

Remarks on the long time behavior of 2D Euler

based on joint works with

P. Constantin, D. Ginsberg
M. Dolce, T. Elgindi

Theodore D. Drivas

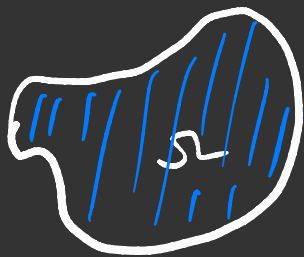
Stony Brook University

October 2022



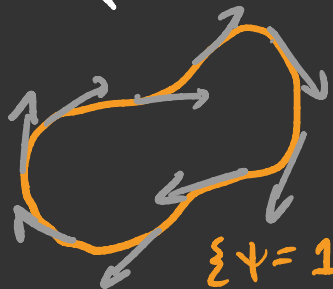
Dynamics of a two dimensional ideal fluid. ①

Ideal fluid in 2D; $u = (u_1, u_2)$



$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p && \text{in } \Omega \\ \nabla \cdot u &= 0 && \text{in } \Omega \\ u \cdot \hat{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

Every solenoidal 2D velocity can be represented by



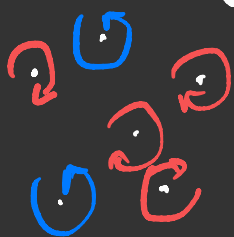
$$u = \nabla^\perp \psi, \quad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$$

↑ streamfunction

Thus, u is pointwise parallel to isolines of ψ .

$$u \cdot \hat{n} = 0 \text{ on } \partial\Omega \iff \psi = \text{const on } \partial\Omega$$

Vorticity has one component $\vec{\omega} = \omega \hat{z}$ with



$$\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$$

in terms of streamfunction $\omega = \Delta \psi$

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 \\ u &= \nabla^\perp \Delta^{-1} \omega \end{aligned}$$

\iff

$$\begin{aligned} \omega(t) &= \omega_0(x_t^{-1}) \\ \dot{x}_t &= u(x_t) \end{aligned}$$

Invariants of the motion:

(2)

Energy: $\frac{d}{dt} \int_{\Omega} \frac{1}{2} |u|^2 dx = 0$ ← uses $u \cdot \vec{n} = 0$ on $\partial\Omega$
 $\nabla \cdot u = 0$ in Ω

Casimirs: $\frac{d}{dt} \int_{\Omega} f(w) dx = 0$ \forall continuous $f: \mathbb{R} \rightarrow \mathbb{R}$

in particular $\|w(t)\|_{L^p} = \|w_0\|_{L^p}$ $\forall p \in [1, \infty]$.

Momentum: $\frac{d}{dt} \int_{\Omega} u dx = 0$ ← on $\Omega = \mathbb{T}^2, \mathbb{T} \times [0, 1], \mathbb{D}$

These facts allow one to define a dynamics

THEOREM: Let $\Omega \subseteq \mathbb{R}^2$ and $w_0 \in C^d(\Omega)$ Then there exists a unique solution $w \in C^d([-\infty, \infty) \times \Omega)$ with

$$\|w(t)\|_{C^d} \leq \left(\frac{\|w_0\|_{C^d}}{\|w_0\|_{L^\infty}} \right) \exp\left(c \|w_0\|_{L^\infty} t / \alpha \right)$$

Gunter, 1926
Lichtenstein, 1925

In fact (Yudovich, 1963), if $w_0 \in L^\infty(\Omega)$ then there exists a unique weak solution $w \in L^\infty([-\infty, \infty) \times \Omega)$.

Moreover the solution depends continuously on the data; i.e.

$$w_0^n \xrightarrow{*} w_0 \text{ then } w(t; w_0^n) \xrightarrow{*} w(t; w_0).$$

Yudovich space is very important for longtime behavior!

Stationary States, Structure and (in)stability

defined by $u \cdot \nabla \omega = 0$ i.e. $\nabla \omega$ and $\nabla \psi$ are colinear
 $u = \nabla \perp \psi$

Here are some special families



$$u(x, y) = v(y) e_x \quad \text{Shear flows}$$



$$u(r, \theta) = v(r) e_\theta \quad \text{Circular flows}$$

Another large subclass of stationary states are

$$\omega = F(\psi) \quad F \in Lip$$

Then, the streamfunction is determined by elliptic problem



$$\begin{aligned} \Delta \psi &= F(\psi) && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial \Omega \end{aligned}$$

This equation can have, none, one or many solutions depending on F . (consider, e.g, $F(x) = \lambda x$)

QUESTION: When must steady states conform to the symmetries of the domain?

Rigidity and Symmetry

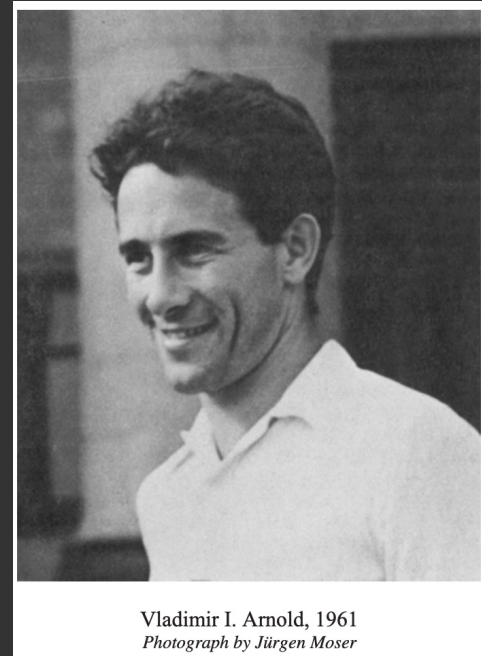
Arnold's Stable steady States

$$\Delta \Psi_0 = F_0(\Psi_0)$$

where F_0 satisfies

$$-\lambda_1 < F_0'(\Psi_0) < 0 \quad \text{or} \quad 0 < F_0'(\Psi_0) < \infty$$

Then, $w_0 = \Delta \Psi_0$ is orbitally stable in L^2 under the Euler dynamics.



Vladimir I. Arnold, 1961
Photograph by Jürgen Moser

Such flows correspond to local maxima or minima of Energy on isovortical sheets, i.e.

$$\mathcal{G}_{w_0} := \{ w : \exists \phi \in \text{Diff}_\mu(\Omega) \text{ s.t. } w = w_0 \circ \phi \}$$

THEOREM: Let (M, g) be a compact two-dimensional Riemannian manifold with smooth bdy ∂M . Let ξ be a Killing field for g tangent to the boundary.

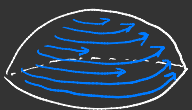
If $u \in C^2(M)$ is Arnold stable, then $\mathcal{L}_\xi u = 0$.



• if M is periodic channel, u is shear



• if M is disk or annulus, u is radial



• if M is spherical cap, u is zonal

• if M has no boundary, $\nabla^2 u$



Other Results: Hamel-Nadirashvili, Gomez-Serrano, Park, Shi, Yao

Steady states come in infinite dimensional families (5)

THEOREM: (Choffrut-Šuvák, 10) Let Ω be an annular domain and consider a non-degenerate Arnold stable steady state. Then each vorticity distribution function in its neighbourhood corresponds to a unique stationary state (on co-adjoint orbit)

$\mathcal{O}_{u_0} = \{ \omega : \omega = \omega_0 \circ \phi, \phi \in \text{Diff}_\mu(\Omega) \}$

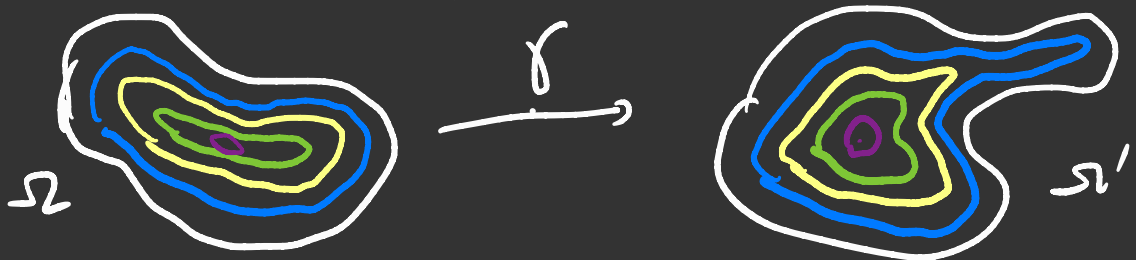
THEOREM: (Constantin-D-Ginsberg, 10) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary and let u_0 be an Arnold stable steady state on Ω with

- u_0 has a single stagnation point in Ω
- $\mu(\Omega) = \oint_{\{\psi=c\}} \frac{d\ell}{|v|} < \infty$

Then, there exists $\varepsilon = \varepsilon(u_0, \Omega)$ such that for all nearby domains $|\Omega' - \Omega| \leq \varepsilon$ and all functions $|p - 1| \leq \varepsilon$ there exists a diffeomorphism $f: \Omega \rightarrow \Omega'$ s.t.

$\det \nabla f = f$

$\int_{\Omega'} p = \int_{\Omega} p$



and $\psi = \psi_0 \circ f^{-1}$ defines Euler solution on Ω' nearby to u_0

$\mathcal{O}_{\psi_0} = \{ \psi : \psi = \psi_0 \circ \phi, \phi \in \text{Diff}_\mu(\Omega) \}$

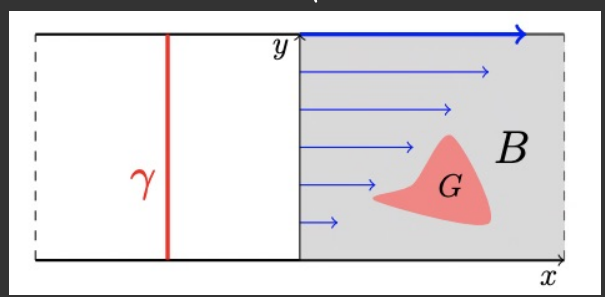
Wandering and infinite time blowup

6

THEOREM (Nadirashvili, 91). Let Ω be the periodic channel. There exists a vorticity $\bar{\omega} \in L^\infty$ and numbers ε_0, T_0 such that for any $\omega \in L^\infty$ with $t \gg T_0$ $\|\omega - \bar{\omega}\|_\infty \leq \varepsilon$ while $\|S_t(\omega) - \bar{\omega}\|_\infty > \varepsilon$

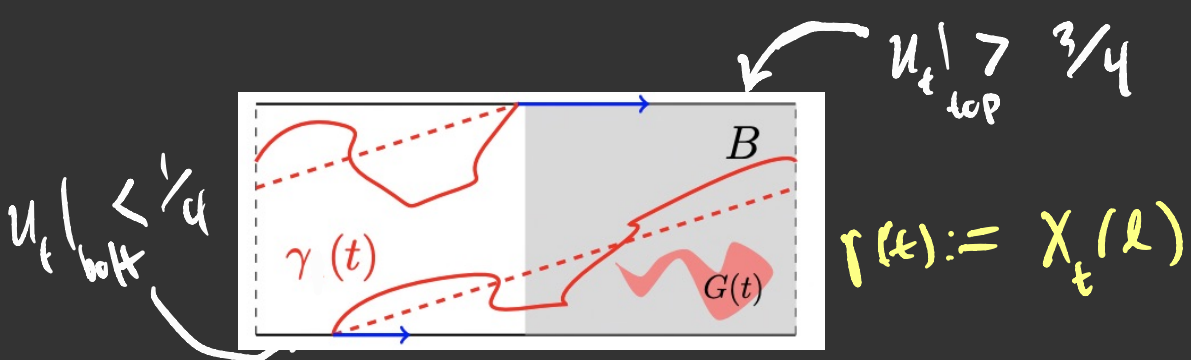


Let v be an Arnold stable steady state with the property that $v|_{\text{top}} \neq v|_{\text{bot}}$. For simplicity take Couette $v = (y, 0)$



Take h s.t. $h \in C_0^\infty$
 $h|_L \approx 1/2$ & $\|h\|_\infty \leq 1$.

Set $\bar{\omega} = -1 + \delta h$ for $\delta \ll 1$. Take $\varepsilon \ll 1$ and $g \in L^\infty$ with $\|g\|_\infty \leq 1$ and set $\omega_0 = \bar{\omega} + \varepsilon g$



Thus $\text{length}(\gamma(t)) \approx t^{1/2}$. Thus, $\ell(t) \cap \text{supp}(h) \neq \emptyset, \forall t \gg T_0$
 Similar construction shows that $\|u(t)\|_{C^{1,\alpha}} \approx t^\alpha$.

Infinite time blowup! Kiselev-Sverak, Denisov, Zlatoš, Elgindi-Tecoul.

Instability in strong norms

7

THEOREM: (Koch, 02; Morozov-Shnirelman-Yudovich, 03)
 Every stationary solution $\bar{w} \in C^{1,\alpha}$ of 2D Euler whose Lagrangian flowmap is not time-periodic (isochronal) is nonlinearly unstable in C^α . Specifically, $\forall M, \varepsilon > 0$ there exists $T = T(M, \varepsilon)$ and a solution $w(t) = S_t(u_0)$ s.t.
 $\|u_0 - \bar{w}\|_{C^\alpha} \leq \varepsilon$ while $\|w(T) - \bar{w}\|_{C^\alpha} \geq M$.

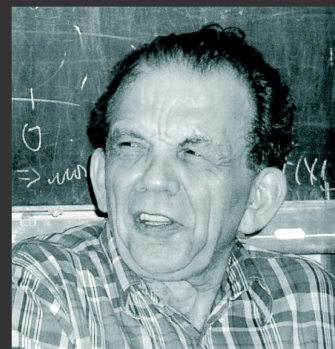
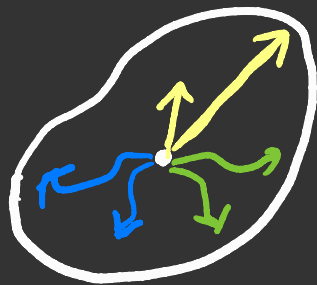
To prove this result, one exploits shearing. Denote

$$\mu(t) = \|\nabla X_t^{-1}\|_\infty, \quad \dot{X}_t = v(X_t)$$

LEMMA 1: If $\mu(t) \leq C$ for all $t \geq 0$, then v is isochronal.



t=0
 t=1
 t=2
 t=3



Example: (Elliptical vortex)

$$v = \nabla^\perp \psi, \quad \psi = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

Remark: characterized in neighborhood of the disk.

LEMMA 2: There exist $w \in C(0, T; C^\alpha)$ such that

$$\|w_0 - \bar{w}\|_{C^\alpha} \leq \varepsilon \quad \text{while} \quad \|w(T) - \bar{w}\|_{C^\alpha} \geq C \mu(T)^\alpha \varepsilon$$

Conjecture 1 (Yudovich (1974), [34, 35], quoted from [23]). *There is a ‘substantial set’ of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.*

Conjecture 1 (Yudovich (1974), [34, 35], quoted from [23]). There is a 'substantial set' of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.

THEOREM (D. Elgindi, 2022): Let ω_f be a L^2 stable steady state on an annular domain Ω with different boundary travel times. Then $\exists \varepsilon > 0$ s.t.

$$\left\{ \omega_0 \in B_\varepsilon(\omega_f) : \text{s.t. } \sup_t \frac{\|\omega(t)\|_{C^\alpha}}{t^{\alpha-1}} = +\infty \right\}$$
contains a dense set in $B_\varepsilon(\omega_f)$.

IDEA: Assume $\|\nabla \Phi\|_{L^\infty} \geq c(t)$ $\forall \omega_0 \in B_\varepsilon(\omega_f)$.
adapt Yudovich's argument get $c(t) = \delta(t) c(t)^\alpha$

Fix $\delta(t)$ s.t. $\delta(t) c(t)^\alpha \rightarrow \infty$. Then \exists a dense set of ω_0 in B_ε s.t. $\sup_t \frac{\|\omega(t)\|_{C^\alpha}}{\delta(t) c(t)^\alpha} = +\infty$. Let

$$U_N = \left\{ \omega_0 \in B_\varepsilon : \sup_t \frac{\|\omega(t)\|_{C^\alpha}}{\delta(t) c(t)^\alpha} > N \right\}$$

By lower semicontinuity of $S_t: C^d \rightarrow C^\alpha$, U_N is open in C^d .

Koch: given $\omega_0 \in B_\varepsilon$, and any $T, \varepsilon > 0$, $\exists \xi$ s.t. $\|\xi\|_{C^\alpha} < \varepsilon$ and $|S_T(\omega_0 + \xi)| \geq \frac{1}{2} K c(T)^\alpha$. Take ε small and T large so $\frac{1}{\delta(T)} \geq 4\varepsilon^{-1} N$. Then $\frac{|S_T(\omega_0 + \xi)|_{C^\alpha}}{\delta(T) c(T)^\alpha} \geq 2N \Rightarrow \omega_0 + \xi \in U_N$

Thus U_N dense in C^d . Let \bar{B} be any ball with $\bar{B} \subseteq B_\varepsilon$. \bar{B} is a complete metric space, $U_N \cap \bar{B}$ are open and dense. $\bigcap_N U_N \cap \bar{B}$ is dense in \bar{B} . by Baire category thm.

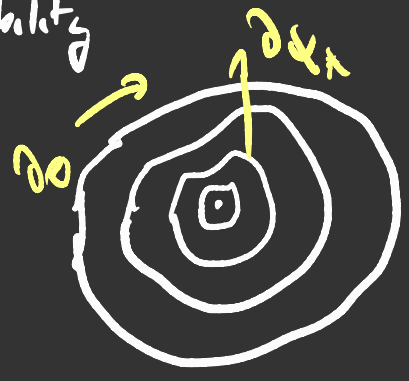
Conjecture 1 (Yudovich (1974), [34, 35], quoted from [23]). There is a 'substantial set' of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.

THEOREM (D. Elgindi, 2022): Let ω_x be a L^2 stable steady state with a region on which the vorticity is a strictly monotone function of streamline near isolated max/min. Then $\exists \varepsilon > 0$
 $D_{\omega_x, \varepsilon} := \{ \omega_0 \in B_\varepsilon(\omega_x) : \|\omega(t)\|_{C^\alpha} \gtrsim t^\alpha \forall t \}$
 is open and dense in $B_\varepsilon(\omega_x)$ in $C^{1, \alpha}$ topology.

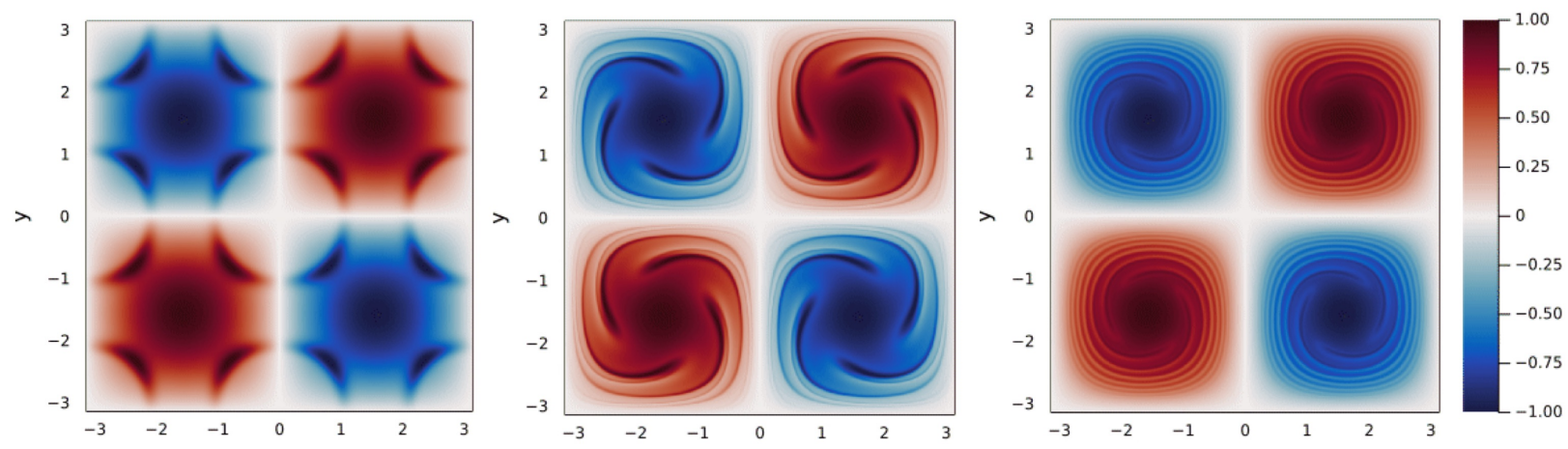
E.g. monotone shears; Poiseuille flow $\omega_x = y$.

Idea: exploit form of Lagrangian stability

$$\begin{aligned} |\psi_x(\phi(t)) - \psi_x(\phi_x(t))|_{L^2} &\lesssim \sqrt{\varepsilon} \\ |\theta_x(\phi(t)) - \theta_x(\phi_x(t))|_{L^2} &\lesssim \sqrt{\varepsilon} t \end{aligned}$$



"2D fluids are rough at infinity" - Dennis Sullivan



The limit of $t \rightarrow \infty$.

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

Since $\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$, the vorticity weak-* converges at long times: $\omega(t_i) \rightarrow \bar{\omega}$.

$$\Omega_+(\omega_0) = \bigcap_{t \geq 0} \text{weak-* closure of } \left\{ \int_T(\omega_0), \tau \geq t \right\}$$

END STATES.

GOAL: Understand the structure of $\Omega_+(\omega_0)$!

Lemma: The kinetic energy is weak-* continuous, i.e.

$$\omega_j \rightarrow \bar{\omega} \Rightarrow E(\omega_j) \rightarrow E(\bar{\omega})$$

Thus, energy is a robust invariant, $E(\bar{\omega}) = E(\omega_0)$
On other hand, the casimirs

$$I_f(\omega) = \int_{\mathbb{R}^2} f(\omega(x)) dx$$

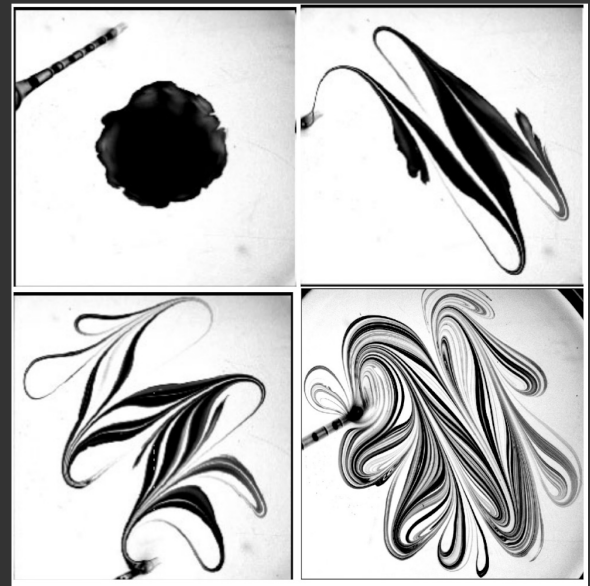
are not weak-* continuous unless f affine.

In general we cannot expect

$$I_f(\bar{\omega}) = \lim_{t \rightarrow \infty} I_f(\omega(t)) = I_f(\omega_0).$$

If f convex, we have

$$I_f(\bar{\omega}) \leq \liminf_{t \rightarrow \infty} I_f(\omega(t)) = I_f(\omega_0)$$



"Enstrophy" is lost due to fine scale mixing.

CONJECTURE: (Sverak 2013): Generically, orbits $\{\omega(t)\}_{t \geq 0}$ are not precompact.

Inverse Cascade of Energy

①

If $\omega_t \rightarrow \bar{\omega}$ then

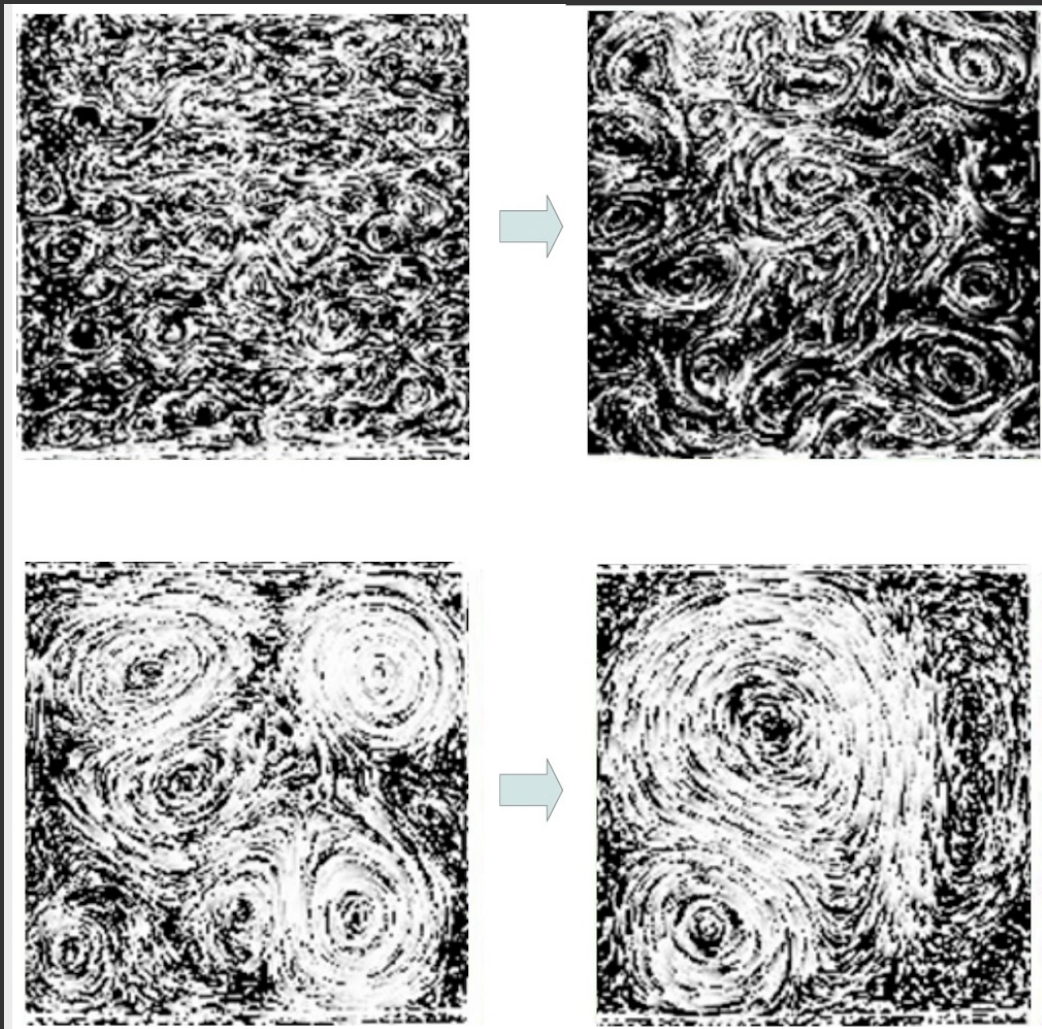
$$E(\omega_0) = E(\bar{\omega}), \quad |\omega_0|_2 \gg |\bar{\omega}|_2.$$

Can the energy go to small scales? (Fjørtoft, 1953)

$$E_k(\bar{\omega}) := \frac{1}{2} \sum_{n \geq k} |\hat{u}(n)|^2 \leq \frac{1}{k^2} \|\bar{\omega}\|_2^2 \leq \frac{1}{k^2} \|\omega_0\|_2^2$$

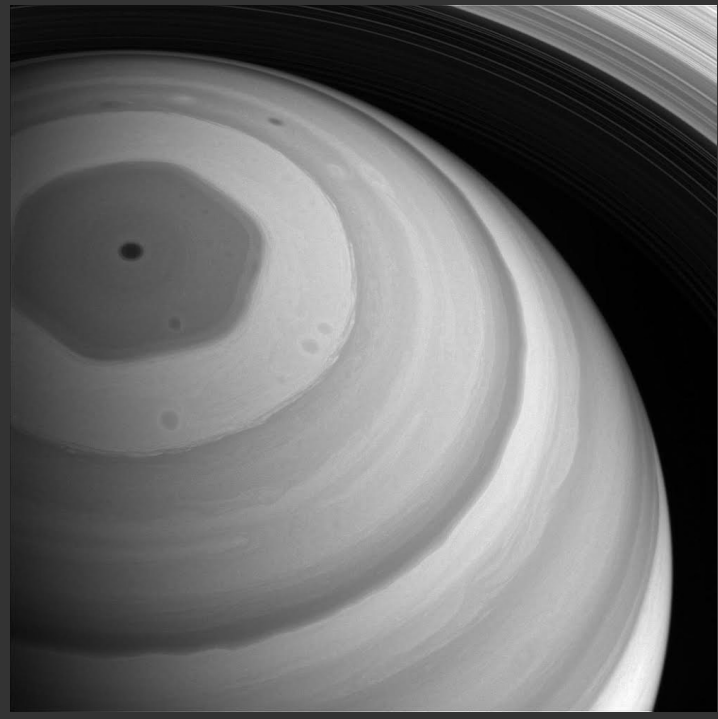
"Spectral blocking": energy cannot pile up at small scales.

In fact, energy is observed to cascade to large scales

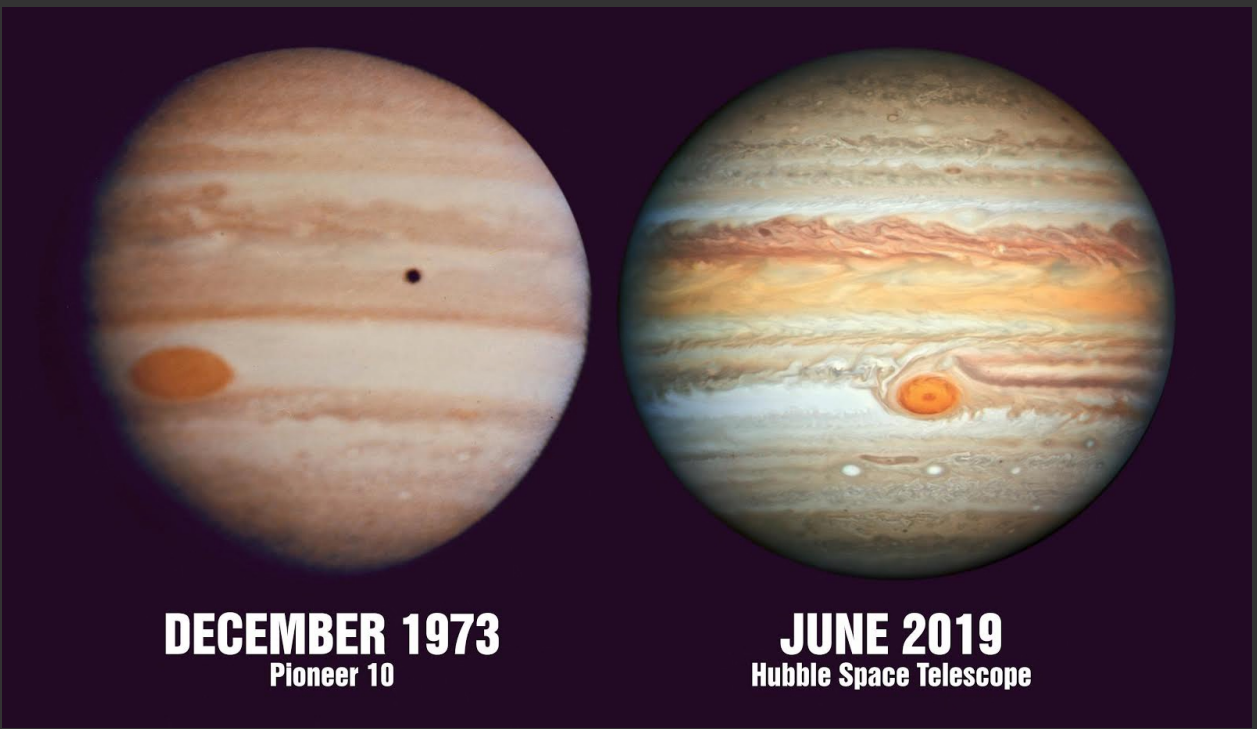


(Massen,
Clerc,
van Heijst,
1999)

Saturn's Hexagon

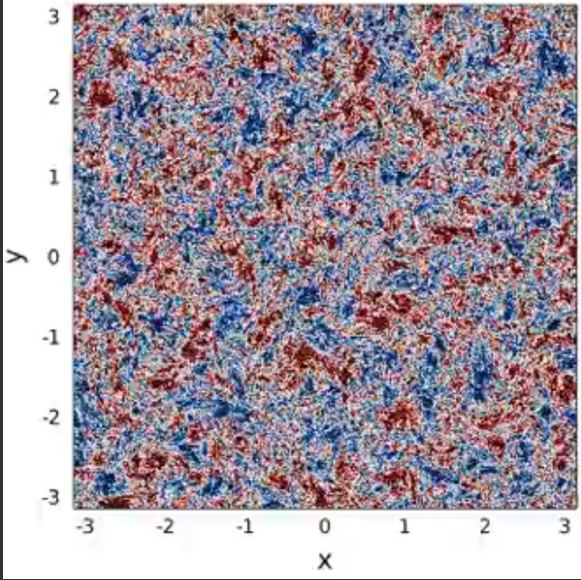


Jupiter's Great Red Spot

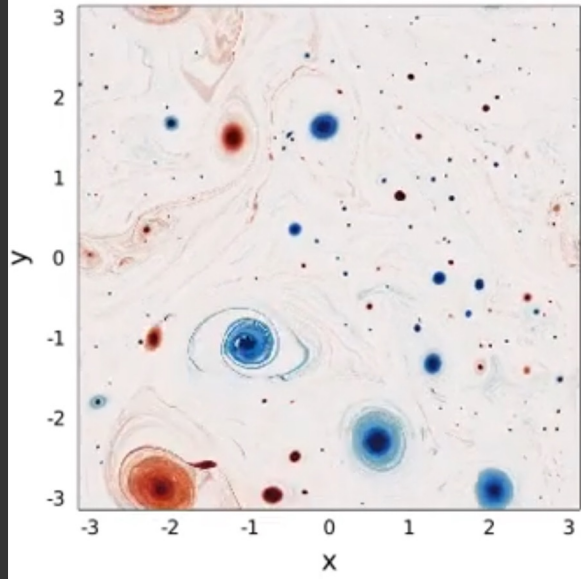


observed first by Galileo in 1665
 exist > 356 earth years!!!

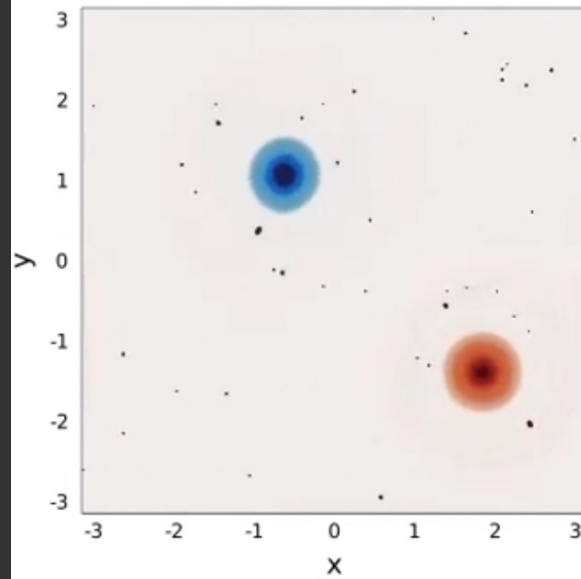
vorticity, $t=0.00$



vorticity, $t=38.80$



vorticity, $t=340.70$



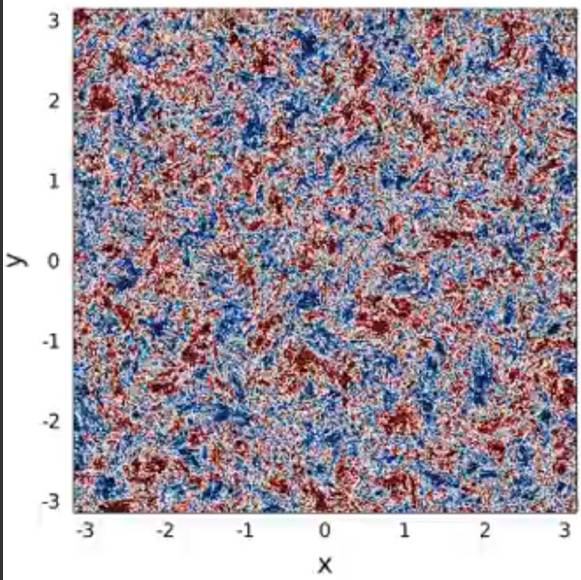
0

t

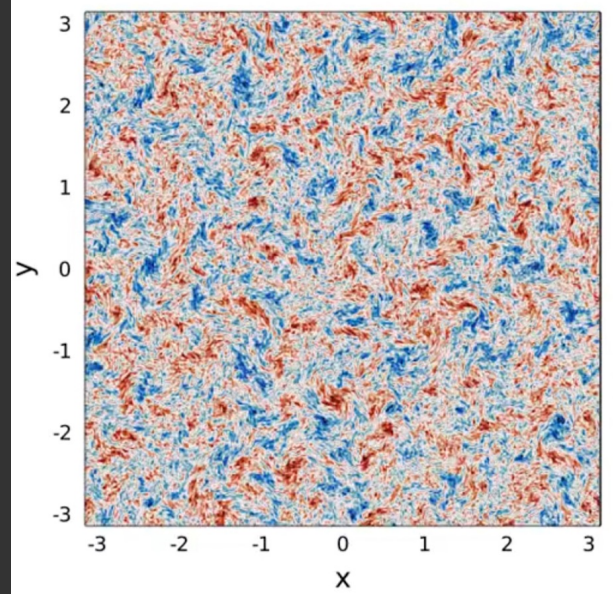


∞

vorticity, $t=0.00$



vorticity, $t=0.00$

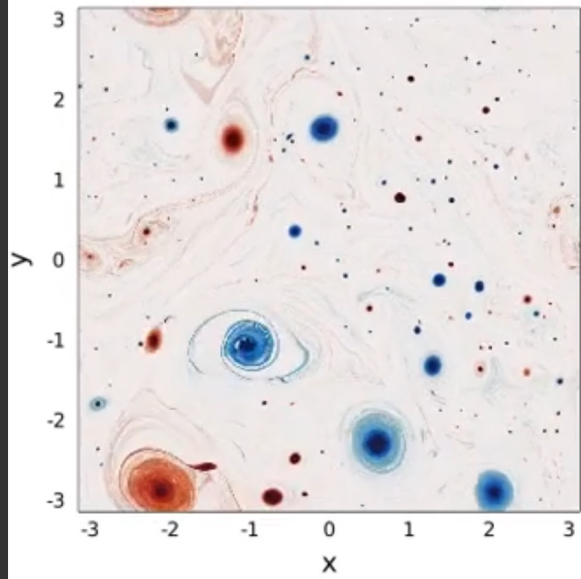


13

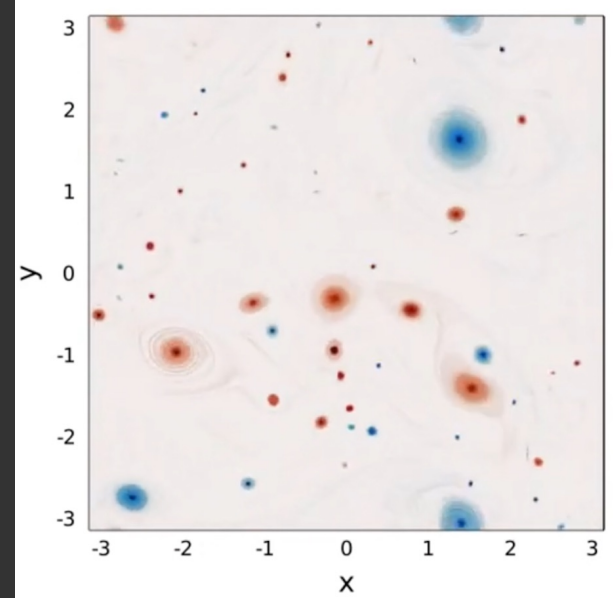
0

t

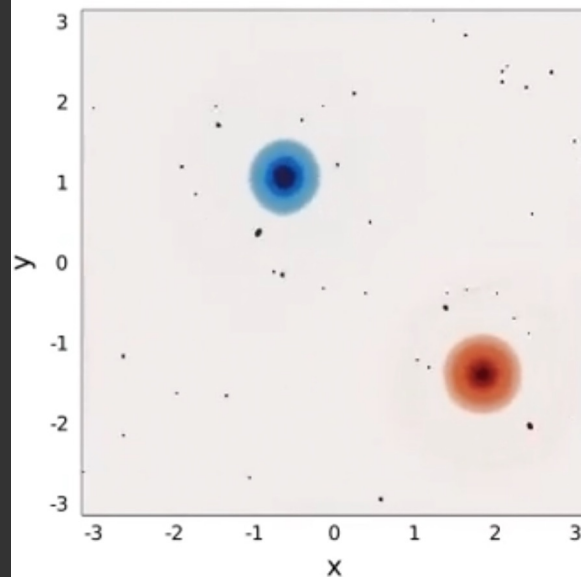
vorticity, $t=38.80$



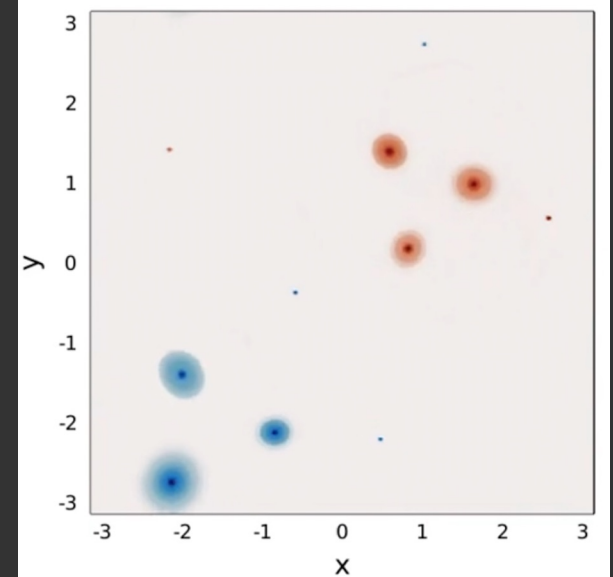
vorticity, $t=38.10$



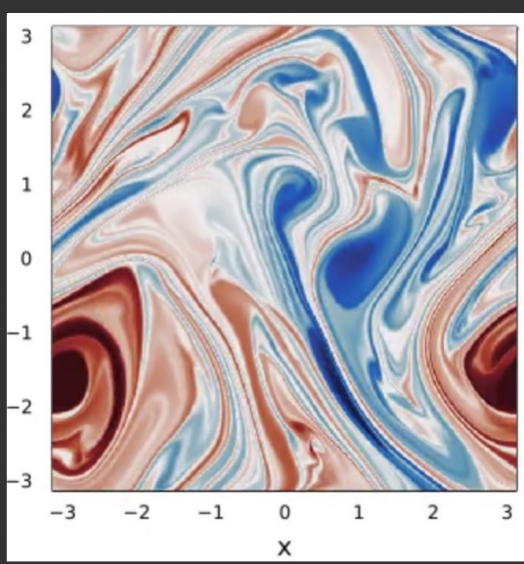
vorticity, $t=340.70$



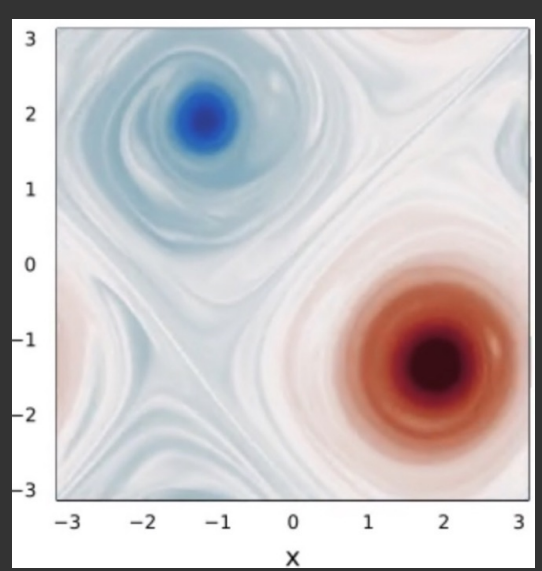
vorticity, $t=295.20$



∞



$t \rightarrow \infty$
 \longrightarrow



14

MYSTERY: How can one understand this apparent decrease of entropy* for the Euler equations at long times, at least if one looks at velocity fields.

* loosely interpreted, some measure of the diversity of a set in the space of all possible velocities.

* While diversity in the space of velocities appears to decrease, the entropy of the corresponding Lagrangian flows are likely increasing (Shnirelman, 1997)

Some Naïve Predictions

(15)

NAIVE CONJECTURE 1: The vorticity is everywhere mixed, i.e. $w(t) \rightarrow \int_{\Omega} w_0 := \bar{w}_0$ as $t \rightarrow \infty$.
i.e. $\Omega_t(w_0) = \{\bar{w}_0\}$.

FALSE: On \mathbb{T}^2 , we must take $\int_{\mathbb{T}^2} w_0 = 0$.
but $E(\bar{w}) = E_0 \neq 0$. Thus $\bar{w} \neq 0$.

NAIVE CONJECTURE 2: The end state minimizes enstrophy subject to fixed energy.

$$I(w) = \frac{1}{2} \int w^2 dx \quad E(w) = E_0 := E(w_0)$$

Both I and E are quadratic in w . Minimizing $I(w) - \lambda E(w)$ yields

$$w = \Delta \psi = -\lambda_1 \psi$$

where λ_1 is the first eigenvalue of Δ . ← large scale "vortex"

Mix (defined by reducing enstrophy) to max degree consistent with fixed energy.

Why choose minimum enstrophy? Theory of selective decay
(Bretherton & Haidvogel, 1976)

Note: for decaying Navier-Stokes, NC2 appears true.
(Foias-Saut, Schneider-Foias, Matthaeus et al...)

Statistical Hydrodynamics

12

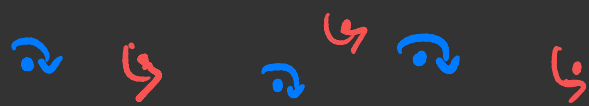
The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow. Its ubiquity suggests an explanation on statistical grounds.



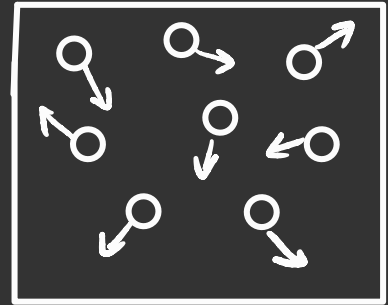
L. Onsager, Statistical Hydrodynamics, 1949

Joyce-Montgomery Considered point vortices

$$w(k) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)} - \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{x}_j(t)} \equiv p_+ - p_-$$



Under assumption of ergodicity of point vortex dynamics on energy surface and hypothesis that entropy



$$S = - \int p_+ \ln p_+ - \int p_- \ln p_-$$

be maximized (property of thermo. equilibrium) predicted

$$-w = -\Delta\psi = e^{-\beta(\psi - \mu_+)} - e^{-\beta(\psi + \mu_-)}$$

where β and μ_{\pm} enforce energy and net +/- circulation.

When $\beta < 0$, describes aggregation of vortices

Negative temperature states

However, if $1/\theta < 0$, then vortices of the same sign will tend to cluster, — preferably the strongest ones —, so as to use up excess energy at the least possible cost in terms of degrees of freedom. It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganised contributions to the flow.

The little vortices who wanted to play

Once upon a time there were n vortices of strengths K_1, \dots, K_n in a two-dimensional frictionless incompressible fluid. They were enclosed by a boundary but could play ring-around-the-rosy otherwise. The rule of that game was 1)

$$K_i dx_i / dt = - \partial W / \partial y_i \quad ; \quad K_i dy_i / dt = \partial W / \partial x_i$$

where $- \rho W(x_1, y_1, \dots, x_n, y_n)$ equals the energy apart from an additive constant (which is infinite on account of the self-energies). The function W is something

like this:

$$W = \frac{1}{2\pi} \sum_{i,j} K_i K_j \log(r_{ij}) + (\text{potential of image forces})$$

and the image forces are finite except near the boundary, --- Now the vortices were very playful like I said and they liked to distribute themselves in completely random fashion but they could not do that because they had too much energy. You see they were not like molecules which have more room in momentum-space the more energy they have. The vortices had only a finite configuration-space. So when they had more energy than the average over that space, they could not play quite the way they wanted to. --- You can describe the ergodic distribution approximately by a canonical distribution

$$f(x_1, y_1, \dots, x_n, y_n) = \exp((\tau + \rho W) / \Theta)$$

with an antitemperature $-\Theta > 0$. You will note that the phase-integral converges for one pair of vortices if and only if $(\rho / 2\pi\Theta) K_i K_j > -2$

For a set of vortices there are further necessary conditions. You can figure out that there is no way to take care of much energy unless you let at least one pair of vortices of the same sign get close together. --- And now you know how the little vortices arranged it so that most of them could play just the way they wanted to. They just pushed the biggest vortices together until the big vortices had all the energy the little ones did not want, and then the little vortices played ring-around-the-rosy until you could not tell which was where, and it make no difference anyway.

Note to L. Pauling (1945)

The case $W > \bar{W}$ is quite different. We now need a negative temperature to get the required energy. The appropriate statistical methods have analogs not in the theory of electrolytes, but in the statistics of stars. In a general way we can foresee what will happen. Vortices of the same sign will tend to move together, more so the stronger the repulsion between them. After this aggregation of the stronger vortices has disposed of the excess energy, the weaker vortices are free to roam at will.

These predicted effects carry some resemblance to familiar habits of vortex sheets. If the rolling up of vortices is to be explained thus on a statistical basis, we may describe it as a process of crystallization, which occurs in response to a prevailing negative "market price" for energy.

Letter to C.C. Lin (1945)

Statistical Hydrodynamics cont.

Miller, Weichman, Cross, Robert, Sommeria...

STEP 1: (Approximation)

Infinite dim phase space \mathcal{H}



finite dim phase space \mathcal{H}_N

e.g. point vortices, Galerkin approx, discrete cells.

STEP 2: (Dynamics)

dynamics are defined which preserve energy and phase space volume (Liouville theorem)

STEP 3: (Entropy Maximization)

let \mathcal{H}_0 be space of states with same energy as u_0 .
A microcanonical ensemble (invariant measure on \mathcal{H}_0) is predicted as a stationary state u_* by demanding available phase space for given state be maximized.

Ergodic hypothesis says finite dim system spends most of its time near u_* .

Criticisms: illegal transposition of limits $T \rightarrow \infty, N \rightarrow \infty$.

Works in systems near-equilibrium, but fluid is far.

- Lyapunov functional (Shnirelman, 1997)
- wandering domains (Nadirashvili, 1991)
- integrable N -vortex motion (Khanin, 1982)

infinite dimensional fluid is an "open system"

- Mixing: $w(t) \rightarrow \bar{w}$ but not strongly. (Bedrossian, Masnick)
- configuration space is unbounded $\dim(\text{Diff}_\mu(\mathbb{T}^2)) = \infty$

Shnirelman's Mixing Theory

Mixing operators
on $L^2(\Omega)$

$$Kf(x) = \int_{\Omega} K(x,y) f(y) dy$$

where

i) $K(x,y) \geq 0$

ii) $\int_{\Omega} K(x,y) dy = 1$

iii) $\int_{\Omega} K(x,y) dx = 1$

Example: a) $K(x,y) = \delta(y - \phi^{-1}(x))$, $\phi \in \text{Diff}_\mu(\Omega)$

b) $K(x,y) = 1$

Set of all mixing operators on $L^2(\Omega)$ called \mathcal{X} is a convex, weakly compact semigroup of contractions in $L^2(\Omega)$. Thus it defines partial order:

$$f \prec g \text{ if } f = Kg$$

$$f \sim g \text{ if } f \prec g \text{ and } g \prec f.$$

For vector fields, say $u \prec v$ if $\nabla \cdot u \geq \nabla \cdot v$.

$$\Omega_{u_0} = \{u \mid u \prec u_0\} \cap \{E(u) = E(u_0)\}.$$

Note that solutions of Euler $u(t) \in \Omega_{u_0}$ and moreover any limit $u(t) \rightarrow \bar{u} \in \Omega_{u_0}$.

A minimal element $v \in \Omega_{u_0}$ is such that for all $w \in \Omega_{u_0}$ with $w \prec v$, we have $v \sim w$.

Mixing is "reversible" on minimal elements!

Lemma: There exists a minimal element $w \in \Omega_{u_0}$. (19)

THEOREM: (Shnirelman, 93)

- (i) any minimal element of Ω_{u_0} is a stationary solution of Euler equations
- (ii) the minimal elements have monotonic $w = F(\psi)$

Namely, all minimal elements satisfy

$$\Delta \psi = F(\psi)$$

for a univalent function F satisfying Arnold stability.

If Euler is "maximally mixing" then long time limits are steady stable states.

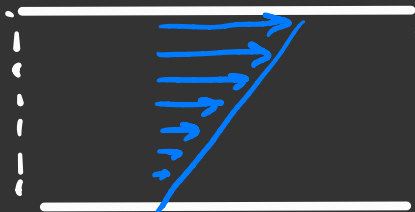
Three types:

Energy-excessive u



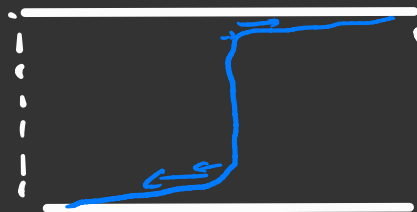
if $v < u$ then
 $E(v) \leq E(u)$

Energy-neutral u



if $v < u$ then
 $E(v) = E(u)$

Energy-deficient u



if $v < u$ then
 $E(v) \geq E(u)$

Shnirelman (93) developed a theory of minimal flows - maximally mixed vorticity at fixed energy.
 → always exist. Constructed using Zorn's Lemma.
 → are steady Euler solns with $w = F(\psi)$, F monotone!

Let $\mathcal{J}_{\omega_0, E_0} = \{ \omega_0 \circ \phi, \phi \in \mathcal{D}_\mu(\mathcal{M}) \} \cap \{ E = E_0 \}$

$\{ \mathcal{D}_E(\omega_0) \}_{E \geq E_0} \subseteq \mathcal{J}_{\omega_0, E_0}, \quad \Omega_+(\omega_0) \subseteq \overline{\mathcal{J}_{\omega_0, E_0}^*} \subseteq \overline{\mathcal{J}_{\omega_0}^*} \cap \{ E = E_0 \}$

E.g: $\omega_0 = \begin{matrix} \text{+} \\ \text{---} \\ \text{-} \end{matrix}$  then $\overline{\mathcal{J}_{\omega_0}^*} = \{ |\omega| \leq 1 : \int_M \omega = \int_M \omega_0 \}$

Theorem: (Dolce-D) Let $f: L^\infty \rightarrow \mathbb{R}$ be strictly convex.

For any $\omega_0 \in L^\infty$ with E_0 , there exists $\omega^* \in L^\infty$

$\min_{\omega \in \mathcal{J}_{\omega_0, E_0}^*} \int_M f(\omega) dx$

These are minimal flows. Moreover, $\exists \Phi$ s.t.

$\omega_{\lambda}^* = \arg \min_{\substack{\omega \in L^\infty \\ |\omega| \leq 1}} \left(\int_M \Phi(\omega) + \lambda(E - E_0) \right)$

No datum $\omega_0 \in X$ is isolated from equilibrium.
 Theorem (Bouchet): extremizers correspond to Mikhaïlov-Robert equilibria.

Excluding Symmetric equilibria

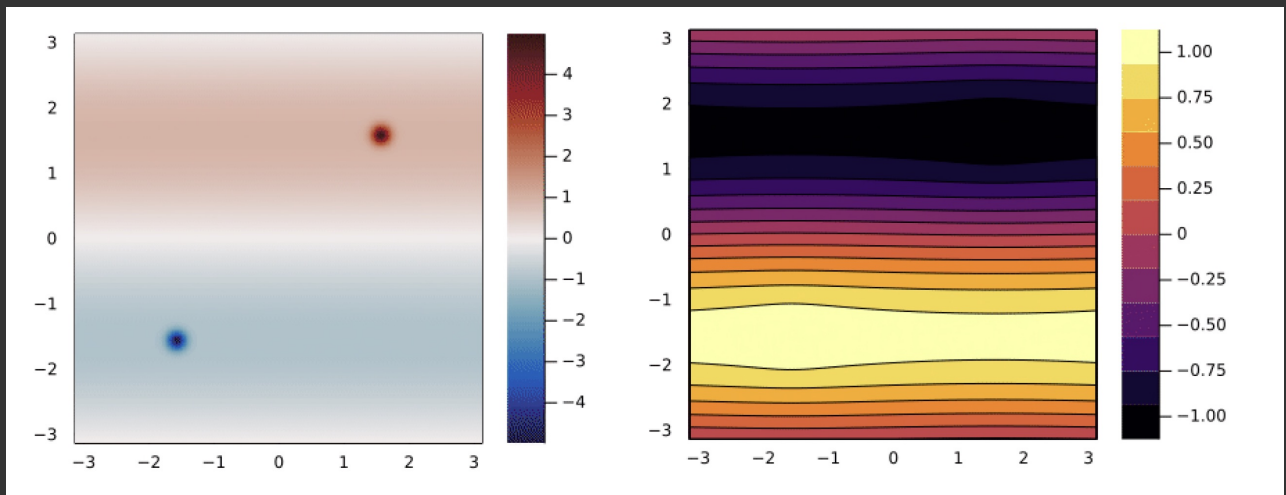
Theorem (Dolce - P.) Let M be \mathbb{T}^2 or $\mathbb{T} \times [0, \pi]$.

For any shear flow $\eta = -v'(y)$ and any $\varepsilon > 0, \delta > 0$

$\exists \zeta \in C_0^\infty$ ^{Greeny-2} such that $\|\zeta - \eta\|_{H^{-1-\delta}} \leq \varepsilon$

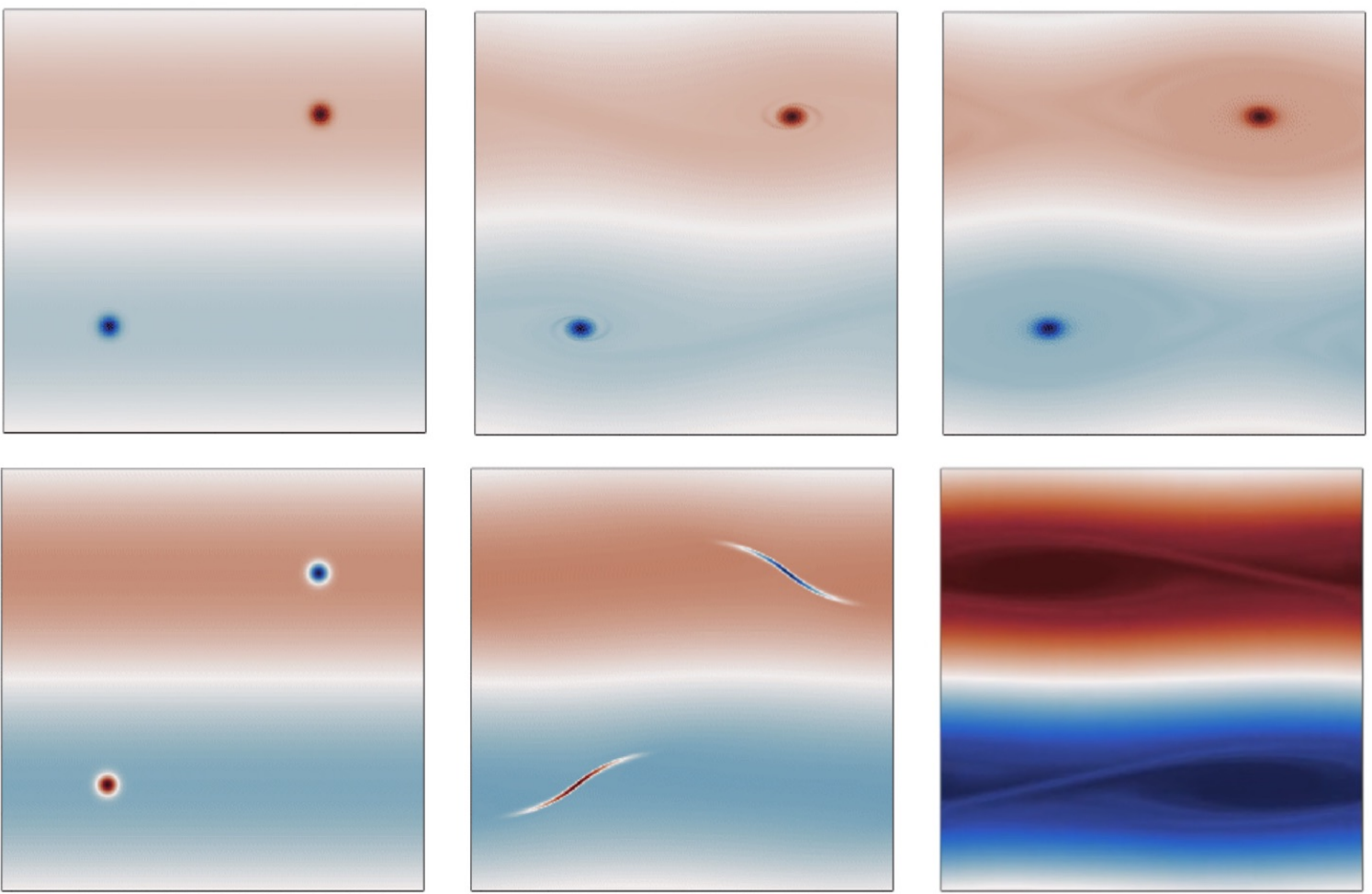
and $\mathcal{O}_{\zeta, E_\zeta, M_\zeta}$ contains no shear flows,
 where E_ζ & M_ζ are energy and momentum of ζ .

Remark: In particular Euler cannot inviscid damp for data of the form:



Idea, have ε -approx point vortices, Energy $\sim \log \frac{1}{\varepsilon}$
 but any shear flow on its orbit has energy = $\mathcal{O}(1)$
 since max value ε^{-2} distributed to a set at most area ε^2 .

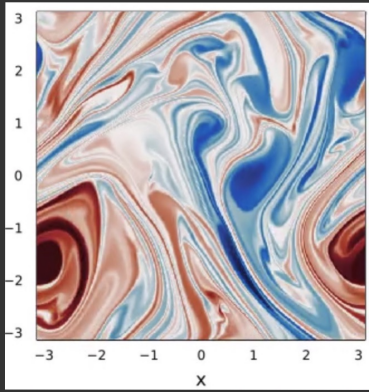
Instead, this is what happens:



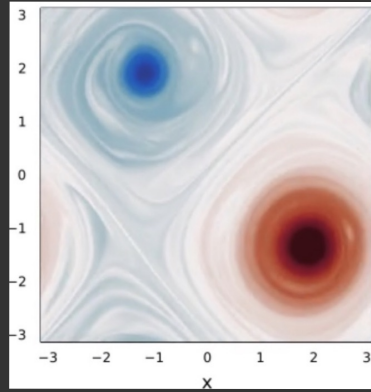
Quasisteady, but may have underlying periodic or quasiperiodic structures.

Returning to the Mystery...

Decrease of "Entropy" for 2D Euler

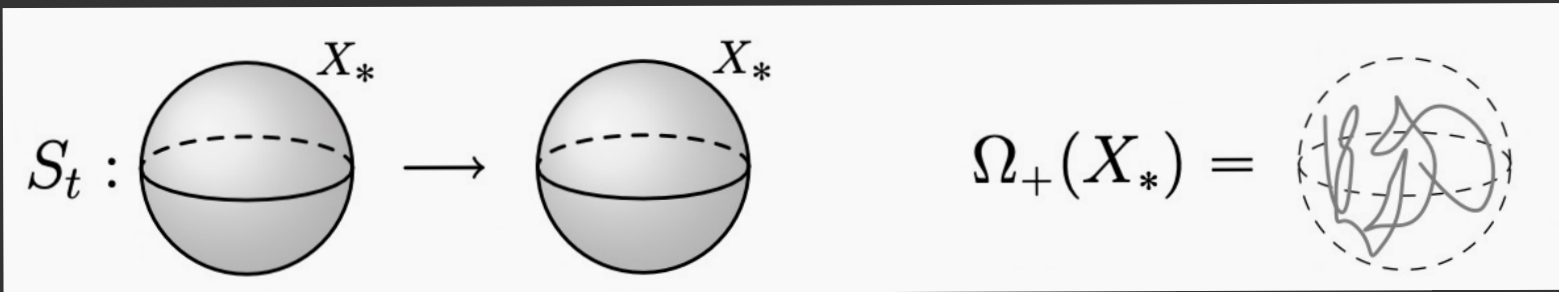


$t \rightarrow \infty$
 \longrightarrow



CONJECTURE: (Sverak, 2013) For generic $w_0 \in L^\infty$, the orbits $\{w(t)\}_{t \geq 0}$ are not precompact in L^2 .

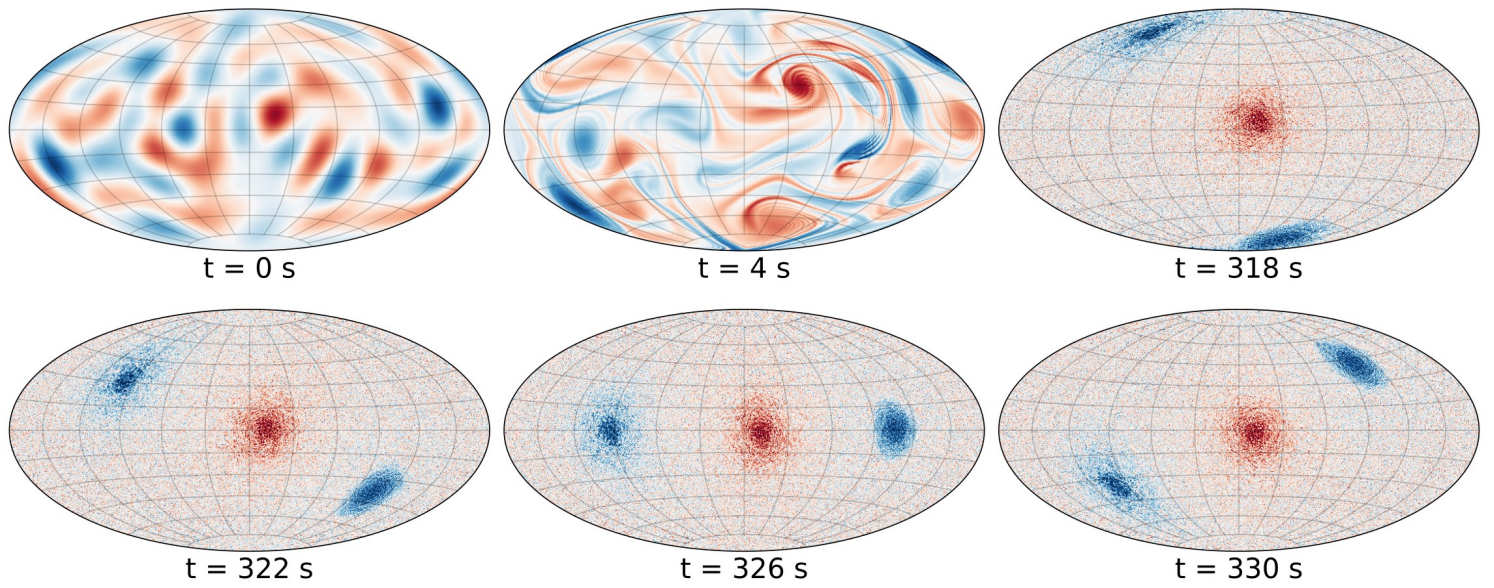
CONJECTURE: (Shnirelman, 2013) For any $w_0 \in L^\infty$, the collection of weak-* limits of the orbit $\{w(t)\}_{t \geq 0}$ consist of vorticities which generate a L^2 -compact orbit.
 e.g. steady, periodic, quasiperiodic...



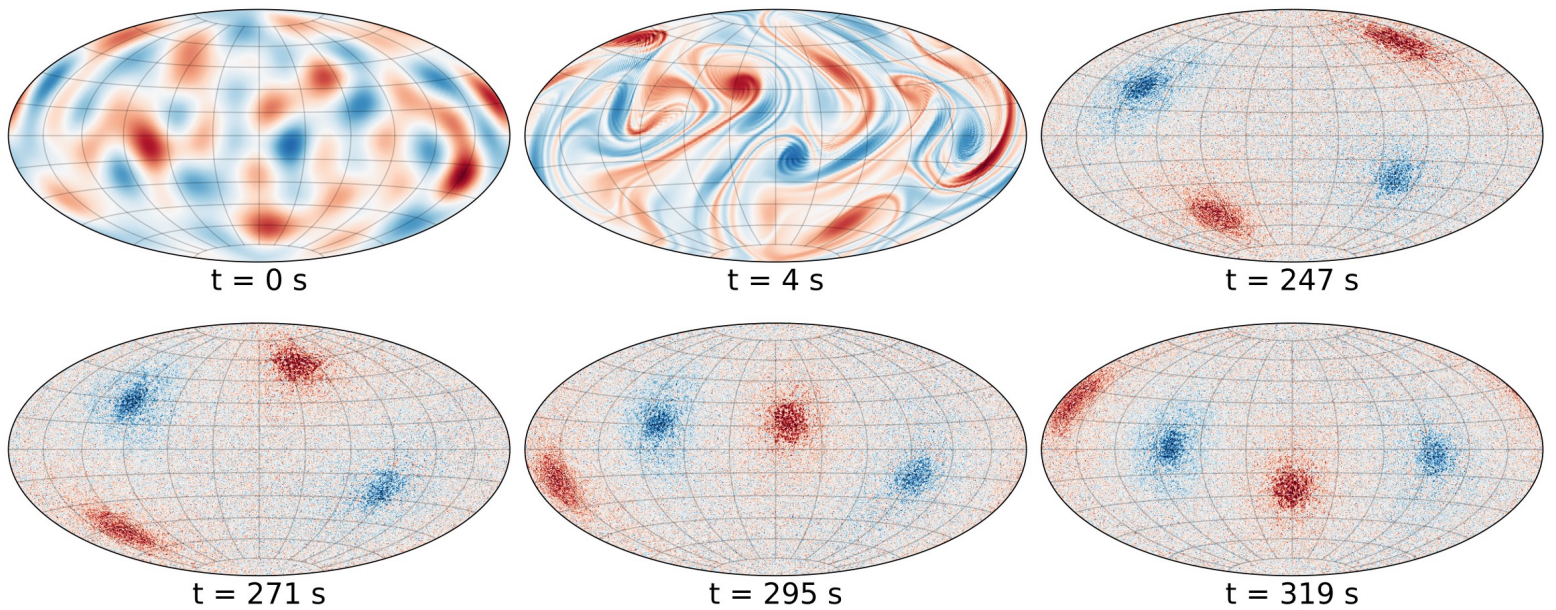
- Theorem: (Sverak) For any $w_0 \in L^\infty$, $\Omega_+(w_0)$ contains an L^2 -compact orbit.

- Both conjectures are "true" in a neighborhood of special equilibria (Bedrossian & Messemidi) Ionescu & Titi

Non-zero net angular momentum.



Zero net angular momentum



Conjecture: (Modin, Virani): long time behavior tracks integrable point vortex motions!

Thank-you!



Detour on Isochronal Flows $w = F(\psi)$ (18)

Conj: (Yudovich): The only constant vorticity isochronal flows are elliptical ($F \rightarrow \text{domain}$).

Conj: (D.-Elgindi): \forall simple connected domains, \exists a unique isochronal flow. ($\text{domain} \rightarrow F$)

Theorem (D.-Elgindi): For slight deformations D' of a disk domain D , \exists a unique isochronal flow.

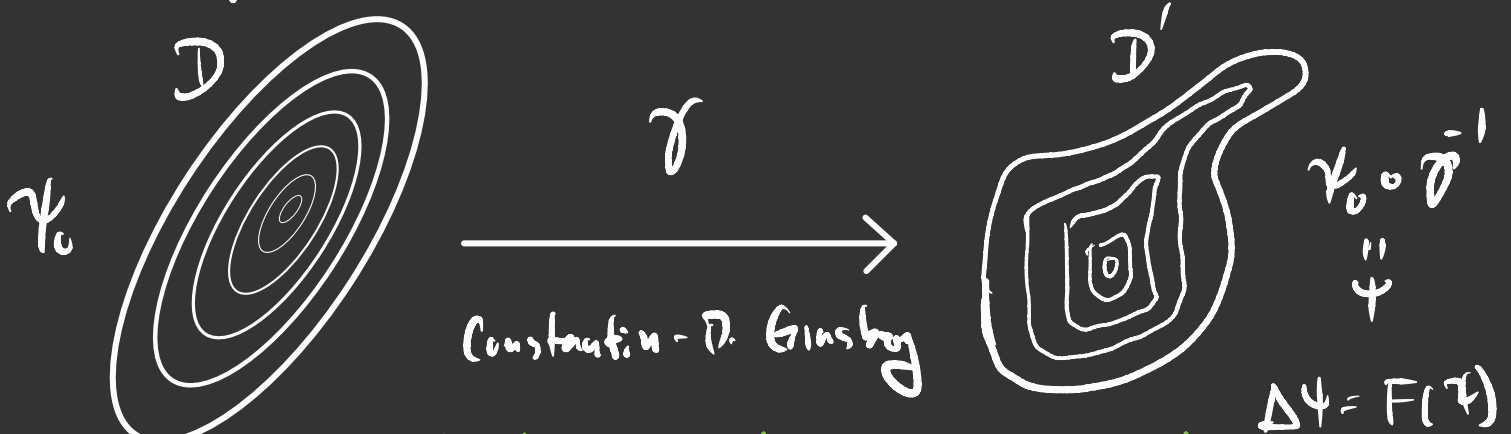
Proof: CoArea formula:

$$\mu(c) = \oint_{\{\psi=c\}} \frac{d\ell}{|\nabla\psi|} = \frac{d}{dc} A(\{\psi \leq c\})$$

Let ψ_0 be isochronal. (e.g. elliptical). Then all $\psi \in \mathcal{O}_{\psi_0}$ have equal enclosed area

$$A(\{\psi \leq c\}) = A(\{\psi_0 \leq c\}) = \mu_0 c$$

Since elliptical isochronal flow is stable, $\Delta\psi = \text{const}$,



Near ellipses, all steady states are unstable!

Variational Approach & consequences (with Michele Dolce) (20)

$$\text{Let } \mathcal{J}_{\omega_0} = \{ \omega_0 \circ \phi, \phi \in \mathcal{D}_r(M) \}$$

$$\mathcal{J}_{\omega_0, E_0} = \mathcal{J}_{\omega_0} \cap \{ E = E_0 \}$$

$$\text{clearly: } \{ S_E(\omega_0) \}_{E > E_0} \subseteq \mathcal{J}_{\omega_0, E_0}$$

$$\Omega_+(\omega_0) \subseteq \overline{\mathcal{J}_{\omega_0, E_0}}^* = \overline{\mathcal{J}_{\omega_0}}^* \cap \{ E = E_0 \}$$

Remark: if ω_0 is Arnold stable then $\overline{\mathcal{J}_{\omega_0, E_0}}^* = \{ \omega_0 \}$.

Theorem: (Dolce-D) Let $f: L^\infty \rightarrow \mathbb{R}$ be strictly convex.

For any $\omega_0 \in L^\infty$ with E_0 , there exists $\omega^* \in L^\infty$

$$\min_{\omega \in \overline{\mathcal{J}_{\omega_0, E_0}}^*} \int_M f(\omega) dx$$

These are minimal flows. Moreover, $\exists \Phi$ convex s.t.

$$\omega_{\lambda} = \text{arg min}_{\omega \in L^\infty, |\omega| \leq 1} \left(\int_M \Phi(\omega) + \lambda(E - E_0) \right)$$

No datum $\omega_0 \in X$ is isolated from equilibria.

Theorem (Bouchet): extremizers correspond to Mikhaïlov-Robert equilibria.

Proof relies on the following characterization

$$X = \{ f : \|f\|_\infty \leq 1 \}$$

Shnirelman, Bauer-Gambu

Sudderth

$$\overline{\mathcal{O}}_{\omega_0, E_0}^* = \{ \omega \in X : \omega = k\omega_0 \text{ for } k \in K \}$$

$$= \{ \omega \in X : \int_M \omega = \int_M \omega_0, \int_M (\omega - c)_+ \leq \int_M (\omega_0 - c)_+ \quad \forall c \in \mathbb{R} \}$$

$$= \{ \omega \in X : \int_M \omega = \int_M \omega_0, \int_M f(\omega) \leq \int_M f(\omega_0) \quad \forall \text{ convex } f \}$$



then $\overline{\mathcal{O}}_{\omega_0, E_0}^* = \{ \omega \in X : \int_M \omega = \int_M \omega_0 \}$

Minimal elements exist by weak* compactness. They are stationary since for any permutation of cubes Q_1 & Q_2 , the mixing $k^\varepsilon \omega = (1-\varepsilon)\omega + \varepsilon \omega \circ \phi$

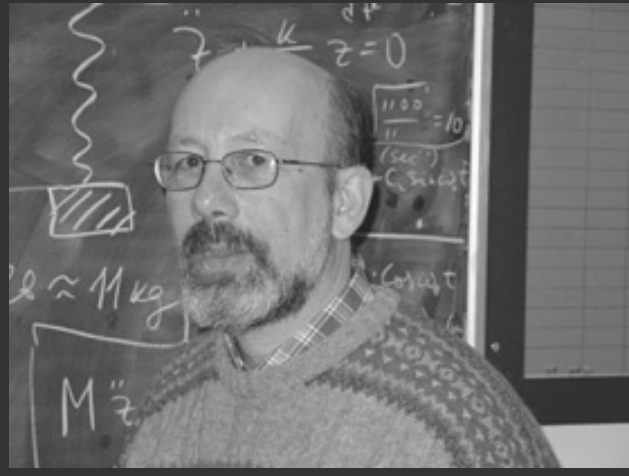
$$\frac{d}{d\varepsilon} E(k^\varepsilon \omega^*) = \int_{Q_1} (\omega^*(x) - \omega^*(\phi(x))) (\psi^*(x) - \psi^*(\phi(x))) \leq 0 \text{ or } \geq 0 \quad \forall \text{ permutations and } Q_1$$

$$\Rightarrow (\omega^*(x) - \omega^*(y)) (\psi^*(x) - \psi^*(y)) \geq 0 \text{ a.e. } x, y \leq 0$$

Monotonic $\omega = F(\psi)$ since

lem: let $\omega \in X, k \in X, \omega_1 = k\omega. \exists \tilde{k} \in X$ st. $\omega = \tilde{k}\omega_1$ iff $I_f(\omega_1) = I_f(\omega)$ for a strict. conv. f .

Shnirelman's Conjecture



- Numerical simulations indicate that the long time behavior is not typically stationary, but rather some time dependent solution. - periodic, quasiperiodic etc.

- This indicates that Euler is **not** a completely effective mixer, leaving solutions "trapped" in time dependent regimes.

CONJECTURE: (Shnirelman, 2013) The space of L^2 -compact (Under Euler evolution) vorticity orbits is the weak- $*$ attractor for the Euler dynamics

- Note, the space of compact orbits is at least not the entire phase space (Inviscid damping) (Bedrossian - Masmoudi, Ionescu - Tataru)
- For any $w_0 \in L^\infty$, at least one long-time limit corresponds to a compact orbit (Šverák)