{(\Delta u)^3}{24L} > 0.$$

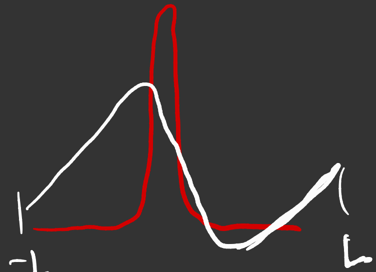
This matches $\varepsilon^\nu = \nu |u^\nu_x|^2$ as $\nu \to 0$. In fact:

An exact viscous solution is available.

$$u^\nu(x,t) = \frac{1}{t}\left[x - L\tanh\left(\frac{Lx}{2\nu t}\right)\right]$$

Whence:

$$\varepsilon^\nu(x,t) = \nu|\partial_x u^\nu|^2 \cong \frac{L^4}{4\nu t^4}\operatorname{sech}^2\left(\frac{Lx}{2\nu t}\right)$$

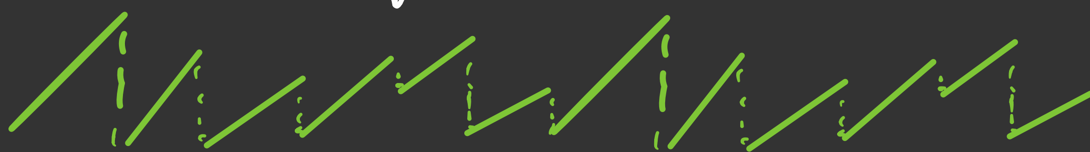


$$\xrightarrow{\nu \to 0} \frac{(\Delta u)^3}{12}\delta(x)$$

Intermittency

Entropic shocks live in:

$$u \in L^\infty_t (L^\infty_x \cap BV_x)$$



Since $L^\infty \cap BV \subseteq B^{1/p,\infty}_p$, $p \geq 1$.

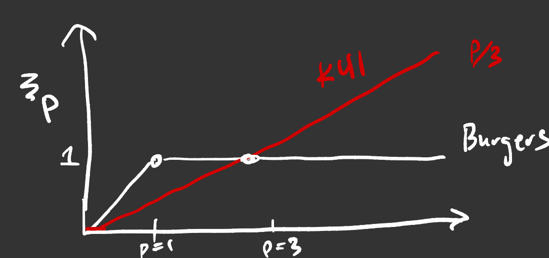
shocks live at the Onsager-critical theshold

They are also <u>intermittent</u>. $u = \begin{cases} \frac{x+L}{t} & -L \leq x \leq 0 \\ \frac{x-L}{t} & 0 < x \leq L \end{cases}$

$$\langle |\delta_\ell u|^p \rangle = \frac{1}{2L} \int_{-L}^{L} |u(x+\ell) - u(x)|^p \, dx$$

$$= \left(1 - \frac{\ell}{2L}\right) \left(\frac{\ell}{t}\right)^p + \frac{\ell}{2L} \left(\frac{2L + \ell}{t}\right)^p$$

$$\sim (\Delta u)^p \begin{cases} (\ell/L)^p & 0 < p < 1 \\ \frac{\ell}{2L} & p \geq 1 \end{cases}$$



$$\partial_t u + u \partial_x u = \nu \partial_x^2 u + \sum_k \sigma(x) \, dW_t^k$$

<u>Mathematical Dream</u>: E-Khanin-Mazel-Sinai (1997, 2000)

Unique invariant measure μ_0 supported on entropic shocks, realized $\mu_\nu \to \mu_0$. displays AD & Intermittency.

Review: Bec - Khanin "Burgers Turbulence" 2007

Connection between intermittency and lower dimensional dissipation

Recall
$$\langle \varepsilon \rangle = \frac{(\Delta u)^3}{24L} \quad \text{and} \quad S_p(\ell) = \langle |\delta_\ell u|^p \rangle \sim (\Delta u)^p \frac{\ell}{2L}$$

$$\Rightarrow \Delta u \sim \langle \varepsilon \rangle^{1/3}$$

Thus
$$S_p(\ell) \sim \underbrace{(\ell \langle \varepsilon \rangle)^{p/3}}_{\text{"K41"}} \underbrace{\left(\frac{\ell}{L}\right)^{-\left(\frac{p-3}{p}\right)}}_{\text{"intermittency" correction}}$$

In fact, we can "derive" this via the "$\frac{4}{5}$-law":

$$\varepsilon[u] = \varepsilon\, \delta_0(x), \qquad \varepsilon_\ell[u] = \varepsilon\, \delta_\ell(x) = \varepsilon \frac{1}{\ell} \varphi\left(\frac{x}{\ell}\right)$$

$$\left\langle \left(\frac{\delta_\ell u}{\ell^{1/3}}\right)^p \right\rangle \simeq \langle \varepsilon_\ell^{p/3} \rangle = \frac{\langle \varepsilon \rangle^{p/3}}{\ell^{p/3}} \int_{-L}^{L} \phi^{p/3}\left(\frac{x}{\ell}\right) dx$$

$$= \frac{\langle \varepsilon \rangle^{p/3}}{\left(\frac{\ell}{L}\right)^{\frac{p}{3}-1}} |\varphi|_{L^{p/3}}^{p/3} \simeq \langle \varepsilon \rangle^{p/3} \left(\frac{\ell}{L}\right)^{-\left(\frac{p-3}{p}\right)}.$$

In fact, starting from $u_0 \in L^\infty$, $\exists !$ dissipative weak solution of Burgers in $BV \cap L^\infty$. In particular
$$u(t) \in B^{1/p}_{p,\infty} \quad \forall\, p \geq 1$$

If $u_0 \in L^2$, immediatly $u(t) \in B^{1/3}_{3,\infty}(\mathbb{T})$!

Source of self-regularization: finite flux!

Consider viscous Burgers: $\partial_t u^\nu + u^\nu u^\nu_x = \nu u^\nu_{xx}$

Theorem: (Goldman–Josien–Otto, 15') Let $\varphi: \mathbb{R} \to \mathbb{R}$ have Lipshitz derivative. Let $\Phi(x)$ be the primative of $\varphi(x)$, and $\tilde{\Phi}(x)$ be the primative of $x\varphi'(x)$.

Then we have

$$\partial_t \varphi(\delta_\ell u) + \partial_x J_\ell[u] + \partial_\ell \Pi[\delta_\ell u] = -\nu \varphi''(\delta_\ell u)\,|\partial_x \delta_\ell u|^2$$

where
$$J_\ell[u] = u(x+\ell)\,\varphi(\delta_\ell u) - \tilde{\Phi}(\delta_\ell u) - \nu \partial_x \varphi(\delta_\ell u)$$
$$\Pi[\delta_\ell u] = \tilde{\Phi}(\delta_\ell u) - \Phi(\delta_\ell u)$$

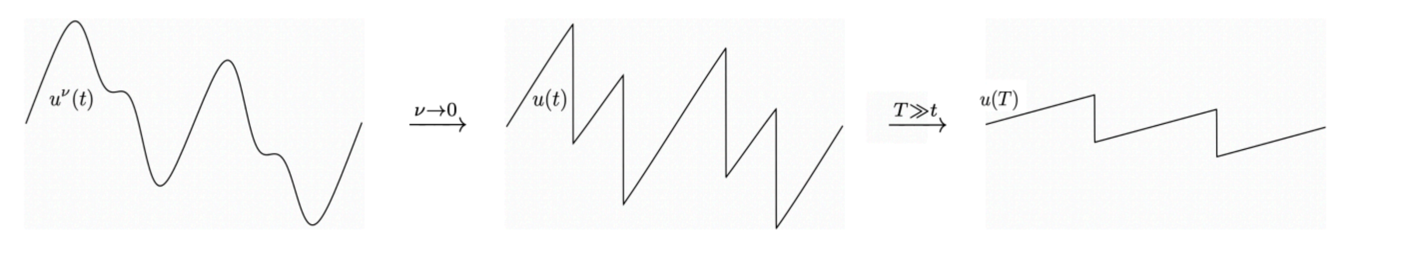
Application: let $p \geq 3, \alpha \in \mathbb{R}$ and $\varphi(x) = \alpha x |x|^{p-2}$. Then
$$\Phi(x) = \frac{\alpha}{p}|x|^p \quad \text{and} \quad \tilde{\Phi}(x) = (p-1)\,\Phi(x).$$

If $\alpha = \frac{p-2}{p}$, we have

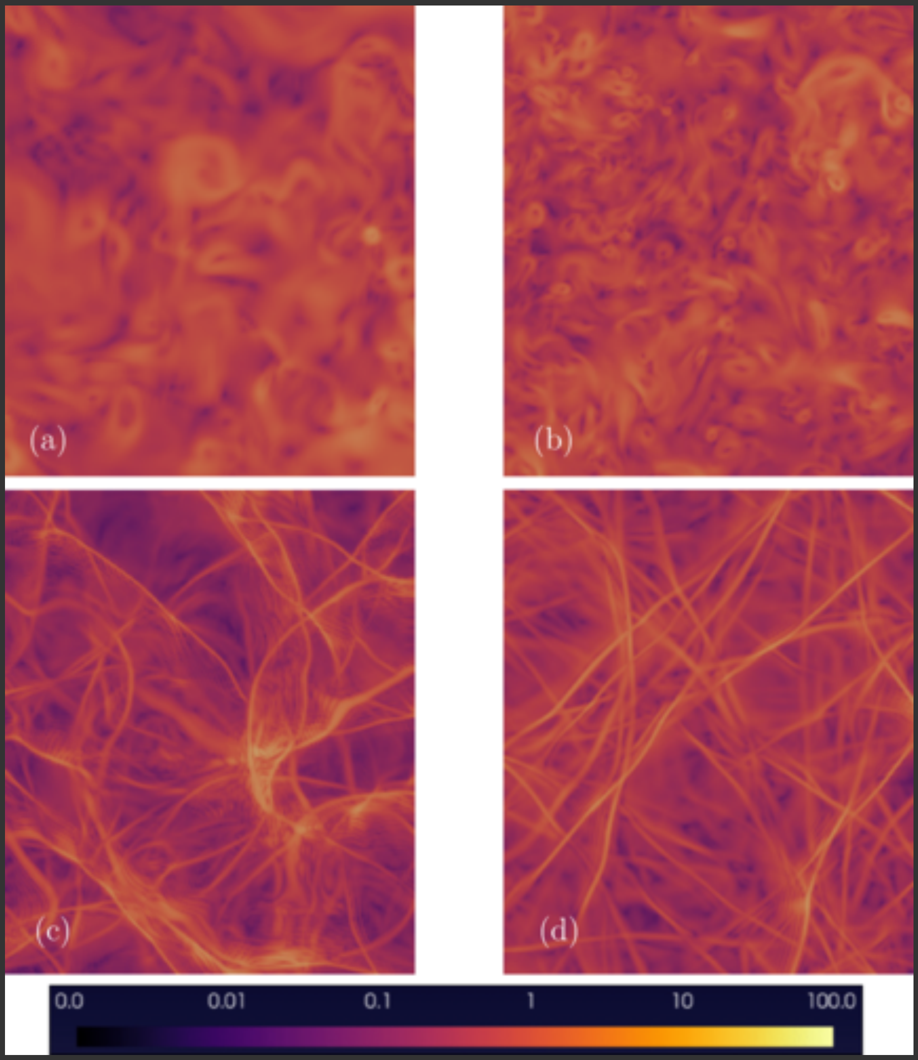
$$\frac{1}{\ell}\fint_0^T \int_{\mathbb{T}} |\delta_\ell u|^p\,dx\,dt = \frac{1}{T}\left[\fint_0^\ell \int_{\mathbb{T}} \varphi(\delta_\ell u_0)\,dx\,d\ell - \fint_0^\ell \int_{\mathbb{T}} \varphi(\delta_\ell u_T)\,dx\,d\ell\right] - \nu \fint_0^\ell \fint_0^T \int_{\mathbb{T}} \varphi''(\delta_\ell u)\,|\partial_x \delta_\ell u|^2\,dx\,dt\,d\ell \lesssim \frac{C(u_0)}{T}$$

$$\boxed{S_p(\ell) \leq C_p \frac{|\ell|}{T}}$$

!



Burgers is a toy model for incompressible turbulence.
But, compressible turbulence in the highly compressive regime is shock dominated...



Contours of $|\nabla p|$ for increasing Mach number, and compressive comp.

(Donzis – John '19)

In fact, they numerically show

$$\frac{\langle \varepsilon_{comp} \rangle}{\langle \varepsilon_{sol} \rangle} \sim \delta^2 \quad \text{where} \quad \delta = \frac{|u_{comp}|_{L^2}}{|u_{sol}|_{L^2}}$$

Case study: Scalar transport

Let $\theta \in \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{R}$ satisfy

$$\partial_t \theta^\kappa + u \cdot \nabla \theta^\kappa = \kappa \Delta \theta^\kappa$$
$$\nabla \cdot u = 0$$
$$\theta^\kappa\big|_{t=0} = \theta_0^\kappa, \quad \fint \theta_0^\kappa \, dx = 0$$

Here, θ^κ represents <u>temperature</u> or <u>dye</u> being stirred by the velocity u. Scalar 'energy' is dissipated:

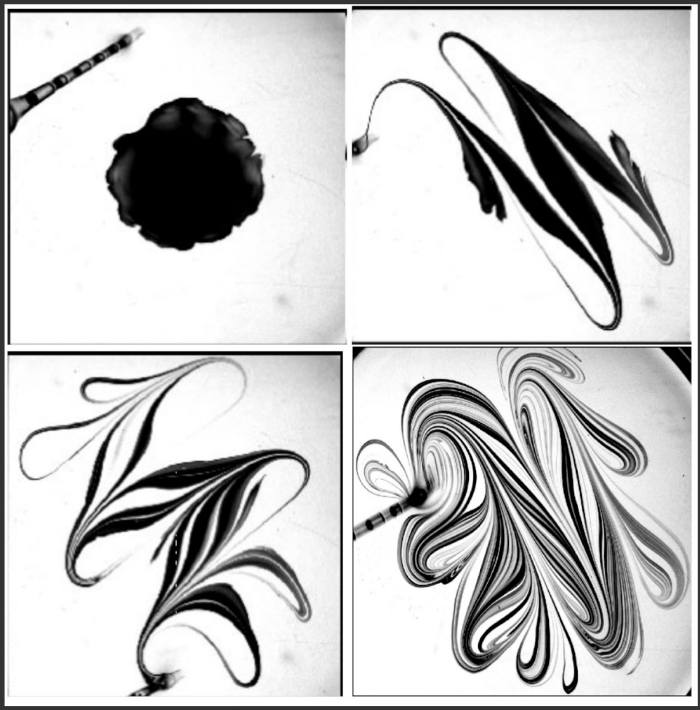
$$\frac{1}{2} \frac{d}{dt} \int |\theta^\kappa|^2 dx = -\kappa \int |\nabla \theta^\kappa|^2 dx$$

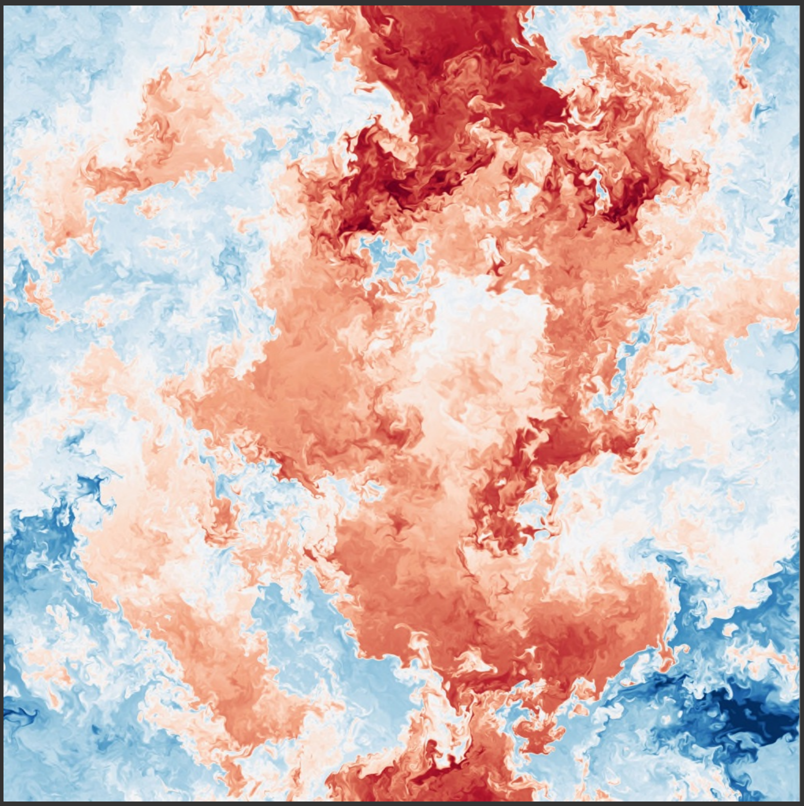
Even though the velocity field does not feature in this balance, it is crucially important to the process

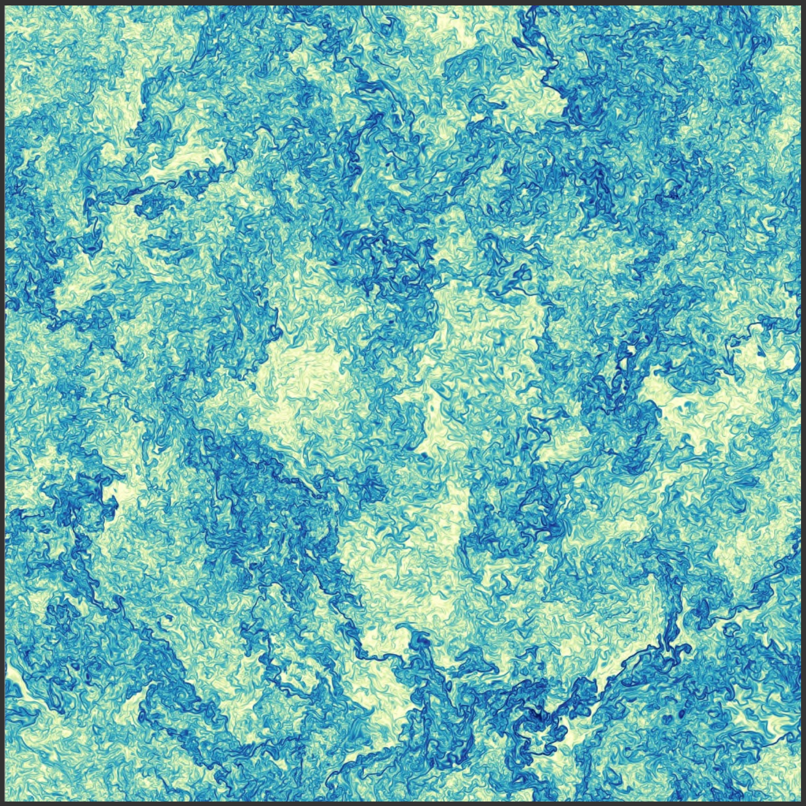
velocity acts to filament the scalar, causing $\nabla \theta^\kappa$ to grow and contribute more to dissipation.

$$\boxed{\text{Anomalous Dissipation:} \quad \kappa \int_0^T \int |\nabla \theta^\kappa|^2 dx\, dt \geq \chi > 0 \qquad \text{Donzis-Sreeni (2005)}}$$

Advection of scalar by smooth velocity field $\Rightarrow$ Mixing







Scalar advected by rough Navier-Stokes and by the Kraichnan model

$\Rightarrow$ Anomalous Dissipation

Now, due to the apriori L^∞ bound,

$$\|\theta^\kappa(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}$$

we have $\underset{\text{weak-}* \ L^\infty}{\theta^\kappa \to \theta}$ to a weak solution of

$$\partial_t \theta + \operatorname{div}(u\theta) = 0.$$

Energy balance holds as soon as $\{\theta^\kappa\}$ cmp in L^2

$$\partial_t\left(\tfrac{1}{2}\theta^2\right) + \operatorname{div}\left(\frac{\theta^2}{2}u\right) = -D_u[\theta]$$

where

$$D_u[\theta] = \lim_{\kappa\to 0} \kappa|\nabla\theta^\kappa|^2$$

and also

$$D_u[\theta] = \lim_{\ell\to 0}\int_{S^{d-1}} \frac{\delta_{\ell z}u(x)\cdot z}{\ell}|\delta_\ell\theta(x)|^2\, d\sigma(z)$$

Note, if $u \in C^\alpha$, $\theta \in B^\beta_{2,\infty}$, then

$$|D_u[\theta]|_{L^1} \lesssim \ell^{\alpha+2\beta-1} \Rightarrow \beta > \frac{1-\alpha}{2} \text{ is conservative}$$

Obukhov (1949) & Corrsin (1951) Theory

'turbulent' velocity "$u \in C^\alpha$" $\alpha \in (0,1]$

gives rise to "$\theta \in C^\beta$" with $\beta = \frac{1-\alpha}{2}$.

THEOREM: (Drivas, Elgindi, Iyer, Jeong, 2020):

Fix $T>0$, $d \geq 2$, $\alpha \in (0,1)$ & $\theta_0 \in C^\infty$. There exist a divergence-free velocity field

$$u \in L^1([0,T]; C^\alpha)$$

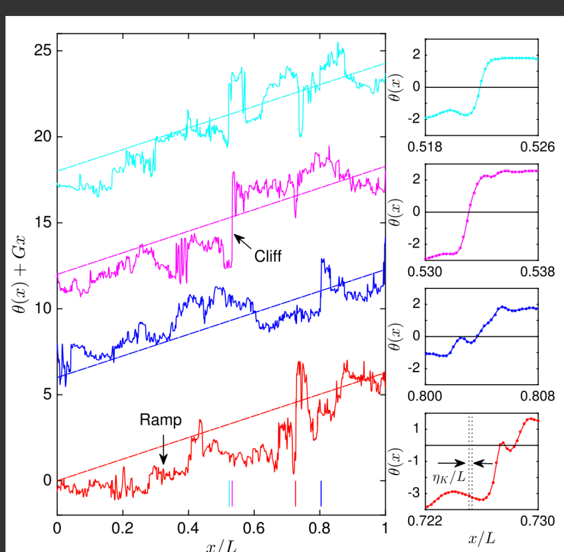
such that we have

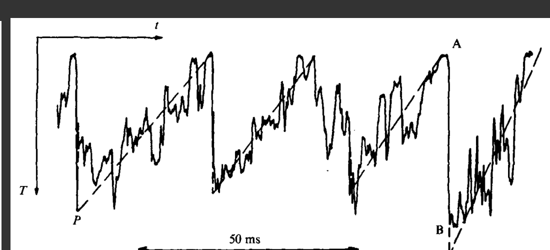
$$\kappa \int_0^T \int |\nabla \theta^\kappa|^2 \, dx \, dt \geq \chi > 0$$

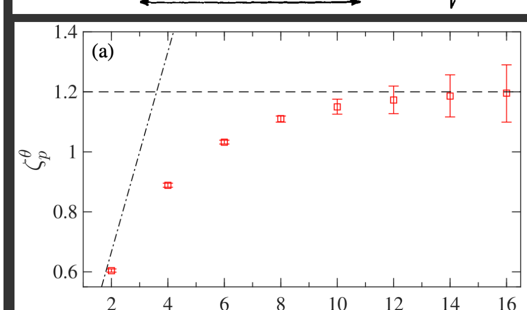
(Crippa–Colombo–Sorella, '23)

Moreover, $\theta \in L^\infty_t C^{\beta-}_x$ where $\beta = \frac{1-\alpha}{2}$

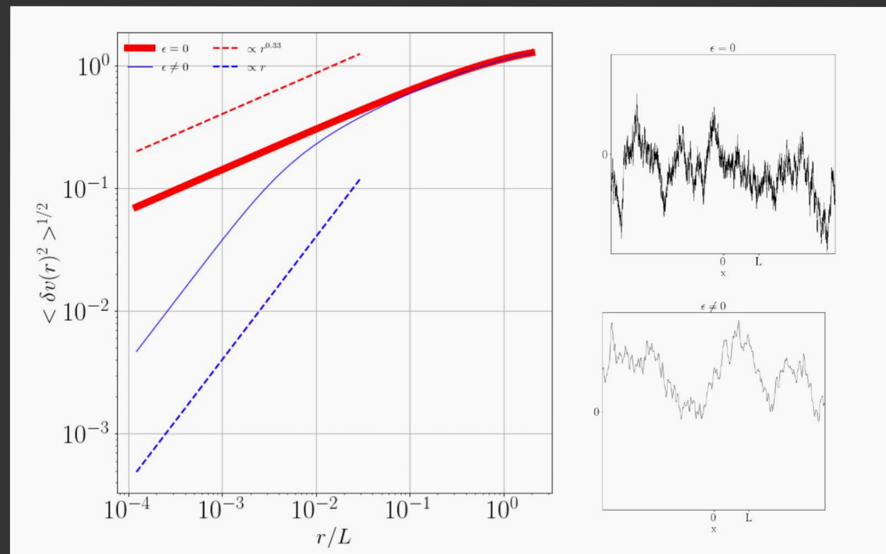
Needs modification again due to Intermittency!







Kraichnan Model



Gaussian Random Field

$$\langle u(x,t) \rangle = 0$$

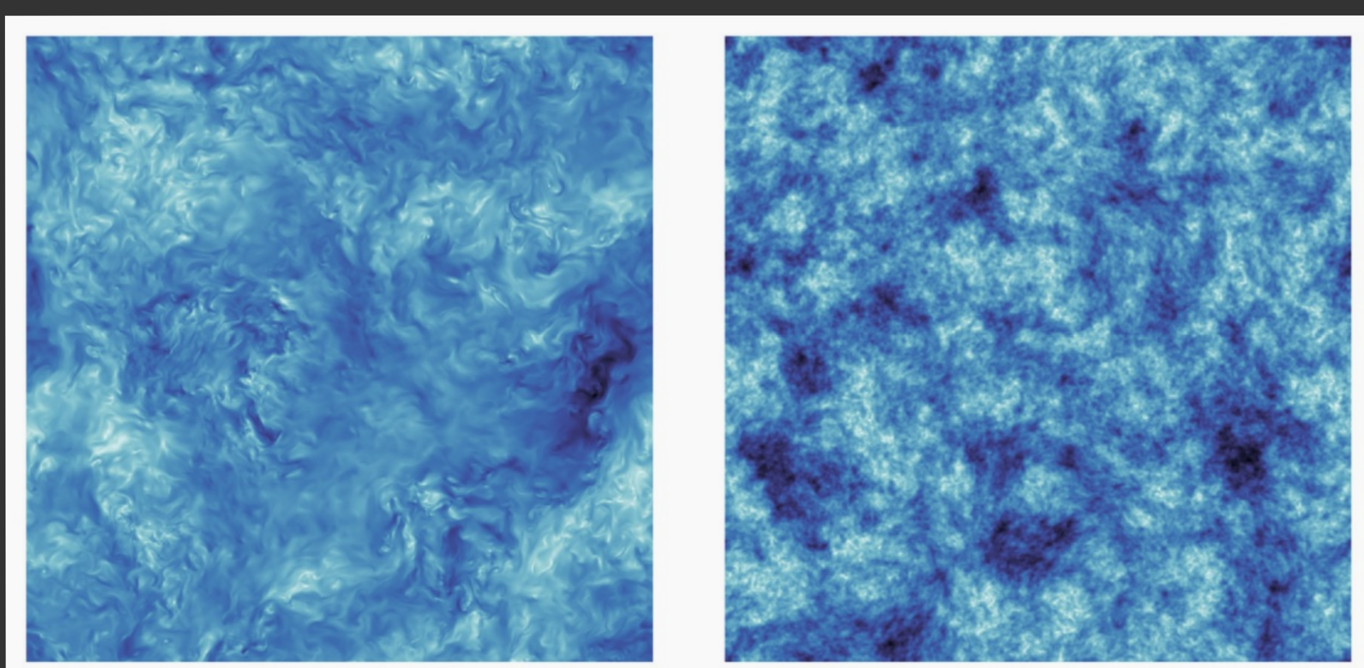
$$\langle |u(x,t) - u(x',t)|^2 \rangle \simeq \delta(t-t') \begin{cases} A\,|x-x'|^{\xi} & \varepsilon \ll |x-x'| \\ B\,|x-x'|^2 & |x-x'| \ll \varepsilon \end{cases}$$

$\xi \in (0,2)$

Roughly, $u \in C^{\xi/2}$ in space.

Navier-Stokes

Kraichnan



$$d\theta_t + du \circ \nabla \theta_t = \kappa \Delta \theta_t \, dt \qquad \xi = 2\alpha \qquad "u \in C^{\alpha}"$$

Studied extensively (Kraichnan, Balkovsky, Falkovich, Bernard–Gawedzki–Kupiainen, Chertkov, Le Jan, Raimond, E, Vanden Eijnden, Vergassola, Lebedev, Fouxon)

Theorem: (Rowan '23): There exists a const $C(d)$ such that $\forall\ \alpha \in (0,1)$:

$$\mathbb{E}\, |\theta_t^{\kappa}|_{L^2}^2 \lesssim e^{-t/C} \|\theta_0\|_{L^2}^2 \qquad \forall\ t > 0$$

for all $\theta_0 \in L^2$. Moreover, it happens conf. in. time. (D-Galeati-Pappalettera)

Theorem: (D-Galeati-Pappalettera)

For all $\theta_0 \in L^2$ $\exists !$ vanishing diffusion solution which is dissipative and is such that

$$\int_0^T \mathbb{E}\, |\theta(t)|_{H^{1-\alpha}} \, dt \lesssim |\theta_0|_{L^2}^2$$

Mechanism: regularization by coercive flux

$$\mathbb{E}\, D[\theta] \propto \lim_{\ell \to 0} \int_{S^{d-1}} \frac{\mathbb{E}\, |\delta_{\ell z} \theta|^2}{\ell^{2(1-\alpha)}} \, d\sigma(z)$$

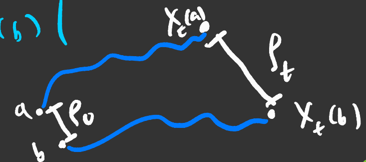
Allows for intermittency!

$$\xi_p = p(1-\alpha) - \frac{p(p-2)}{d+2}\alpha + \mathcal{O}(\alpha^2)$$

(Bernard–Gawedzki–Kupiainen, 1989)

"Source" of Dissipation

$\rho_t(a,b) = |X_t(a) - X_t(b)|$



(Le-Jan 1985)

$$d\rho_t = b(\rho_t)\,dt + \sqrt{2\sigma(\rho_t)}\,dW_t + \sqrt{2\kappa}\,dB_t$$

$$\rho_0 = |a-b|$$

closed equation for particle separation!

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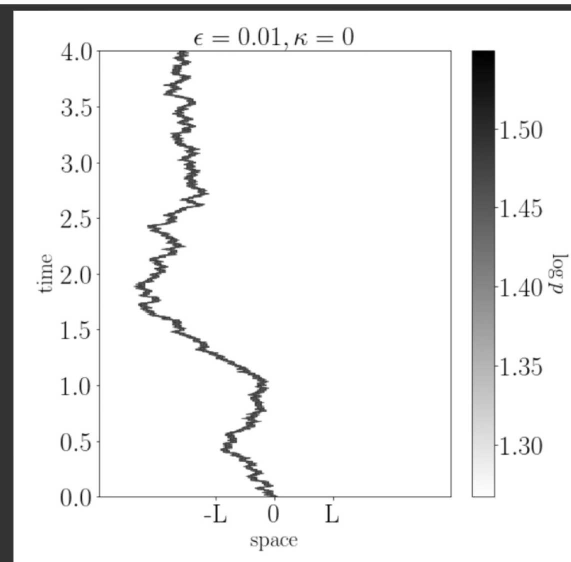
when Gaussian field is Lipshitz, then $\rho = 0$ is an absorbing point for the diffusion $\Longrightarrow$ unique particle trajectories



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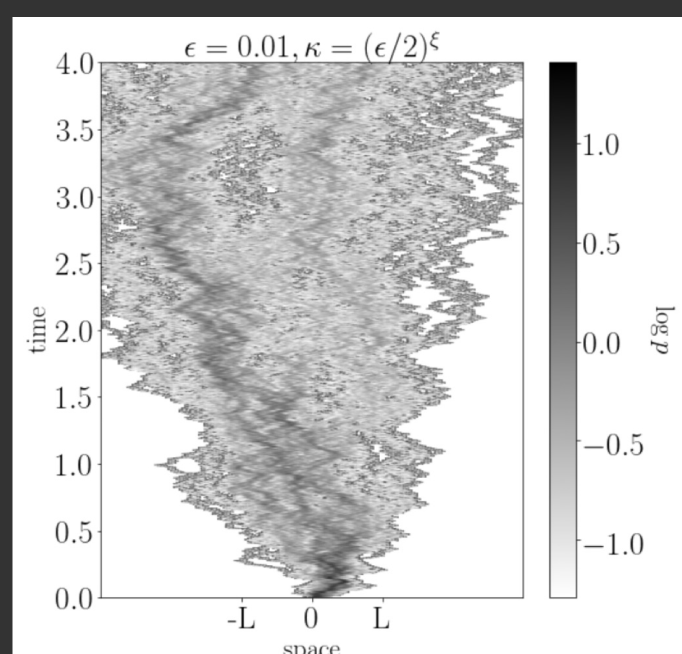
when Hölder e.g. $\xi = \frac{4}{3}$:

$$d\rho_t = \rho_t^{1/3}\,dt + \rho_t^{2/3}\,dW_t + \sqrt{2\kappa}\,dB_t$$

The process ρ_t is instantaneously reflecting at $\rho = 0$

Le-Jan Raimond (2002, 2004)

Non-uniqueness of trajectories!



Connection between Anomalous dissipation of passive scalar and non-uniqueness of particle trajectories

(Bernard – Gawedzki – Kupiainen, 1998)

Let $u \in L^\infty_{t,x}$, $\nabla \cdot u = 0$, $\theta_0 \in L^2$ and let θ^κ be unique sln of

$$\partial_t \theta^\kappa + u \cdot \nabla \theta^\kappa = \kappa \Delta \theta^\kappa$$

Recall that we say anomalous dissipation occurs if

$$\liminf_{\kappa \to 0} \int_0^T \int_{\mathbb{T}^d} \kappa |\nabla \theta^\kappa|^2 \, dx\, dt > 0$$

Equivalently: $\liminf_{\kappa \to 0} \|\theta^\kappa(t)\|_{L^2} < \|\theta_0\|_{L^2}$.

Theorem (D-Eyink '17, Rowan '23) Suppose the passive scalar driven by u exhibits anomalous dissipation. Then, the integral curves of u (backwards in time) are non-unique for a positive measure set of initial conditions.

Fluctuation Dissipation-relation: $\kappa \int_0^T \int_{\mathbb{T}^d} |\nabla \theta^\kappa|^2 \, dx\, dt = \int_{\mathbb{T}^d} \mathrm{Var}\left(\theta_0 (X_{T,0}(a))\right) da$

Proof: We will show that $\exists$ final data θ_f such that

$$\partial_t \theta + u \cdot \nabla \theta = 0$$
$$\theta(T) = \theta_f$$

has non-unique positive solutions

This will imply the statement by Ambrosio's superposition principle.

Idea of superposition principle: Given a non-negative solution $\partial_t \theta + u\cdot\nabla\theta = 0$ we aim to produce a path measure. Favre average with a mollifier supported everywhere:

$$\partial_t \bar{\theta}_\ell + \nabla\cdot\left(\tilde{b}_\ell \bar{\theta}_\ell\right) = 0 \qquad \tilde{b}_\ell := \overline{(u\theta)}_\ell / \bar{\theta}_\ell$$

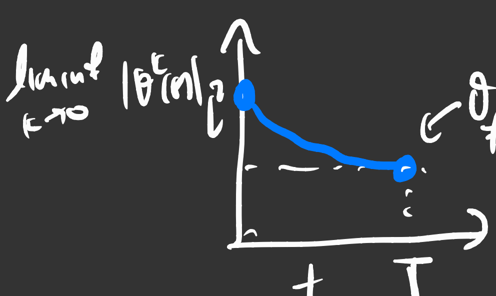
Here $\tilde{b}_\ell$ is smooth, since $\bar{\theta}_\ell > 0$. Then

$$(\varphi, \bar{\theta}_\ell)_{L^2} = \int_{\mathbb{T}^d} \varphi(X_t^\ell(a))\,\theta_0^\ell(a)\,da =: \int_{\text{paths}} \varphi\, d\mu_t^\ell$$

$$d\mu_t^\ell = \theta_0 \cdot \delta_{X_t^\ell} \xrightarrow{\ell\to 0} d\mu$$

The measure $d\mu$ can be shown to give zero mass to any Lipshitz curve not solving integral form of $\dot{x} = u(x)$.

Assume anomalous dissipation

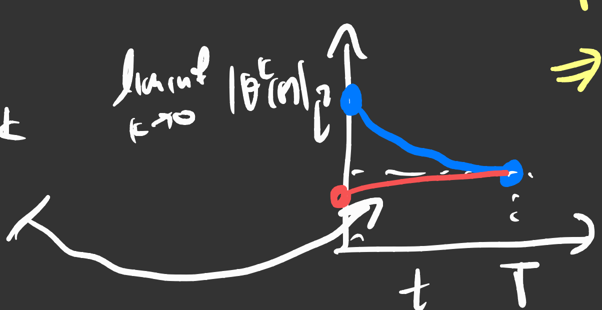


$\theta_f = \theta(T) = \lim_{\kappa\to0} \theta^\kappa(T)$.

Now solve on $t\in[0,T]$

$$\partial_t \tilde{\theta}^\kappa + u\cdot\nabla\tilde{\theta}^\kappa = -\kappa\Delta\tilde{\theta}^\kappa$$

$$\tilde{\theta}^\kappa\big|_T = \theta_f$$

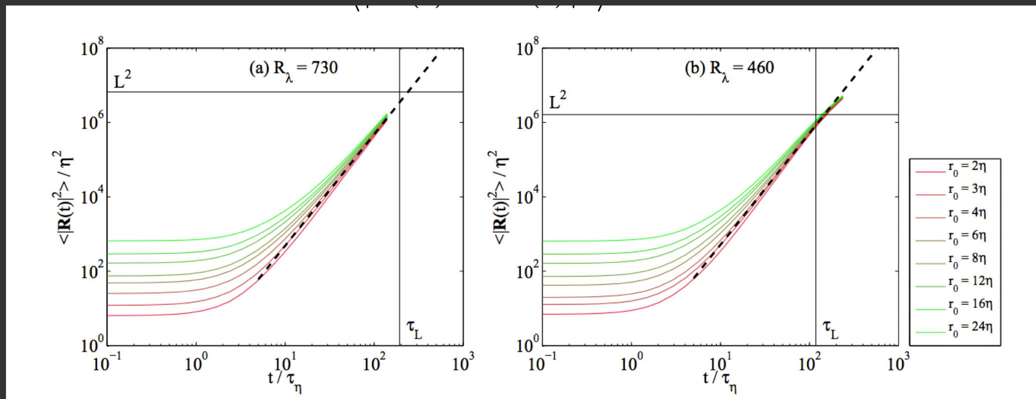


Thus we have two distinct non-neg solutions

$\Rightarrow$ two distinct path measures.

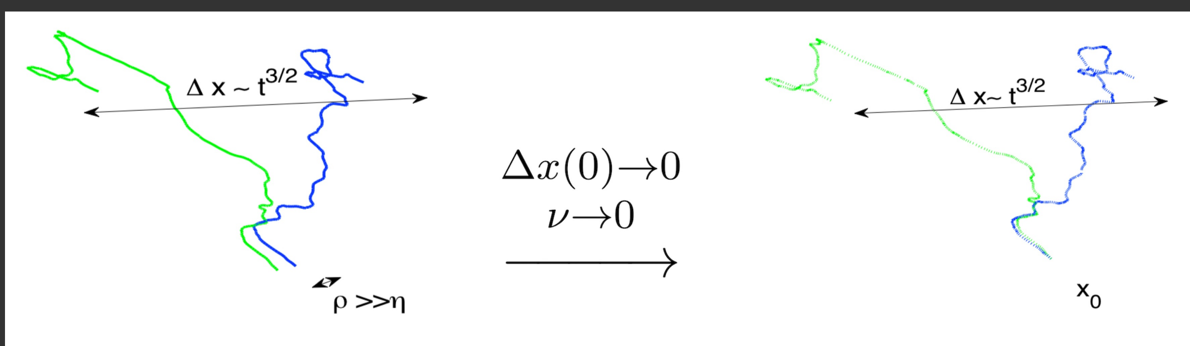
$\Rightarrow$ non-uniqueness

Richardson (1926) : $\langle |x_1(t) - x_2(t)|^2 \rangle \sim G\langle \varepsilon \rangle t^3$



Toy understanding: K41 velocity $\delta_\ell u \sim (\varepsilon \ell)^{1/3}$.

$$\dot{\delta x} = (\varepsilon \delta x)^{1/3} \Rightarrow |\delta x|^2 = \left[|\delta x_0|^{2/3} + \varepsilon^{1/3} t\right]^3 \sim \varepsilon t^3$$



Spontaneous Stochasticity:

Bernard, Gawedzki-Kupiainen (1998) ...

Gawedzki - Vergassola (2000) ...

Eyink (2006...)

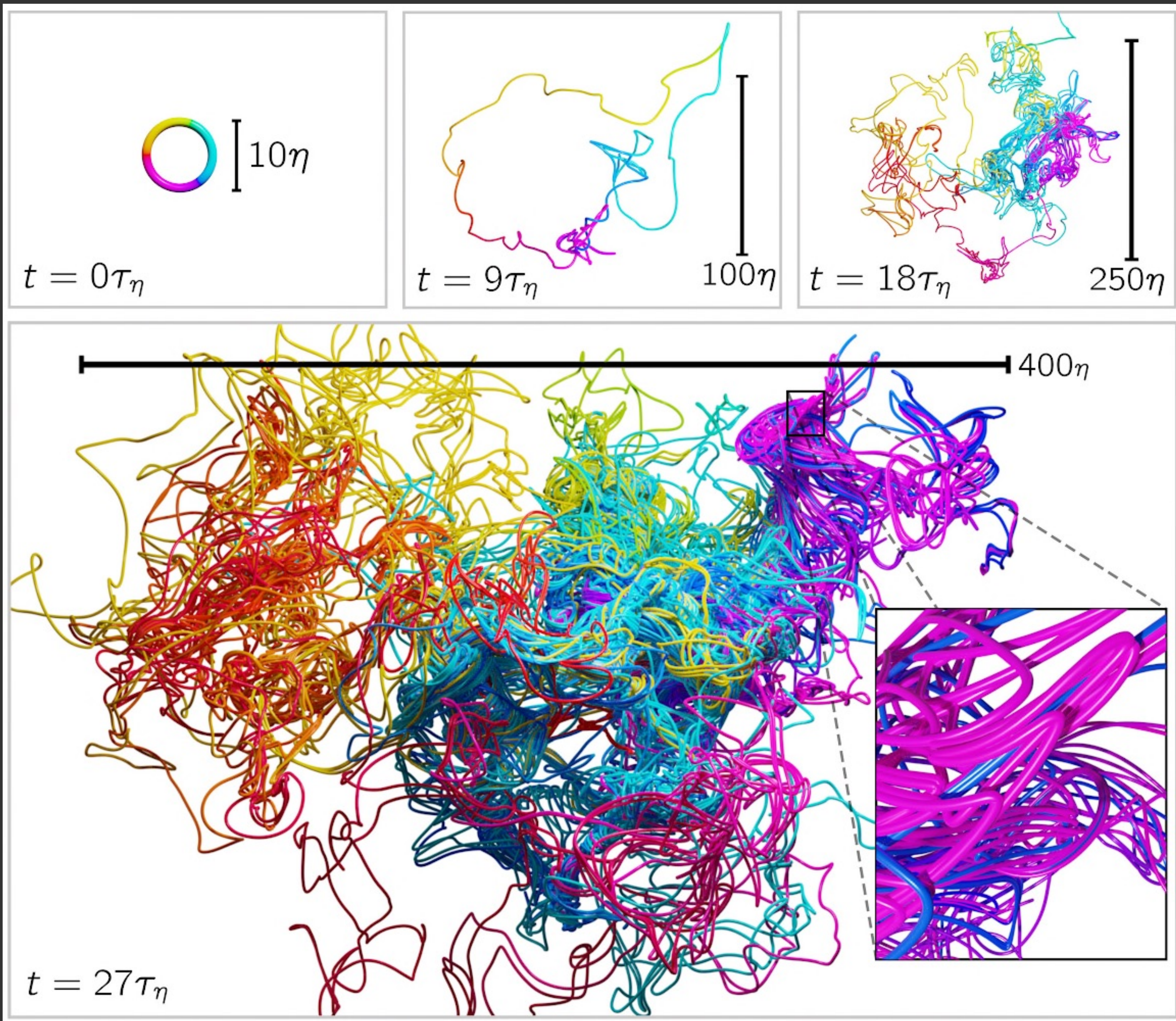
Drivas - Mailybaev, Drivas - Mailybaev - Raibekas (2018, 2020)

ALSO IN SPACE OF VELOCITIES! Kraichnan-Leith

Kolmogorov 1958-1959 seminar

8. *Consideration (at least on models) of the conjecture that, in the situation at the end of 5 above, in the limit the dynamical system turns into a random process (the conjecture of the practical impossibility of a long-term weather forecast).*

Arnol'd - Khesin



(Brentkampt-Drivas-Lalescu-Wilczek '22)

Lagrangian Formulae for Dissipation Anomaly

- Link between Lagrangian reversibility & Anomalous Dissipation

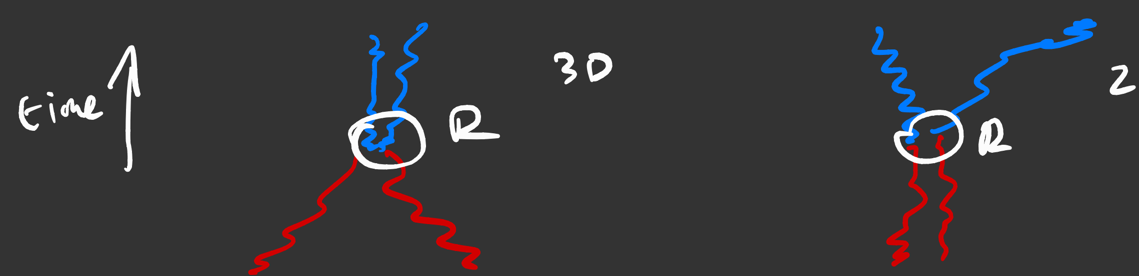
THEOREM: (Drivas, 2019) Let $u \in L^3(0,T;L^3)$ be a weak Euler solution. Then

$$D[u] = \lim_{\substack{\tau\to 0\\ \ell\to 0\\ R\to 0}} \left[\frac{\langle |\delta X^{\ell}_{t,t-\tau}(r;x) - r|^2 \rangle_{R,x,t} - \langle |\delta X^{\ell}_{t,t+\tau}(r;x) - r|^2 \rangle_{R,x,t}}{4\tau^3} \right]$$

where $\dot{X}^{\ell}_{t,s} = \bar{u}_{\ell}(X^{\ell}_{t,s}, s)$.

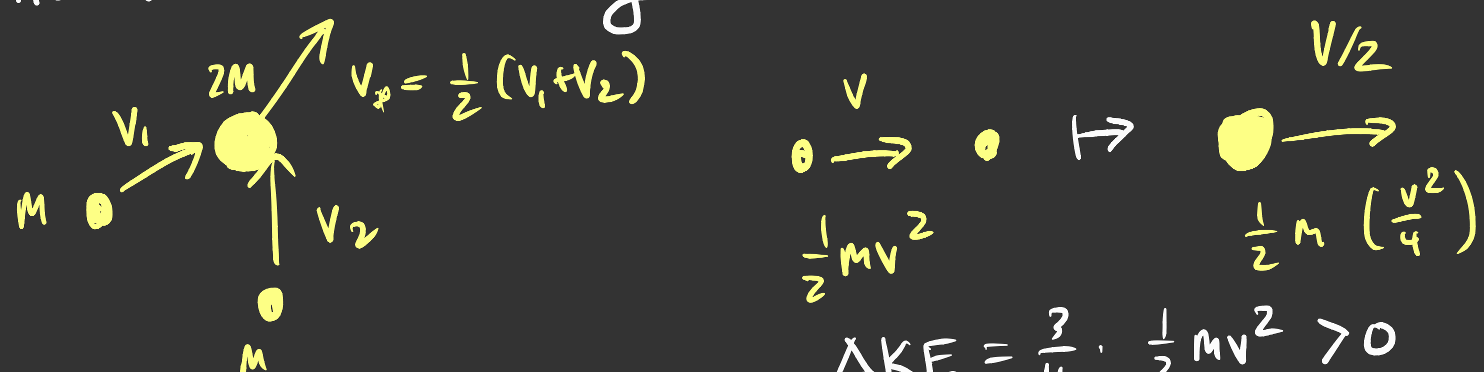
Note: $D[u] = \varepsilon[u] > 0$ in 3d

$D[u] = -I[u] < 0$ in 2d, small-scale forced



Uses rigorous version of Ott-Mann-Gawedzki relation, noted by Jucha et al (2014) to link with irreversibility.

Heuristic understanding: inelastic collision



$$\Delta KE = \frac{3}{4}\cdot\frac{1}{2}mv^2 > 0$$

Scaling laws for Lagrangian Velocity

$\dot{X}_t(a) = V(t,a)$ $\qquad$ $V(t,a) = u(X_t(a),t)$

Let us determine the order of magnitude v_λ of the turbulent velocity variation over distances of the order of λ. It must be determined only by ε and, of course, the distance λ itself.† From these two quantities we can form only one having the dimensions of velocity, namely $(\varepsilon\lambda)^{1/3}$. Hence we can say that the relation

$$v_\lambda \propto (\varepsilon\lambda)^{1/3} \tag{33.6}$$

must hold. We thus find that the velocity variation over a small distance is proportional to the cube root of the distance (*Kolmogorov and Obukhov's law*). The quantity v_λ may also be regarded as the velocity of turbulent eddies whose size is of the order of λ: the variation of the mean velocity over small distances is small compared with the variation of the fluctuating velocity over those distances, and may be neglected.

The relation (33.6) may be obtained in another way by expressing a constant quantity, the dissipation ε, in terms of quantities characterizing the eddies of size λ; ε must be proportional to the squared gradient of the velocity v_λ and to the appropriate turbulent viscosity coefficient $\nu_{\mathrm{turb},\lambda} \propto v_\lambda\lambda$:

$$\varepsilon \propto \nu_{\mathrm{turb},\lambda}(v_\lambda/\lambda)^2 \propto v_\lambda{}^3/\lambda,$$

whence we obtain (33.6).

Let us now put the problem somewhat differently, and determine the order of magnitude v_τ of the velocity variation at a given point over a time interval τ which is short compared with the time $T \sim l/u$ characterizing the flow as a whole. To do this, we notice that, since there is a net mean flow, any given portion of the fluid is displaced, during the interval τ, over a distance of the order of τu, u being the mean velocity. Hence the portion of fluid which is at a given point at time τ will have been at a distance τu from that point at the initial instant. We can therefore obtain the required quantity v_τ by direct substitution of τu for λ in (33.6):

$$v_\tau \propto (\varepsilon\tau u)^{1/3}. \tag{33.7}$$

The quantity v_τ must be distinguished from v_τ', the variation in velocity of a portion of fluid as it moves about. This variation can evidently depend only on ε, which determines the local properties of the turbulence, and of course on τ itself. Forming the only combination of ε and τ that has the dimensions of velocity, we obtain

$$v_\tau' \propto (\varepsilon\tau)^{1/2}. \tag{33.8}$$

Unlike the velocity variation at a given point, it is proportional to the square root of τ, not to the cube root. It is easy to see that, for τ small compared with T, v_τ' is always less than v_τ.‡

Landau - Lifshitz Prediction:

$$\boxed{\delta_\tau V(a) := V(t+\tau, a) - V(t,a) \sim (\varepsilon\tau)^{1/2}}$$

Theorem (P. Isett, 25') Let $u \in L^\infty_t C^\alpha_x$ be a weak solution of Euler. For each $a \in \mathbb{T}^d$, consider <u>any</u> solution of

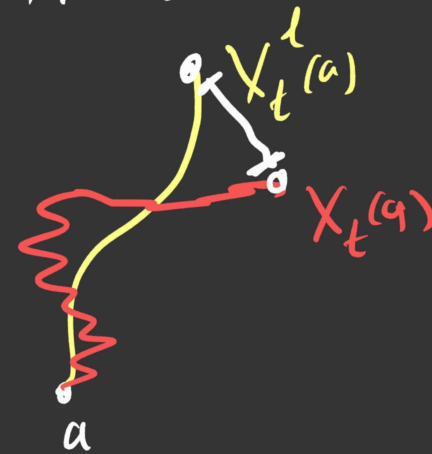
$$\frac{d}{dt} X_t(a) = u(X_t(a), t)$$
$$X_0(a) = a$$

Then $X_t \in L^\infty_a C_t^{\frac{1}{1-\alpha}}$. Thus $\dot{X}_t = L_a C_t^{\frac{\alpha}{1-\alpha}}$

Remark: If $\alpha = 1/3$ (K41 value), then $\frac{\alpha}{1-\alpha} = 1/2$ (Landau)

Proof: Let us introduce trajectories in the coarse-grained field:

$$\frac{d}{dt} X^\ell_t(a) = \bar{u}_\ell(X^\ell_t(a), t)$$
$$X^\ell_0(a) = a$$



We shall estimate:

$$|\dot{X}_{t+\tau}(a) - \dot{X}_t(a)| \leq |\dot{X}^\ell_{t+\tau}(a) - \dot{X}^\ell_t(a)|$$
$$+ |\dot{X}_{t+\tau}(a) - \dot{X}^\ell_{t+\tau}(a)|$$
$$+ |\dot{X}^\ell_t(a) - \dot{X}_t(a)|$$

We assume $\alpha \in [0, 1/2]$. Others similar.

Double Hölder regularity of p
$[p]_{C^{2\alpha}} \leq [u]^2_{C^\alpha}$

Let $\tau_\ell := "\frac{\ell}{\delta_\ell u}" = \ell^{1-\alpha}$,

$$|\dot{X}^\ell_{t+\tau}(a) - \dot{X}^\ell_t(a)| = -\int_t^{t+\tau} [\nabla p_\ell + \nabla\cdot\tau_\ell](X_s(a), s)\, ds$$

$$\leq \tau \ell^{2\alpha-1} [u]^2_{C^\alpha} \lesssim \ell^\alpha \frac{\tau}{\tau_\ell} = \left(\frac{\tau}{\tau_\ell}\right)\cdot \tau_\ell^{\frac{\alpha}{1-\alpha}}$$

Next, we find $|X^\ell_t(a) - X_t(a)| \lesssim \ell e^{t/\tau_\ell}$. Indeed:

$$|X^\ell_t(a) - X_t(a)| \leq \int_0^t |\bar{u}_\ell(X^\ell_s(a), s) - u(X_s(a), s)|\, ds$$

$$\leq \int_0^t |\bar{u}_\ell(X^\ell_s(a)) - \bar{u}_\ell(X_s(a))|\, ds$$

$$+ \int_0^t |\bar{u}_\ell(X_s(a)) - u(X_s(a))|\, ds$$

$$\leq |\nabla \bar{u}_\ell|_{L^\infty} \int_0^t |X^\ell_s(a) - X_s(a)|\, ds + [u]_{L^\infty_t C^\alpha_x} \ell^\alpha$$

$$\lesssim \ell^{\alpha-1}\left[\int_0^t |X^\ell_s(a) - X_s(a)|\, ds + \ell\right]$$

Grönwall's inequality gives the result.

Finally, we must contend with two terms of form

$$|\dot{X}_t^\ell(a) - \dot{X}_t(a)| = |\bar{u}_\ell(X_t^\ell, t) - u(X_t, t)|$$

$t \in [0, \tau_\ell]$

$$\leq |\bar{u}_\ell(X_t^\ell, t) - u(X_t^\ell, t)| + |u(X_t^\ell, t) - u(X_t, t)|$$

$$\leq [u]_{C^\alpha}\left[\ell^\alpha + |X_t^\ell - X_t|^\alpha\right]$$

$$\lesssim \ell^\alpha e^{\alpha t/\tau_\ell}$$

Thus, combining the above results.

$$\boxed{|\dot{X}_{t+\tau}(a) - \dot{X}_t(a)| \lesssim \tau_\ell^{\frac{\alpha}{1-\alpha}}\left[1 + e^{\alpha t/\tau_\ell} + e^{\alpha(t+\tau)/\tau_\ell}\right]}$$

Thus, provided $t = 0$ and $\tau \leq \tau_\ell := h$, we find

arbitrary, could have labelled at any time T.

$$\sup_{\tau \leq h} |\dot{X}_\tau(a) - \dot{X}_0(a)| \leq h^{\frac{\alpha}{1-\alpha}} \implies X_t \text{ is Hölder at } t=0.$$

Arbitrariness of "$t=0$" proves paths are Hölder cont. $\blacksquare$

Questions:

① Is there a Lagrangian analogue of Onsager's conjecture in C^α?

Namely, if all trajectories of a weak solution of Euler are $L^\infty_a C^{1/2+}_t$, does this imply conservation?

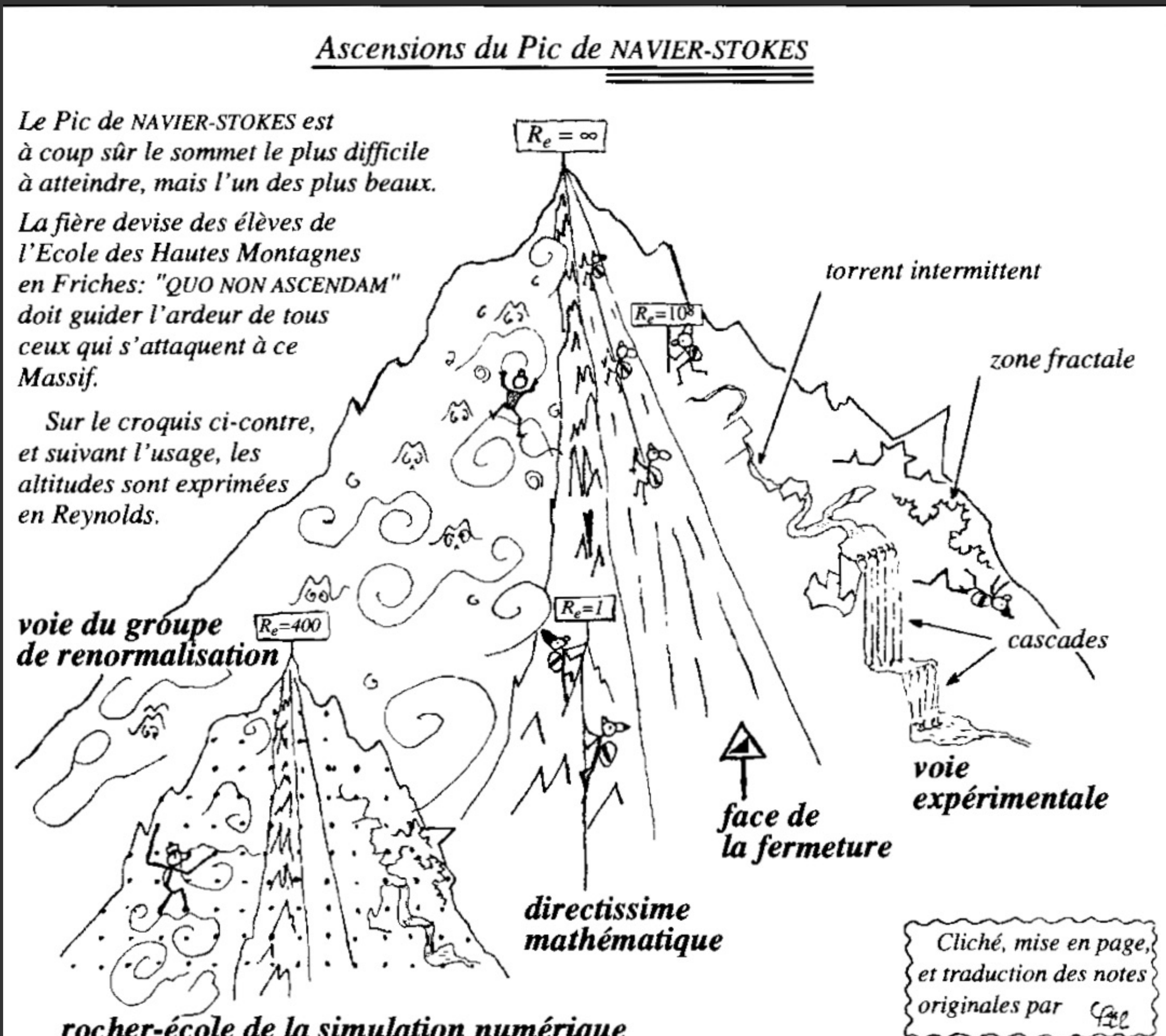
Note: $L^\infty_a C^{\alpha+}_t$ regularity of all trajectories does not imply any regularity of Eulerian vector field (think shear flows).

② Is there a Besov analogue of the trajectory regularity theorem? Namely can one show if $u \in B^{\alpha}_{3,\infty}$

$$\| X_{t+\tau}(a) - X_t(a) \|_{L^2_{t,a}} \lesssim \tau^{\frac{\alpha}{1-\alpha}},$$

or, at least a uniform bound along approximations?

Thank you !!



Ascensions du Pic de NAVIER-STOKES
Le Pic de NAVIER-STOKES est à coup sûr le sommet le plus difficile à atteindre, mais l'un des plus beaux.
La fière devise des élèves de l'Ecole des Hautes Montagnes en Friches: "QUO NON ASCENDAM" doit guider l'ardeur de tous ceux qui s'attaquent à ce Massif.
Sur le croquis ci-contre, et suivant l'usage, les altitudes sont exprimées en Reynolds.
$R_e = \infty$
torrent intermittent
$R_e = 10^8$
zone fractale
$R_e = 1$
$R_e = 400$
voie du groupe de renormalisation
cascades
voie expérimentale
face de la fermeture
directissime mathématique
rocher-école de la simulation numérique
Cliché, mise en page, et traduction des notes originales par

Turbulence is a life force. It is opportunity. Let's love turbulence and use it for change.

Lucky Numbers 34, 15, 28, 4, 19, 20