On the development of shocks and cusps

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Compressible Euler System

$\rho : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$, $\quad u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $\quad E : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$

mass:	$\partial_t \rho + \operatorname{div}(u\rho) = 0$
momentum:	$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = 0$
energy:	$\partial_t E + \operatorname{div}((p+E)u) = 0$

The energy is decomposed

$$E = \frac{1}{2}\rho|u|^2 + e$$

($\frac{1}{2}\rho|u|^2$: kinetic, e: internal)

This system is closed by declaring the internal energy is related to the pressure

$$e = \frac{1}{\gamma-1} p, \qquad \gamma = \frac{c_p}{c_v} > 0 \text{ : adiabatic index}$$

This is the ideal gas law: $pV = nRT$, $\rho = \frac{n}{V}$

$R = c_p - c_v$, $\quad e = c_v \rho T$. Thus

$$p = (c_p - c_v)\rho T = \left(\frac{c_p}{c_v} - 1\right)e = (\gamma-1)e.$$

Define the entropy (per unit mass)

$$s(\rho, e) = \log\left(\frac{p}{\rho^\gamma}\right)$$

Remarkably, for classical solutions

$$\partial_t s + u \cdot \nabla s = 0.$$

Then the entropy satisfies the conservation law

$$\partial_t (\rho s) + \nabla \cdot (\rho s u) = 0$$ and ≥ 0 for nonideal fluid model

As long as the solution (ρ, u, E) remains smooth, it can be replaced by the system

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0 \\ &\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + pI) = 0 \\ &\partial_t (\rho s) + \nabla \cdot (\rho s u) = 0 \end{aligned}$$

WARNING: not right after shocks!

this is what is implicitly done when propagating isentropic shock.

with pressure function $p(\rho, s) = \rho^\gamma e^s$

1D Theorem (Buckmaster, D., Shkoller, Vicol, 2021)

From open set of smooth initial conditions at $t=0$ (1D; isentropic)

- There forms a Hölderian preshock at some $t_* > 0$ where $u_*, \rho_* \in C^{1/3}$.
- The blowup enjoys a fractional series expansion

$$\rho_*(x), u_*(x) = c_0 + c_1 x^{1/3} + c_2 x^{2/3} + c_3 x + O(x^{4/3})$$

After the preshock, the solution is continued as an entropy producing shock with properties

- Shock front $\{x = y(t)\}$ along which

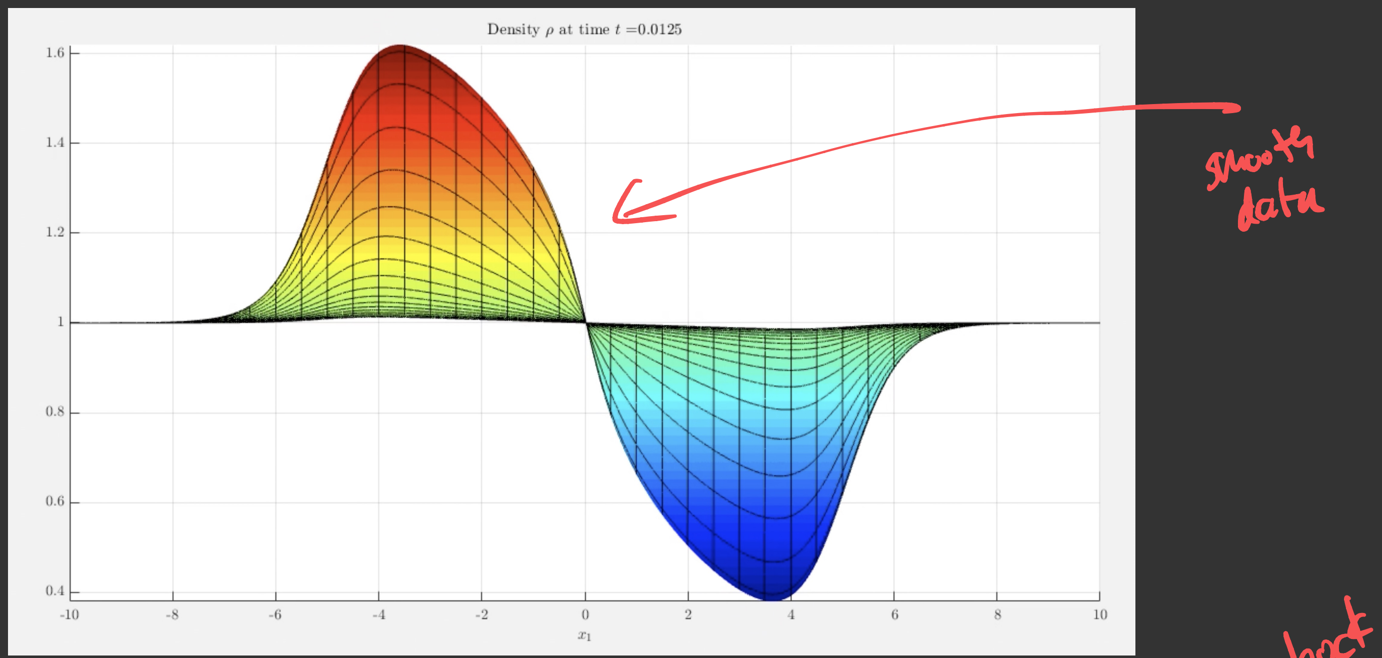
$$[\![u]\!] \sim (t-t_*)^{1/2} \quad [\![\rho]\!] \sim (t-t_*)^{1/2} \quad [\![S]\!] \sim (t-t_*)^{3/2}$$

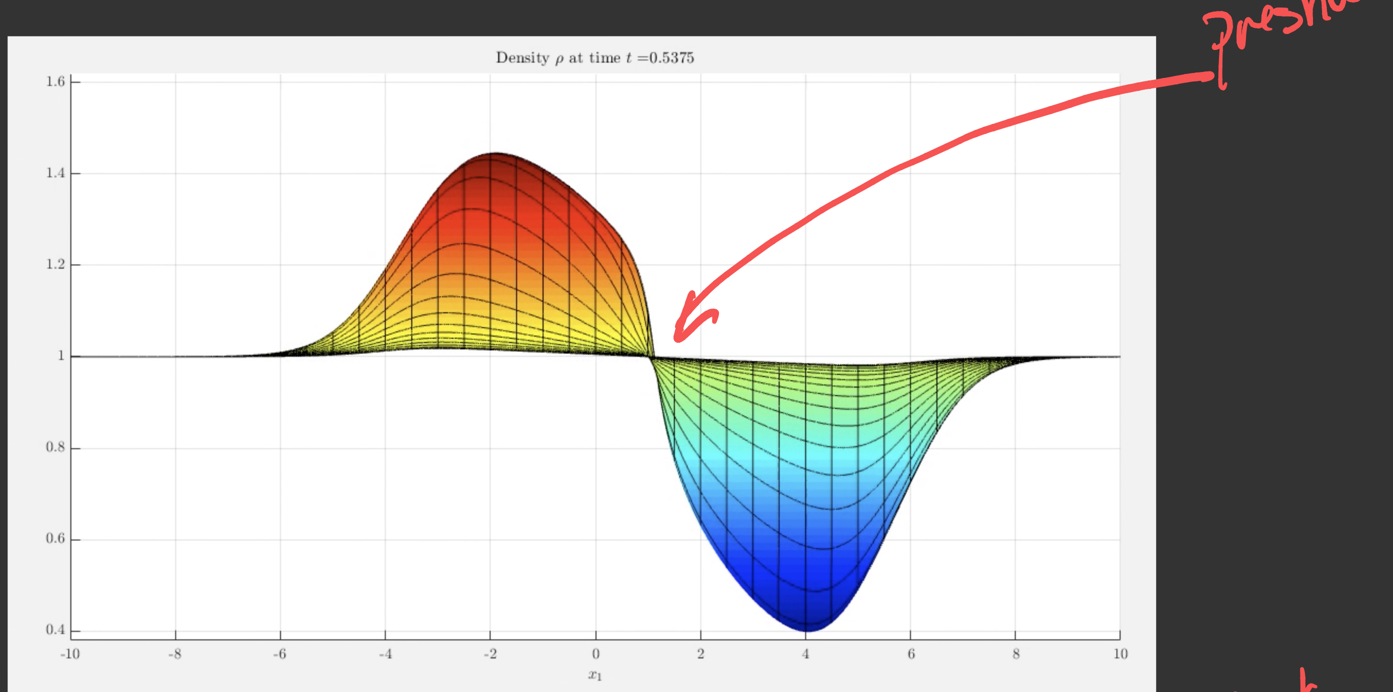
- A characteristic surface for u, $\{x = y_2(t)\}$, has $P, \rho, S \in C^{3/2}$ cusp whereas $u \in C^2$

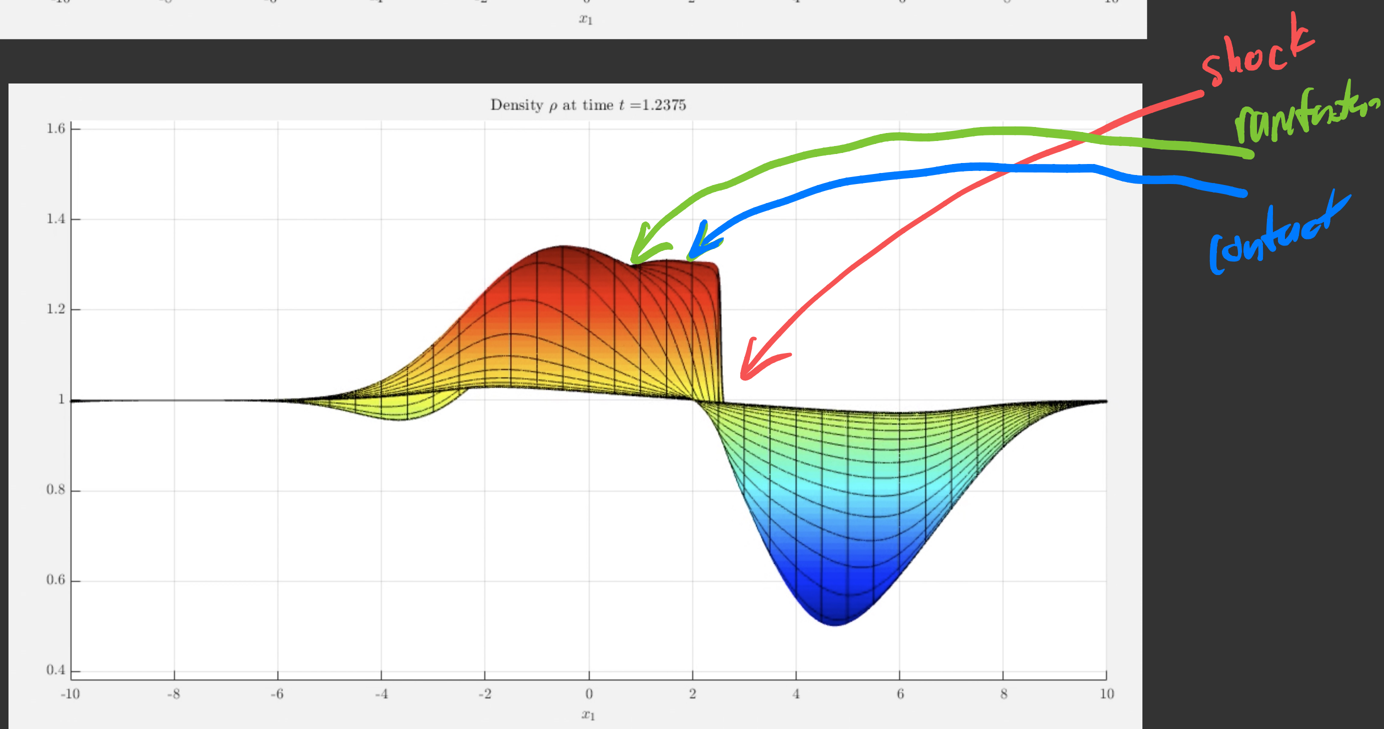
weak contact

- A characteristic surface of $u - c$, $\{x = y_1(t)\}$ has $\rho, u \in C^{3/2}$ cusp, $S=0$ and $P \in C^2$.

weak rarefaction


Density ρ at time t =0.0125
x_1
smooth data


Density ρ at time t =0.5375
x_1
preshock


Density ρ at time t =1.2375
x_1
shock
rarefaction
contact

Previous works (in brief)

- Landau and Lifshitz asserted existence of weak singularities, no examples.
- 1d hyperbolic conservation laws (scalar and systems) have a long history. See Dafermos (2010). These methods allow to continue solutions past singularity but do not give detailed structural information.
- Majda (1983) evolves a preexisting shock.
- Lebaud (1994) did first study of this for a reduced 1D model (isentropic), but did not investigate cusps. Follow ups, Chen & Dong, Kong (2001, 2002), Christodoulou-Lisibach (2016)
- Christodoulou (2019) studies irrotational development outside symmetry completely. Discovered cusps, but not the right jump conditions and thus physics.

Shock formation

Let $c = \sqrt{\partial p / \partial \rho}$ be the speed of sound

Introduce Riemann variables:

$$w := u + \frac{1}{\alpha} c, \quad z := u - \frac{1}{\alpha} c, \quad \alpha := \frac{\gamma - 1}{2}$$

Then the system for (ρ, u, S) is equivalent to

$$\begin{aligned} \partial_t w + \lambda_3 \partial_x w &= \frac{\alpha}{8\gamma} (w-z)^2 \partial_x S \\ \partial_t S + \lambda_2 \partial_x S &= 0 \\ \partial_t z + \lambda_1 \partial_x z &= \frac{\alpha}{8\gamma} (w-z)^2 \partial_x S \end{aligned}$$

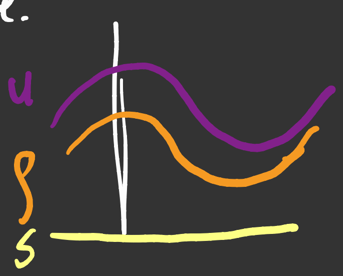
where $\lambda_3 = u + c, \quad \lambda_2 = u, \quad \lambda_1 = u - c$

Original system is recovered by

$$u = u(w, z, S) \quad \rho = \rho(w, z, S) \quad E = E(w, z, S)$$

If $S_0 = \text{constant}$, $S(t) = \text{const.}$ Thus if $z_0 = 0$, $z(t) = 0$

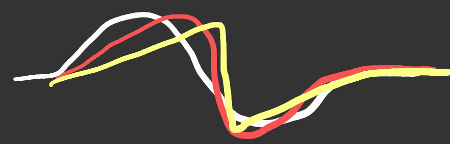
i.e.



Then $u + c = \frac{1+\alpha}{2} w$ and set $\frac{1+\alpha}{2} = 1$ by changing time

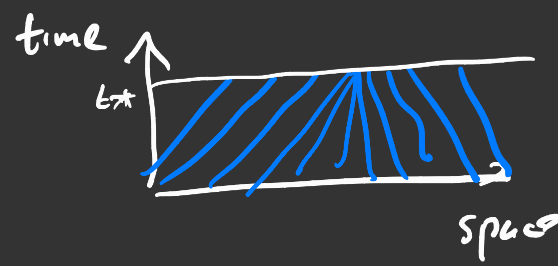
$$\partial_t w + w \partial_x w = 0$$

Burgers equation



$$\partial_t w + w \partial_x w = 0$$

$$w\big|_{t=0} = w_0$$



Let η be characteristics

$$\frac{d}{dt}\eta(t,x) = w(\eta(t,x),t), \qquad \eta_0(x) = x$$

Then $\frac{d}{dt} w(\eta(t,x),t) = 0$ so $w(\eta(t,x),t) = w_0(x)$

and thus $\eta(t,x) = x + t w_0(x)$

Since $w_x = \partial_x w$ satisfies

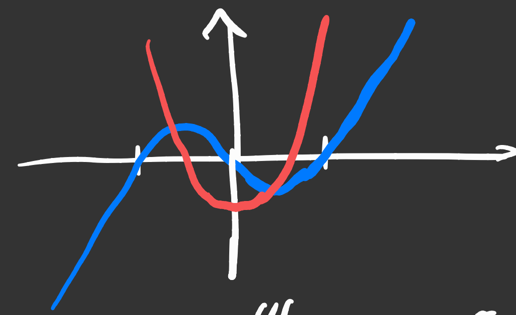
$$\partial_t w_x + w \partial_x w_x = -w_x^2$$

$$\boxed{w_x(\eta(t,x), t) = \frac{w_0'(x)}{1 + t w_0'(x)}}$$

First singularity emerges from label x_* at which $w_0'(x_*)$ is <u>most negative</u>. Time of blowup is $t_* = 1/-w_0'(x_*)$.

Typical situation: $w_0(x) = -x + x^3$

$w_0'(x) = -1 + 3x^2$



$x_* = 0$, $w_0'(x_*) = -1$, $w_0''(x_*) = 0$, $w_0'''(x_*) = 6 > 0$

generic in that it is stable under C^3 perturbation.

Blowup occurs at time $t_* = 1/-w_0'(0) = 1$.

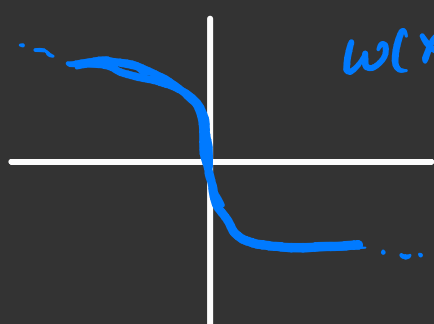
Recall $\eta(t,x) = (1-t)x + tx^3$

Consider $|x| \ll 1$. Since $w(x,1) = w_0(\eta^{-1}(1,x))$

and $\eta(1,x) = x^3$, $\eta^{-1}(1,x) = x^{1/3} = \mathrm{sgn}(x)\,|x|^{1/3}$.

Thus $w(x,1) = w_0(x^{1/3}) = -x^{1/3} + x$

↑ ceases to be a diffeo but still a homeo.

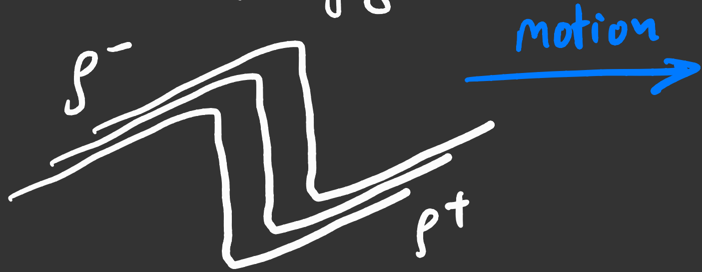


we should expect precisely $C^{1/3}$ cusps to arise from generic initial conditions! [Chr. 07] [BSV, 20, 21]

NOTE: $w_0 = -x + x^n$ gives $C^{1/n}$ cusp for any $n \geq 2$.
Can be smooth, but not generic.

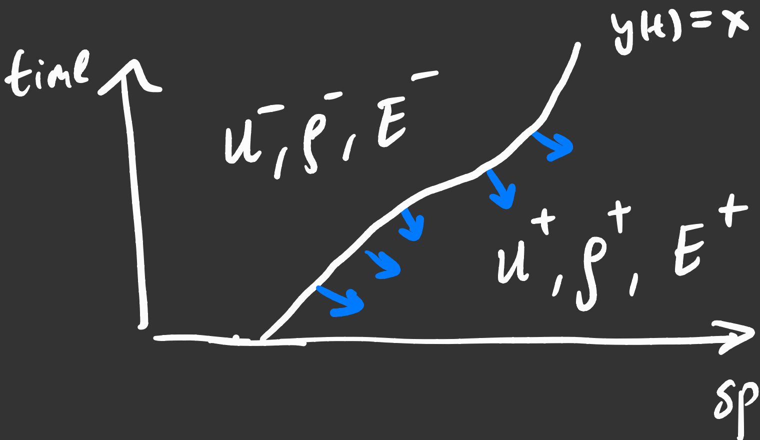
Life after the first singularity

... shock forms, the equations hold classically away from the shock front and weakly across, to conserve mass, momentum and total energy.



Shock front: $\Sigma \subset \mathbb{R}^d \times (T_1, T_2]$

orientable hypersurface



$\Sigma = \{ x = y(t) \}$

spacetime normal

: $(-\dot{y}(t), 1)$.

across the shock, the solution jumps

$$[\![\rho]\!] = \rho^- - \rho^+, \quad \dots \quad \rho^{\pm} = \rho|_{y^{\pm}}$$

in a way consistent with mass, momentum, energy cons.

Note if

$$\partial_t \rho + \partial_x f = 0 \quad , \qquad f = \rho u$$

$$\frac{d}{dt}\int_{\pi} \rho \, dx = \frac{d}{dt}\int_{-\pi}^{y(t)} \rho \, dx + \frac{d}{dt}\int_{y(t)}^{\pi} \rho \, dx$$

$$= \dot{y}(t)(\rho^- - \rho^+) + \int_{\pi} \partial_t \rho \, dx$$

$$= \dot{y}(t)[[\rho]] - \int_{-\pi}^{y(t)} \partial_x f \, dx - \int_{y(t)}^{\pi} \partial_x f \, dx$$

$$= \dot{y}(t)[[\rho]] - [[f]] = 0$$

Thus we arrive at the following conditions

mass :	$\dot{y}(t)\,[[\rho]] = [[\rho u]]$
momentum :	$\dot{y}(t)\,[[\rho u]] = [[\rho u^2 + p]]$
energy :	$\dot{y}(t)\,[[E]] = [[(p+E)u]]$

speed of shock determined to fix one of these.

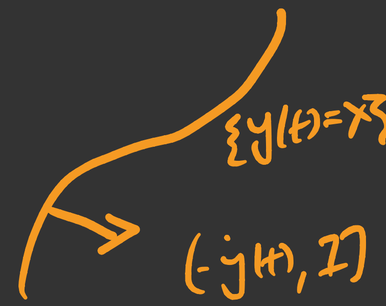
Other two?

Rankine - Hugoniot conditions

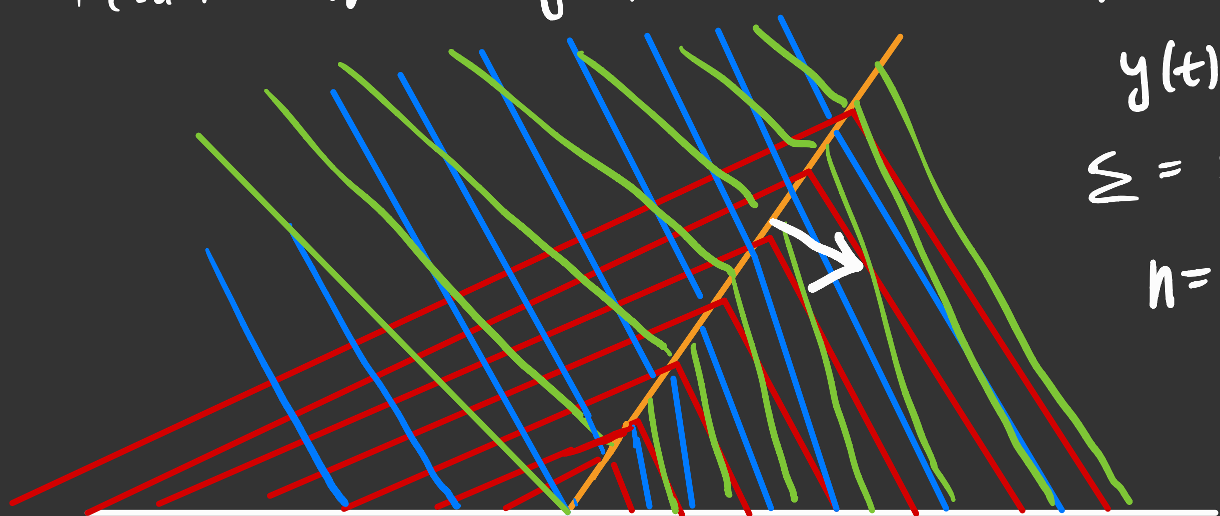
Deterministic development: Lax entropy conditions

movement of information

$$\begin{cases} \partial_t + (u+c)\,\partial_x = (1, u+c)\cdot\nabla_{t,x} \\ \partial_t + u\,\partial_x = (1, u)\cdot\nabla_{t,x} \\ \partial_t + (u-c)\,\partial_x = (1, u-c)\cdot\nabla_{t,x} \end{cases}$$



Recall $\hat{n} = (-\dot{y}(t), 1)$ is shock spacetime normal



$y(t) = kt$

$\Sigma = \{x - tk = 0\}$

$n = (-k, 1)$

$n\cdot(1, u^+ - c^+) < 0$, $n\cdot(1, u^+) < 0$, $n\cdot(1, u^+ + c^+) < 0$

$n\cdot(1, u^- - c^-) < 0$, $n\cdot(1, u^-) < 0$, $n\cdot(1, u^- + c^-) > 0$

Physical entropy conditions:

$$\frac{d}{dt}\int_{\mathbb{T}} \rho S\,dx > 0 \iff [\![S]\!] > 0$$

Theorem: For weak shocks, Lax $\iff$ physical entropy

mass :	$j(t)\,[\![\rho]\!] = [\![\rho u]\!]$	← speed of shock determined
momentum :	$j(t)\,[\![\rho u]\!] = [\![\rho u^2 + p]\!]$	← production of s, z on shock
energy :	$j(t)\,[\![E]\!] = [\![(p+E)u]\!]$	←

Recall the Riemann variable system:

$$\partial_t w + (u+c)\partial_x w = \frac{\alpha}{8\gamma}(w-z)^2\partial_x S$$

$$\partial_t s + u\,\partial_x S = 0$$

$$\partial_t z + (u-c)\partial_x z = \frac{\alpha}{8\gamma}(w-z)^2\partial_x S$$

holds to either side of the shock.

$$[\![\rho u]\!]^2 = [\![\rho u^2 + p]\!]\,[\![\rho]\!] \qquad E_1(w^-, w^+, s^-, s^+, z^-, z^+) = 0$$

$$[\![\rho u]\!]\,[\![E]\!] = [\![(p+E)u]\!]\,[\![\rho]\!] \iff E_1(w^-, w^+, s^-, s^+, z^-, z^+) = 0$$

Regard $w^{\pm}$, s^+, z^+ as <u>given</u>, solve coupled 6th order polynomial system for s^- and z^-.

We express

$$\langle\langle w \rangle\rangle = \frac{w^- + w^+}{2}, \qquad [[w]] = w^- - w^+$$

Lemma: there exists exactly one real root consistent with entropy production. Moreover, if $|[[w]]| \ll 1$

$$[[z]] = -C_z(\gamma, \langle\langle w \rangle\rangle, z_+)\, [[w]]^3 + \mathcal{O}([[w]]^5)$$
$$[[s]] = C_s(\gamma, \langle\langle w \rangle\rangle, z_+)\, [[w]]^3 + \mathcal{O}([[w]]^5)$$
$$\dot{\jmath} = \underbrace{C_j(\gamma, \langle\langle w \rangle\rangle, z_+)}_{\left(\frac{1+\alpha}{2}\right) K} + \mathcal{O}([[w]]^2)$$

$\left(\frac{1+\alpha}{2}\right) K \leftarrow$ speed of sound at $t=0, x=0$

Recall

$$\partial_t w + \Big(\overbrace{\left(\tfrac{1+\alpha}{2}\right) w + \left(\tfrac{1-\alpha}{2}\right) z}^{u+c}\Big)\partial_x w = \frac{\alpha}{8\gamma}(w-z)^2 \partial_x S$$

Assuming time is short so $|[[w]]| \ll 1$ and thus $|z| \ll 1$ and $|s| \ll 1$ so

$$\partial_t w + \left(w + \text{small error}\right)\partial_x w = \left(\text{small entropic error}\right)$$
$$w_0(x) = K - b x^{1/3} + \left(\text{small error } |x| \ll 1\right)$$

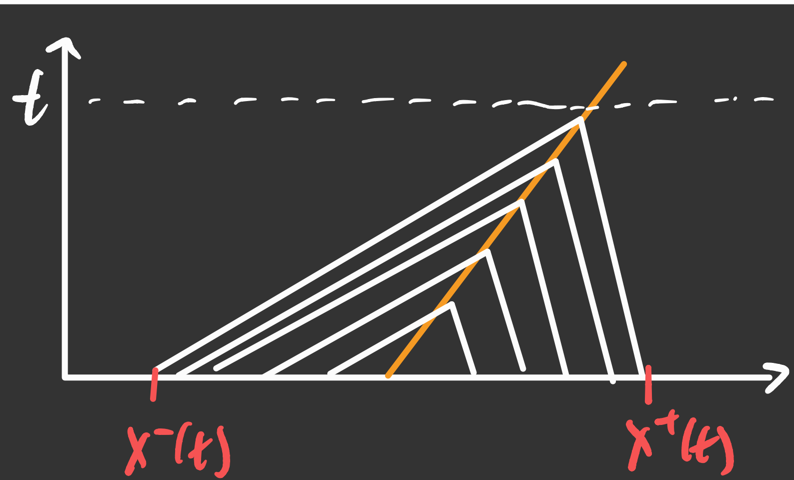
Thus, for short time we say

$$w(t,x) \approx w_B(t,x) = w_0(\eta^{-1}(t,x))$$

where

$$\eta(t,x) = x + t\, w_0(x)$$
$$\approx x + t(k - b x^{1/3}) + \cdots$$

> QUESTION: Which labels $x^{\pm}(t)$ fall into the shock at time $t > 0$?



We look for a such that $X_t(a) = y(t)$, at shock. Recall:

$$y(t) = kt + \left(\begin{matrix}\text{small error}\\ \text{for } |t| \ll 1\end{matrix}\right)$$

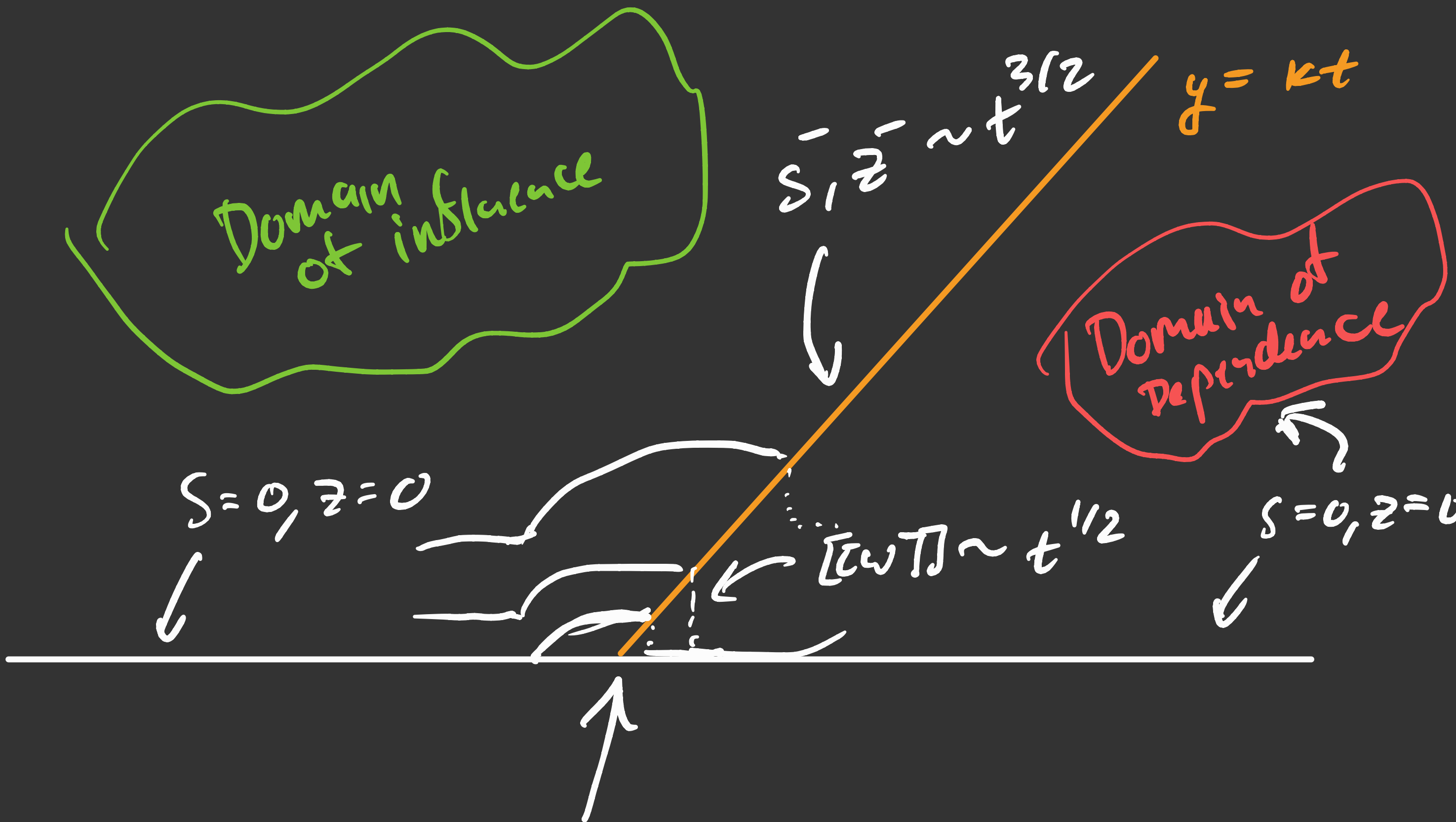
Then $X^{\pm}(t)$ solve $X^{\pm}(t) = \left(\frac{1 \pm \alpha}{2}\right) t\, b\, X^{\pm}(t)^{1/3}$, thus

$$x^{\pm}(t) = \pm (bt)^{3/2} \sim t^{3/2}$$

Thus

$$[[w]] = w_0(a^-) - w_0(a^+) \approx 2b(bt)^{1/2} \sim t^{1/2}$$

Thus, since $[[s]] \sim -[[z]] \sim [[w]]^3$, we find



shock front serves as Cauchy hypersurface for entropy and z in the domain of influence of the shock.

Initial data on $\{x = kt\}$ of type

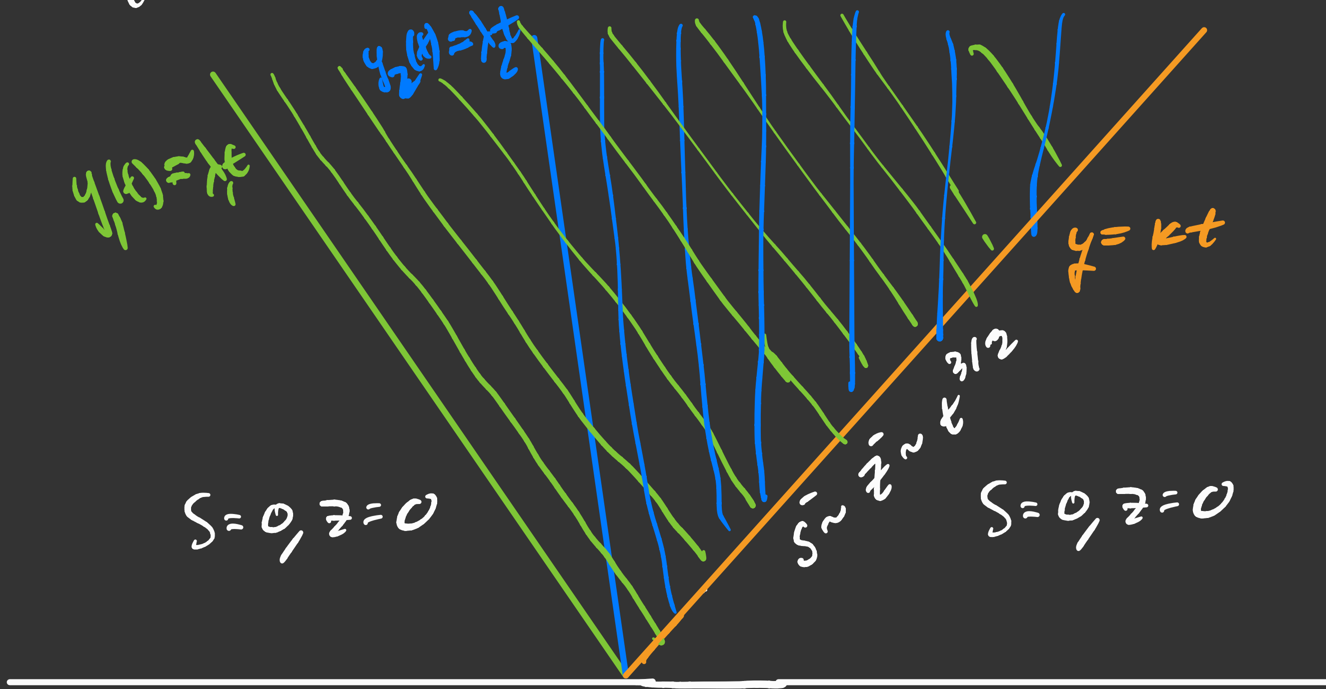
$$z_0(x), s_0(x) = \begin{cases} 0 & x \leq 0 \\ c\, x^{3/2} & x = kt \end{cases}$$

Cusp singularity at $x = 0$.

$$\partial_t w + (u+c)\,\partial_x w = \frac{\alpha}{8\gamma}(w-z)^2\,\partial_x S$$

$$\partial_t S + u\,\partial_x S = 0$$

$$\partial_t z + (u-c)\,\partial_x z = \frac{\alpha}{8\gamma}(w-z)^2\,\partial_x S$$



For short times, near $x=0$

$$(u+c) \approx \kappa, \quad u \approx \frac{\kappa}{1+\alpha}, \quad u+c \approx \left(\frac{1-\alpha}{1+\alpha}\right)\kappa.$$

Flows

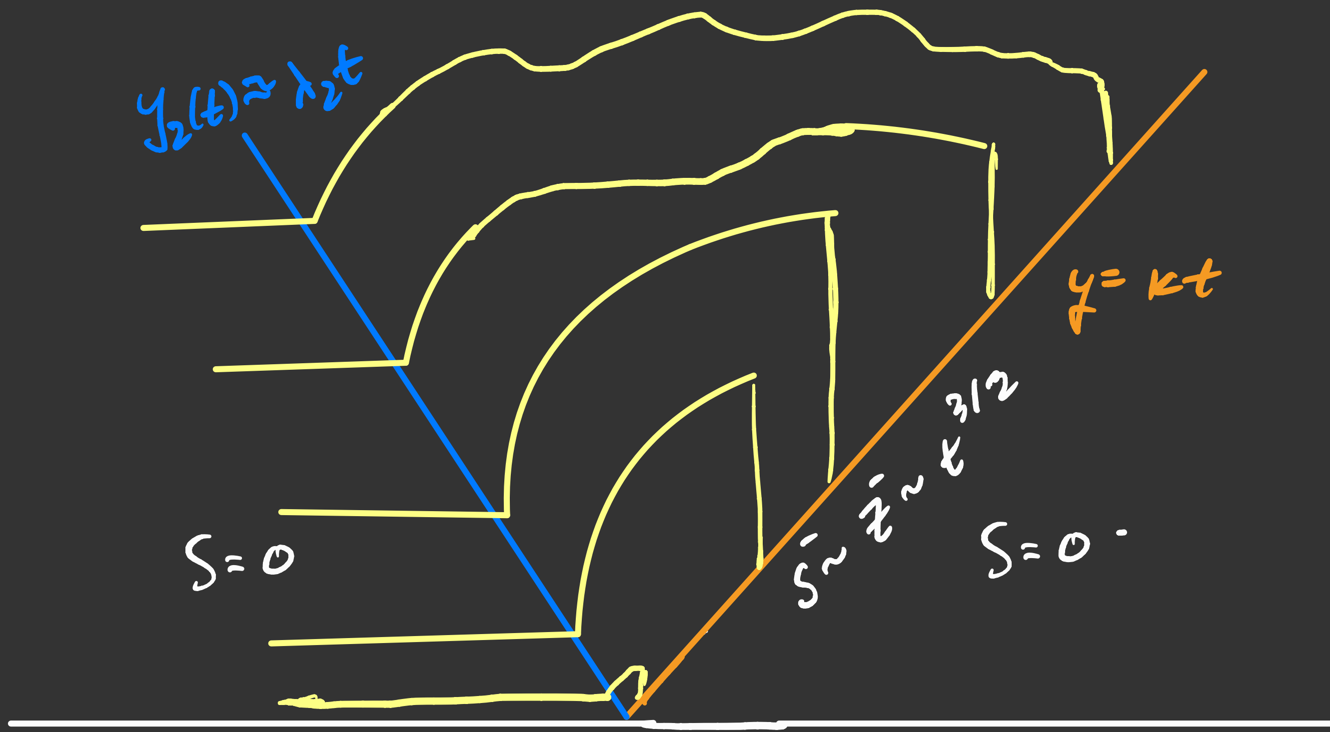
$$\eta_3(t,x) = x + \lambda_3 t$$

$$\eta_2(t,x) = x + \lambda_2 t$$

$$\eta_1(t,x) = x + \lambda_1 t$$

$$0 < \lambda_1 < \lambda_2 < \lambda_3$$

Since entropy is transported: $\partial_t S + \lambda_2 \partial_x S = 0$.



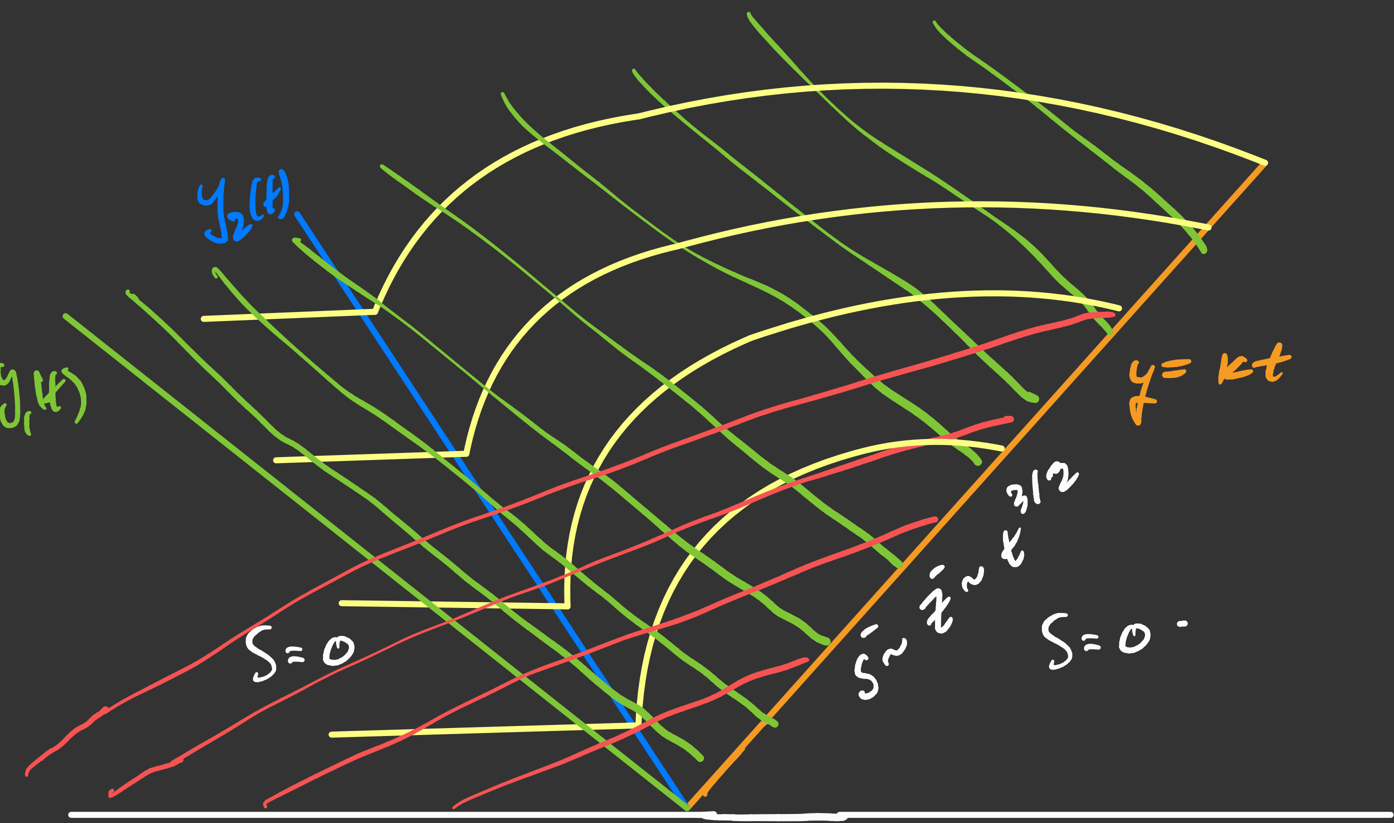
$$S(t,x) \sim \begin{cases} 0 & x < y_2(t) \\ (x - y_2(t))^{3/2} & y_2(t) \leq x \leq y(t) \\ 0 & y(t) \leq x \end{cases}$$

$C^{3/2}$ cusp in entropy propagates with the fluid velocity from the preshock.

$$\partial_t w + (u+c)\partial_x w = \frac{\alpha}{8\gamma}(w-z)^2\partial_x S$$

$$\partial_t z + (u-c)\partial_x z = \frac{\alpha}{8\gamma}(w-z)^2\partial_x S$$

naively $C^{1/2}$ across $y_2(t)$



But $\partial_x S$ acts as a force which is integrated along transversal characteristics

Gain in regularity!

$$z(a + y_2(t), t) - z_0(a + (\lambda_2 - \lambda_1)t)$$

$$\approx \frac{\alpha}{8\gamma}\int_0^t (\sigma^2 S)(a + (\lambda_2-\lambda_1)t + \lambda_1\tau, \tau)\,d\tau$$

$$\approx C\int_{t + a/(\lambda_2-\lambda_1)}^{t} (y - \lambda_1(t-\tau))^{1/2}\,d\tau \sim a^{3/2}$$

Thus, z and w both make $C^{3/2}$ cusp singularities along $\{x = q_2(t)\}$.

$$S \in C^{3/2}, \quad z \in C^{3/2}, \quad w \in C^{3/2}$$

Miraculously, there is a cancellation in the quantity $u = \frac{1}{2}(w+z)$ which makes

$$u \in C^2 \qquad \left(\text{actually should be } C^{5/2}\right)$$

Uses "good unknowns"

$$q_w = \partial_x w - \gamma c \, \partial_x S \qquad q_z = \partial_x z - \gamma c \, \partial_x S.$$

Thus we call this a weak contact discontinuity, since the velocity is slightly smoother than density & pressure.

Across $\{t = y_1(t)\}$ a similar phenomenon happens for z,

$$z(x,t) \approx c \begin{cases} 0 & x < y_1(t) \\ (x - y_1(t))^{3/2} & y_1(t) < x \ll y_2(t) \end{cases}$$

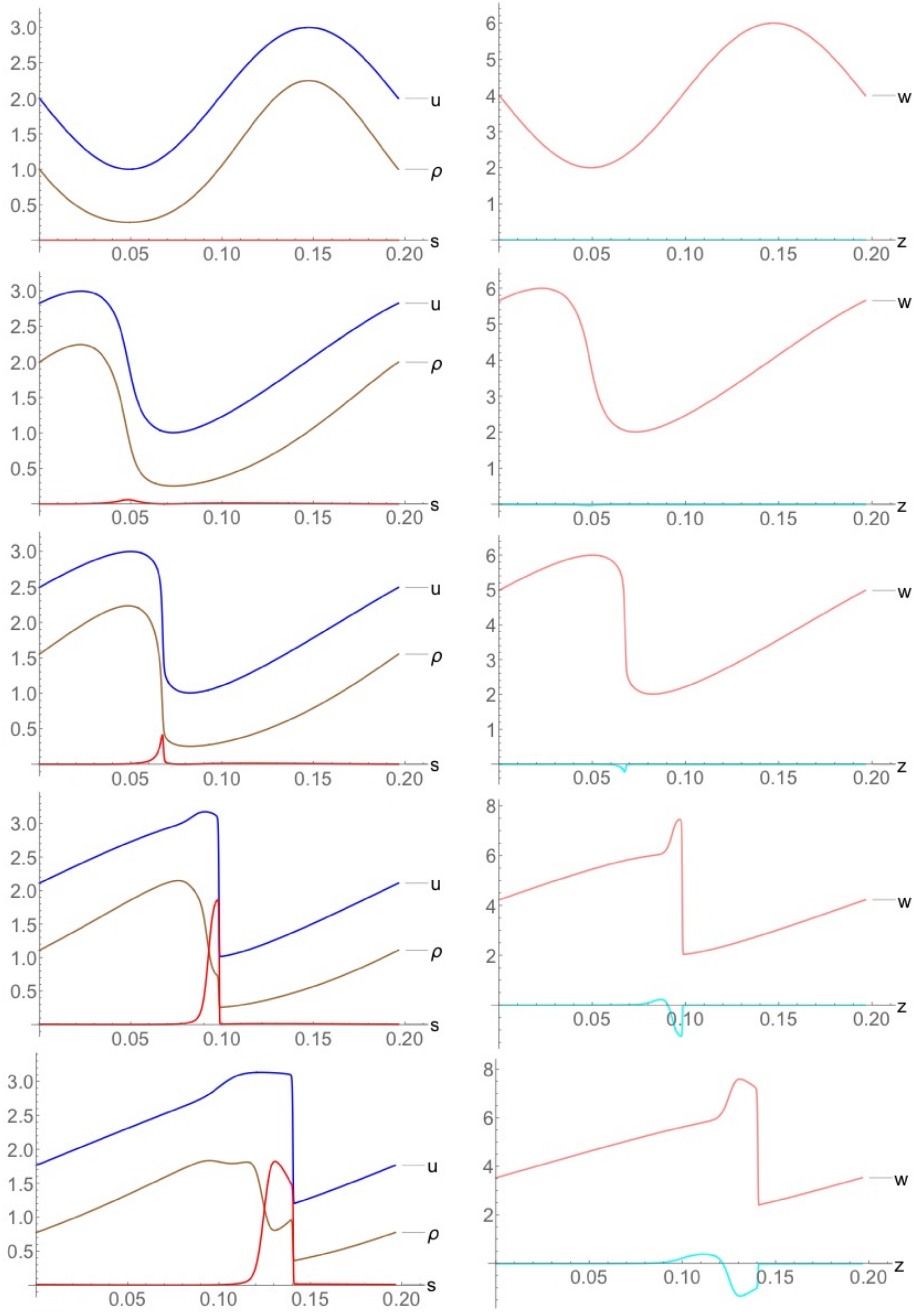
However, here entropy is vanishing, so

$$z \in C^{3/2}, \quad w \in C^{3/2}, \quad s = 0$$

Miraculously, the pressure again smoothen

$$p \in C^2$$

Thus we call this a weak rarefaction wave.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ρ
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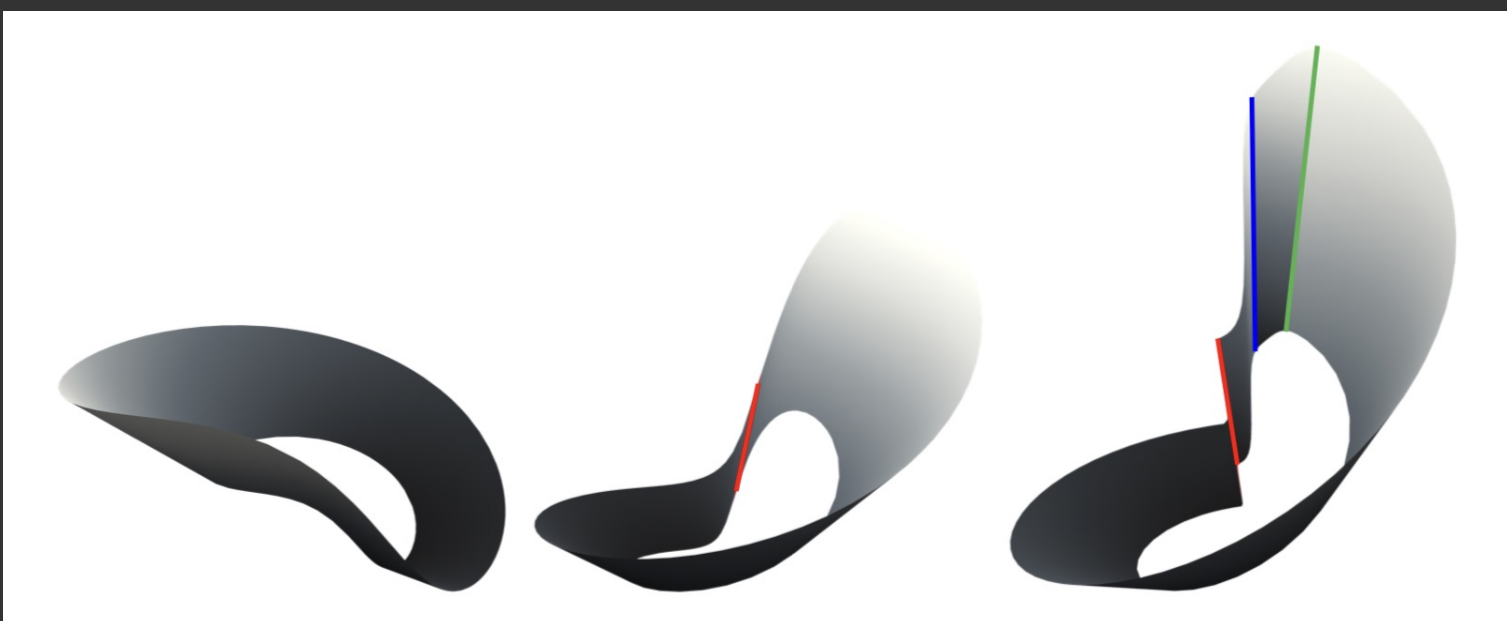
THANK-You!

real story...



2D Azimutal shock development