Mathematical Aspects of Turbulence

Outline:

		Date
Lecture 1 & 2:	Phenomenology and Mathematics of fully developed turbulence	2/3, 2/5
Lecture 3:	Transition to turbulence and a problem of Kolmogorov	2/10, 2/11
Lecture 4:	Formation of small scales in ideal (inviscid) fluids	2/11

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Detailed Outline:

Lecture 1:
- Equations of motion and basic properties
- High Reynolds number behavior, exp. & numerics
- Kolmogorov 1941 theory

Lecture 2:
- Landau's Kazan remark & intermittency
- Onsager's Conjecture and dissipative weak Euler Solutions.
- Examples of "anomalous dissipation"
  - convex integration constructions
  - Burgers
  - Passive scalar constructions.

Lecture 3:
- Transition to turbulence
- Kolmogorov's flow problem.
- Stability of laminar state at "low Re"
- Instability of laminar state at "high Re" (Meshalkin & Sinai)

Lecture 4:
- Bifurcation to stable secondary flows near onset (Yudovich).
- Long time behavior of ideal fluids near laminar states.
- Mixing, instability and infinite time growth (Koch, Nadirashvili)

Transistion (Dynamical progression) to Turbulence

> *"A. N. Kolmogorov considered his work on turbulence to be non-mathematical. He wanted to explain observed phenomena from first principles, i.e. from the Navier-Stokes equations. This dream was one of the sources of his interest in the theory of dynamical systems, in which he perceived a set of tools for the description of turbulent chaos."* – V.I. Arnol'd [22]

Consider the ODE

$$\frac{d}{dt} u(t) = F(u(t))$$
$$u(0) = u_0$$

The variable $u = u(t)$ belongs to a linear space $\mathcal{H}$ called the phase space and $F: H \to H$.

Two cases:

- finite dimensional $H = \mathbb{R}^d$
- infinite dimensional $H =$ a Hilbert space.

2D fluids fall into the latter category with $H = L^2_\sigma$ or $H = H^1_\sigma$ (depending on regularity of external force)

$$L^2_\sigma = \{ u \in L^2(\mathbb{T}^2) : \nabla \cdot u = 0 \}$$

$$H^1_\sigma = \{ u \in H^1(\mathbb{T}^2) : \nabla \cdot u = 0 \}$$

$$\frac{d}{dt}u(t) = F(u(t)), \qquad u(0) = u_0$$

- We are interested in the $t \to \infty$ behavior.
- Generally $F(u) = F_\mu(u)$ for some parameter μ.

Long time behavior will depend strongly on μ.
The role of μ will depend on details of system.
Here is a schematic of STAGES: (Landau)

1. For μ small, $\mu < \mu_1$, there exists a unique stationary solution $u = u_1^s$, ie.

$$(*) \qquad F_\mu(u_1^s) = 0.$$

This solution is stable and attracts all orbits

$$u(t) \to u_1^s \quad \text{as} \quad t \to \infty.$$

2. For larger μ, $\mu_1 < \mu < \mu_2$, other solutions of $(*)$ appear, $u_2^s, u_3^s, \dots$ and u_1^s loses stability. We say a bifurcation has occurred at $\mu = \mu_1$.
The new solutions are typically stable and each have a basin so $u(t) \to u_i^s$ for some $i = 2, 3 \dots$.

3. When μ is further increased, $\mu_2 < \mu < \mu_3$, a Hopf bifurcation can occur. In this cas the flow doesn't become stationary but rather

$$u(t) - \phi(t) \to 0 \quad \text{as} \quad t \to \infty$$

where $\phi(t)$ is a time-periodic solution

$$(\star\star) \qquad \frac{d}{dt}\phi(t) = F(\phi(t)) \qquad t \in \mathbb{R}$$

$$\phi(t+T) = \phi(t).$$

Apriori, the value μ_2, solution ϕ and period T are all unknown and a detailed analysis is required.

From this point, we can either a Feigenbaum cascade of period doubling: ϕ_j with periods $2T, 2^2T, \ldots 2^jT$
or

4. For larger μ, $\mu_3 < \mu < \mu_4$ invariant tori can appear, i.e.

$$u(t) - \phi(t) \to 0$$

where ϕ is a quasiperiodic solution of $(\star\star)$

$$\phi(t) = g(\omega_1 t, \ldots, \omega_n t)$$

where g is periodic with period 2π in each variable and $\omega_i = 1/T_i$ are rationally independent real numbers.

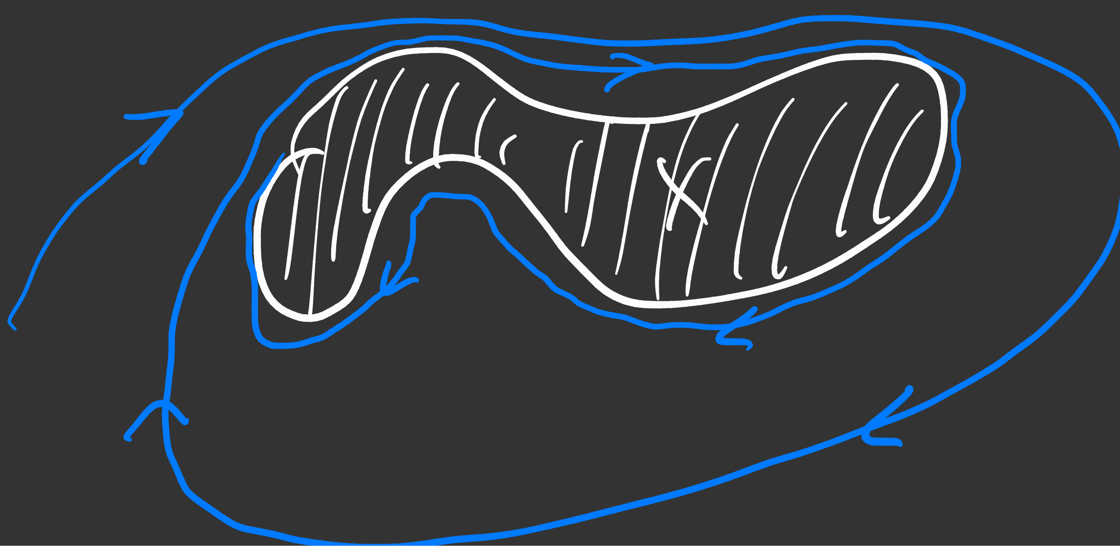
Flow looks chaotic, but fourier analysis shows system has discrete frequencies.

5. For $\mu > \mu_4$, the flow is chaotic, a stage where $u(t)$ looks completely random for all time. Fourier reveals a wide-band spectrum. One can show

$$u(t) \to X \quad \text{as} \quad t \to \infty$$

in the sense

$$\mathrm{dist}(u(t), X) \to 0 \quad \text{as} \quad t \to \infty$$



Here, X is an invariant subspace of H, i.e.

$$S_t(X) = X \qquad \forall\, t > 0$$

where S_t is the semigroup of $\dot{u} = F(u)$.

The set X may be extremely complicated (Fractal) with complex geometry (non-smooth).

The Kolmogorov Model

Analysis Seminar (1958) "Theory of dynamical systems and hydrodynamic stability".

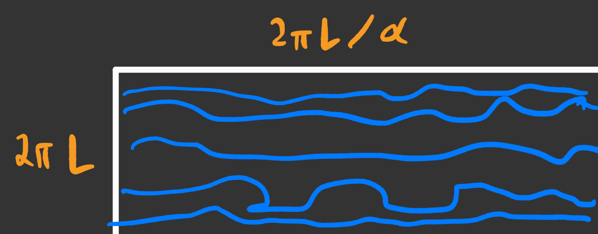
Consider the motion of a two-dimentional fluid, periodic with period $2\pi L/\alpha$ in x and period $2\pi L$ in y: Fix $\alpha>0$, $L>0$, $\gamma>0$

On $\mathbb{T}^2_\alpha := [0, 2\pi L/\alpha) \times [0, 2\pi L)$, consider

$$\partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u + \gamma f$$

$$\nabla\cdot u = 0$$

$$\int_{\mathbb{T}^2_\alpha} u\, dx\, dy = 0$$



The forcing f is taken to be an eigenfunction of the Stokes operator: $A = -\mathbb{P}\Delta$

$$A f_\lambda = \lambda f_\lambda$$

Can be constructed from eigenfunctions of Laplacian

$$f_\lambda = \nabla^\perp q_\lambda \quad \text{where} \quad -\Delta q_\lambda = \lambda q_\lambda, \quad \int q_\lambda = 0$$

example

$$\begin{pmatrix} 0 \\ \sin(\alpha n x) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sin(my) \\ 0 \end{pmatrix} \qquad n, m \in \mathbb{N}$$

⑥

А. Н. Колмогоров предложил рассмотреть решение системы

$$\left.\begin{aligned} \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \nu\Delta u + \gamma \sin y, \\ \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \nu\Delta v, \\ \frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} = 0, \end{aligned}\right\}$$

(Arnol'd and Meshalkin, 1960)

Vorticity equation: $\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$

$\Delta q_\lambda = \lambda q_\lambda$

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega - \gamma \lambda q_\lambda$$

Representing $u = \nabla^\perp \psi$ by streamfunction ψ, $\omega = \Delta\psi$

$$\partial_t \Delta\psi + J(\psi, \Delta\psi) = \nu \Delta^2 \psi - \gamma \lambda q_\lambda$$

$$J(a,b) = \nabla^\perp a \cdot \nabla b$$

Note, this equation admits a laminar solution:

$$\bar\psi = \frac{1}{\nu\lambda} q_\lambda$$

Since, $\nu \Delta^2 \bar\Psi = \lambda q_\lambda$, $J(\bar\Psi, \Delta\bar\Psi) = \lambda J(\bar\Psi, \bar\Psi) = 0$.

The velocity corresponding to this solution is

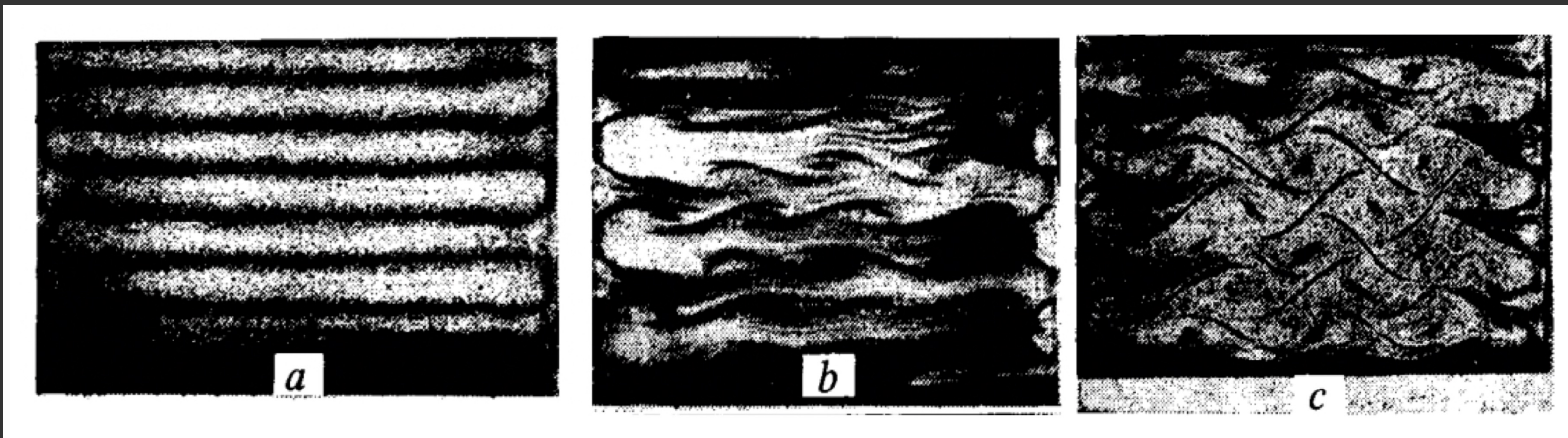
$$\bar u = \frac{1}{\nu\lambda} f_\lambda \overset{\text{for example}}{=} \frac{\gamma}{\nu} \begin{pmatrix} \sin y \\ 0 \end{pmatrix}$$

This laminar flow is the unique steady state when $\mu = \gamma/\nu^2$ is small, and is the global attractor for all data.

μ is order parameter: instability, periodic-in-time, turbulence...

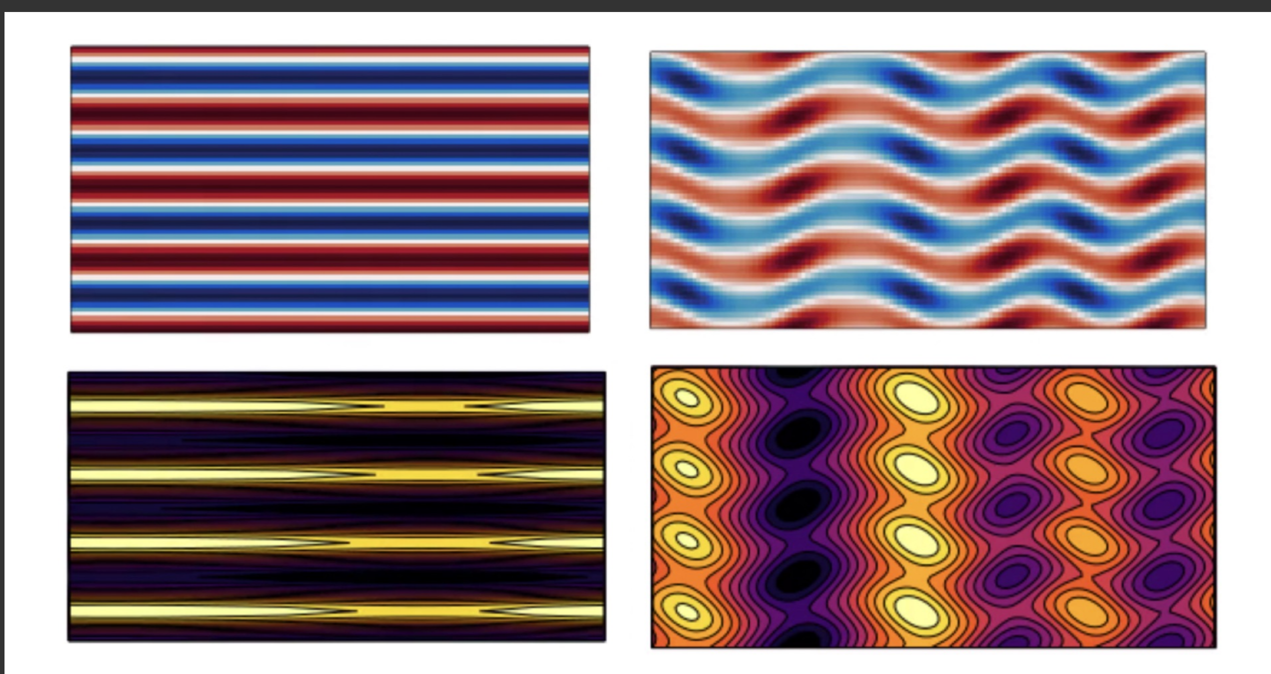
Exceptional (trivial case): when λ is smallest eigvalue, "gravest mode" is unique global attractor for all $\mu > 0$!

Real World



Obukhov, "Kolmogorov flow and laboratory simulation of it" (1983)

Virtual Reality



simulations done on my personal computer using publically available code writen by

Navid Constantinou

$$f = \begin{pmatrix} \cos(5y) \\ 0 \end{pmatrix}$$

Arnold, "Kolmogorov's hydrodynamic attractors" 1991

The flow in this 'Kolmogorov model' is usually supposed to be two-dimensional and the exterior force field has usually the form of the first harmonic. If the Reynolds number is small, the steady laminar flow emerges: its velocity profile has a sinusoidal form. This flow loses its stability as the Reynolds number grows. The Kolmogorov 1958 programme was to study rigorously the bifurcations of the attractors in this system and their dependence on the growing Reynolds number. This programme was partially fulfilled in the work of Mechalkin & Sinai (1961). As far as I know, the bifurcations in such systems have not been completely understood even now, even in the two-dimensional case and even numerically, using the Galerkin approximation for *moderate* Reynolds number (but see Nicolaenko & She 1990).

Upper Bounds Kolmogorov viewed the sequence of bifurcation picture as one possible "route to turbulence"

quasiperiodic flows with 2, then 3, 4 ...

(2-tori, 3-tori ... in phase space) appear subsequently.

Kolmogorov asked if an infinite sequence of such bifurcations could occur within a finite parameter range (as in logistic map picture)

$\boxed{\text{NO}}$: Ladyzhenskaya (1982), Il'yashenko (1983).

$$\boxed{\dim_H(\text{Attractor}) \lesssim \frac{1}{\alpha} G^{1/2} \left(1 + \log \frac{G}{\alpha^{1/4}}\right)^{1/3}}$$

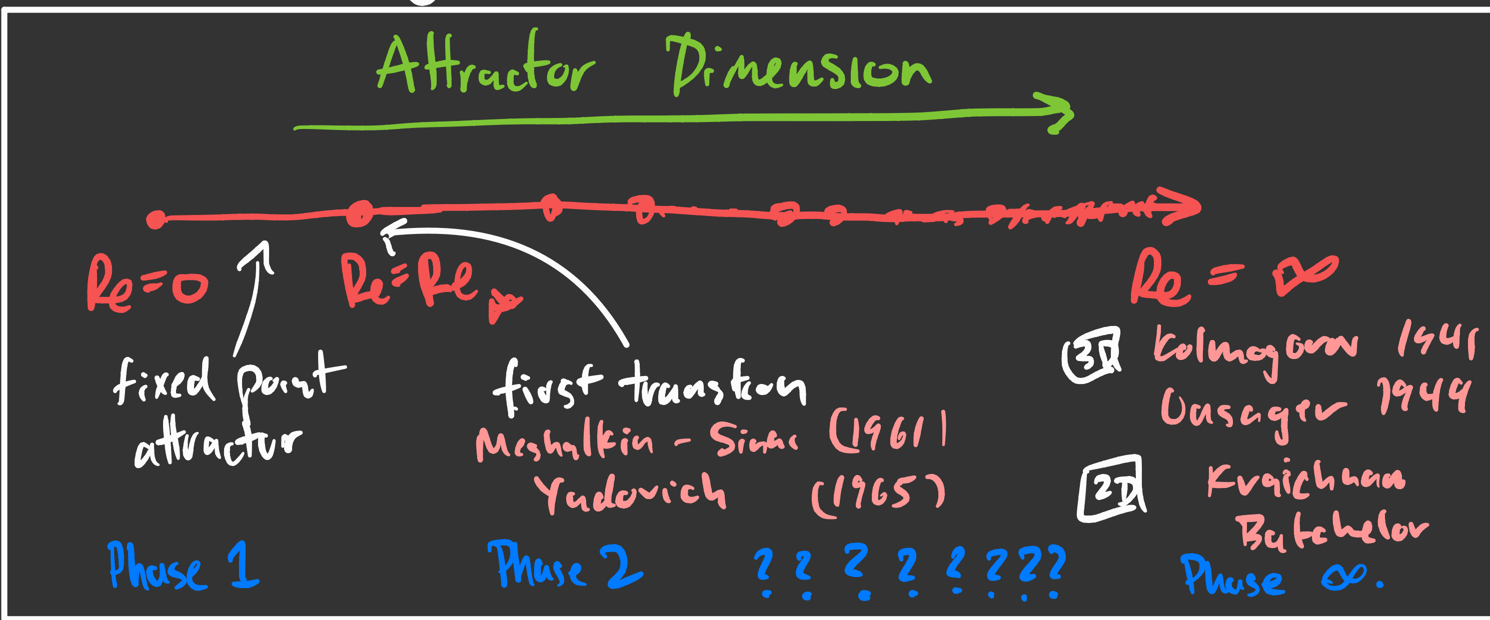
where $G = \|f\|_{L^2} \frac{L^2}{\nu^2}$. Constantin - Foias - Temam (1988).

dependence expected sharp (agree's with Landau's D.O.F arg)

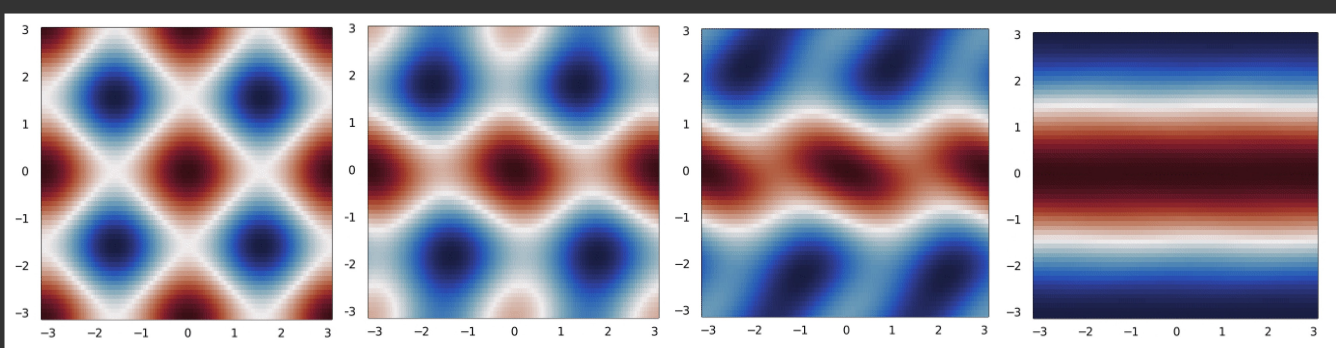
Arnol'd, "Kolmogorov's hydrodynamic attractors" 1991

Does the minimum of the Hausdorff dimensions of the attractors grow with Reynolds number for the Navier–Stokes system on a compact two-dimensional manifold (for, say, the Kolmogorov model)?

Here, we try to build up the picture:



PHASE: I



Theorem: There exists $\mu^* = \frac{\gamma}{r_*^2}$ such that for all $\mu < \mu_*$, the laminar solution is the unique stationary solution and attracts all orbits as $t \to \infty$. If $\lambda = \lambda_1$, true for all μ.

PHASE II:

THEOREM: (Yudovich 1965). Let $\alpha \in (0,1)$. Consider Navier-stokes forced by $\gamma \cos y$ with $\gamma > 0$ on $\mathbb{T}^2_\alpha$. Then, there exists a critical Grashof number $\mu_*(\alpha)$ which is a monotonically increasing function of α satisfying

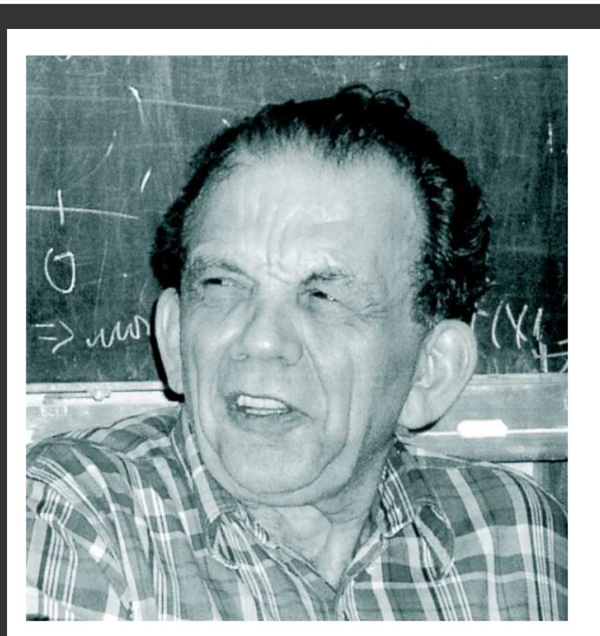
$$\lim_{\alpha \to 0} \mu_*(\alpha) \to \sqrt{2}, \qquad \lim_{\alpha \to 1} \mu_\alpha(\alpha) \to \infty.$$

and is such that the following hold:

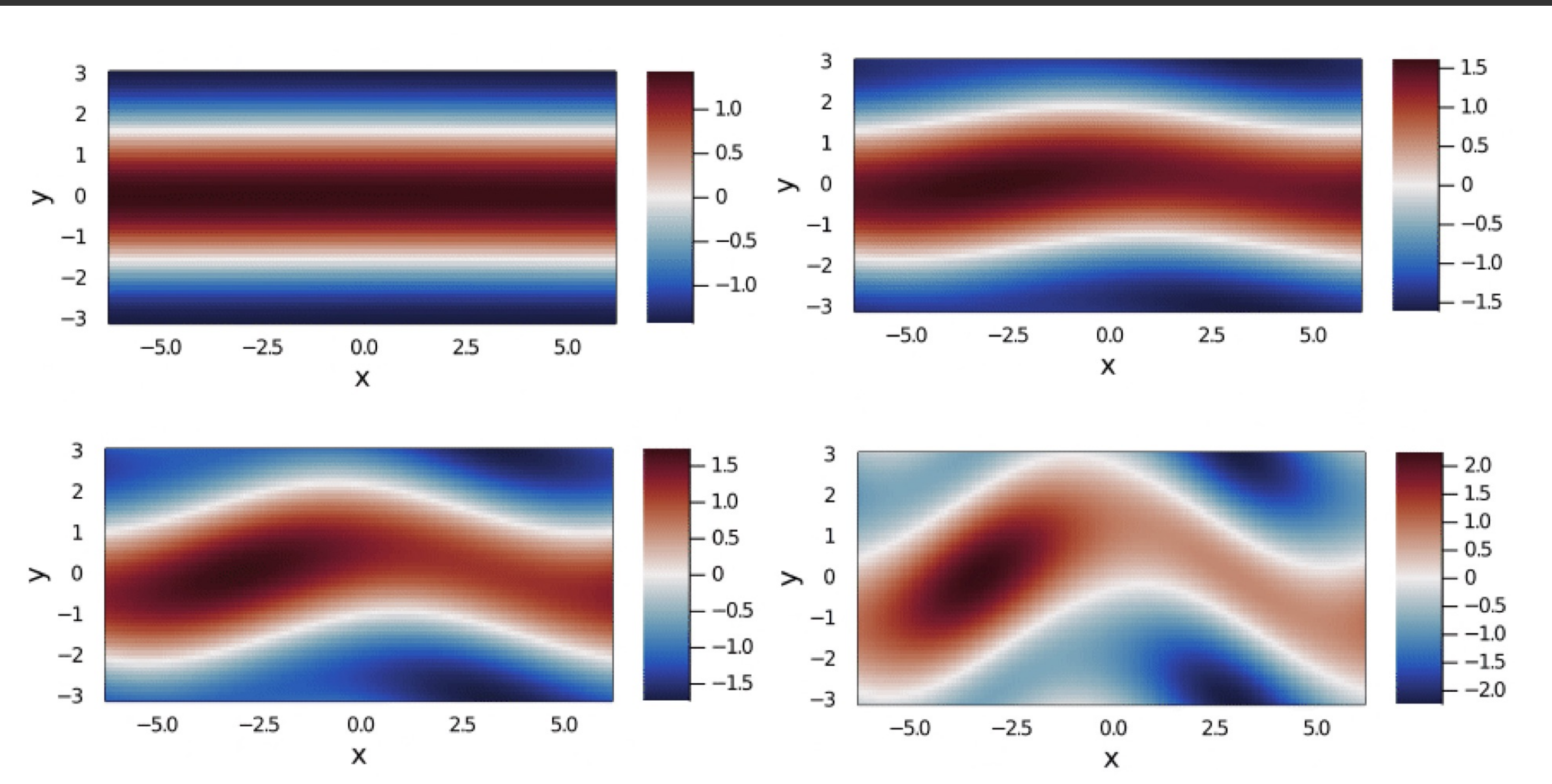
(1) If $\mu < \mu_*$, the laminar state is unique steady solution in $H^2(\mathbb{T}^2_\alpha)$ and is attractor for all data.

(2) For $\mu > \mu_*$, the laminar state becomes unstable (Meshalkin - Sinai (1961))

(3) For $|\mu - \mu_*| \ll 1$, there exists a two-dimensional manifold of stationary states which bifurcate from the laminar state and are linearly stable.



Yudovich's Bifurcation, and its fate



Elements of the proof:

Step 1: express nonlinear problem and its linearization as eigenvalue problems on the Hilbert space H^2

$$\psi = \nu\mu\,(\phi - \cos y)$$

Stationary.

Then the perturbation ϕ satisfies nonlinear equation

$$\Delta^2\phi = \mu \sin y\, \partial_x(\phi + \Delta\phi) + \mu J(\phi, \Delta\phi)$$

Its linearization reads

$$\Delta^2\phi = \mu \sin y\, \partial_x(\phi + \Delta\phi)$$

These read $N[\phi] = \frac{1}{\mu}\phi$, $L[\phi] = \frac{1}{\mu}\phi$ in H_2, $(f,g)_{H^2} = \int_{\mathbb{T}^2_\alpha} \Delta f\, \Delta g$.

Review of Bifurcation Theory

$G[u] = \lambda u$ eqn in Banach space X

(a) $G[0] = 0$
(b) G is compact
(c) G is differentiable at $u=0$ with compact $A = G'[0]$.

Consider $S = \{(\lambda, u) \in \mathbb{R} \times X : u \neq 0, \lambda u = G[u]\}$

DEF: A bifurcation point λ_* is such that any neighborhood of $(\lambda_*, 0)$ there is a point $(\lambda, u) \in S$

Bifurcation points must be eigenvalues of A by implicit function theorem. Reverse not true but

THM (Krasnoselskii) Suppose (a) – (c) hold and that λ_* is an eigenvalue of A with odd multiplicity. Then λ_* is a bifurcation point for $F[u] = \lambda u - G[u]$

Global refinement by Rabinowitz.

- To obtain information about multiplicity of eigenvalue, we study also adjoint problem and find eigenfunc with same λ and not orthogonal to original eigenfunction
  $\Rightarrow$ λ simple
- (Simplicity in subspace). Consider $\frac{d^2}{dx^2} h = -\lambda h$

  eigenvalues: $\lambda = k^2$ eigenfunctions $\sin(kx)$, $\cos(kx)$
  geo = alg mult = 2 odd even
  simple in even and odd subspaces.

- we restrict to even and odd subspaces to construct a simple eigenvalue
- We apply these theorems in the setting of

$$X = \mathbb{H}^2_e, \mathbb{H}^2_o, \quad G = N, \quad A = L.$$

- To construct eigenfunction, note any such is a linear combination of

$$\phi_n(x,y) = e^{i\alpha n x} \sum_{m \in \mathbb{Z}} c_m(\alpha n)\, e^{imy}$$

Then $c_n = d_m / (k^2 + m^2 - 1)$ with $k = \alpha n$ &

$$(*) \qquad a_m d_m + d_{m-1} - d_{m+1} = 0$$

with

$$a_m = \frac{2(k^2+m^2)^2}{\mu k (k^2+m^2-1)}.$$

One derives characteristic eqn:

$$(**) \qquad F(\mu) := \frac{1}{[a_1, a_2, \dots]} = -\frac{d_0}{2}$$

$$\frac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

converges by Van Vleck convergence theorem and lower bnd on a_n.

μ is a real root of $(*)$ iff d_n solves $(**)$, $d_n \neq 0$

PHASE ∞: (aspects of it)

Kolmogorov's conjecture more precisely

1992-11. Consider the Navier–Stokes equation (say, on the 2-torus) with external force proportional to the viscosity (Kolmogorov's model). Is it true that, as the viscosity tends to 0 (i. e., the Reynolds number grows), there appear attractors of dimensions increasing with the Reynolds number (and containing no smaller attractors)?

Is it true that, moreover, the minimum dimension of all attractors unboundedly increases with the Reynolds number?

A. N. Kolmogorov suggested (in 1958) *that the answer to the first question was affirmative, but he doubted that so was the answer to the second because of the experiments on delaying loss of stability.*

There are two Kolmogorov's 1958 conjectures on the behavior of the dimension of minimal attractors (i. e., the attractor which does not contain a smaller attractor) when its Reynolds number R tends to the infinity:

a) weak

$$\max_{\min \mathrm{Attr}} \dim \min \mathrm{Attr} \to \infty \quad \text{for } \nu \to 0;$$

b) strong

$$\min_{\min \mathrm{Attr}} \dim \min \mathrm{Attr} \to \infty \quad \text{for } \nu \to 0;$$

(see [1], Ch. I) where ν is the (kinematic) viscosity of a current.

V. I. Arnol'd 1991, 1999, 2004.

THEOREM: (Babin-Vishik 1983) For $\alpha \in (0,1)$ and $\mu > \mu_* = \mu_*(\alpha)$

$$2\left\lfloor \frac{1}{\alpha} \right\rfloor \leq \dim_H(\text{Attractor}) \leq \frac{1}{\alpha}\left(1 + \alpha^{1/2}\mu\right).$$

dimension of unstable manifold of laminar steady state.

2π 

# of boxes of side length 2π in $2\pi/\alpha$ by 2π rectangle.

Consider only steady states $\tilde{u} = \frac{\nu}{\gamma} u$, $\mu = \frac{\gamma}{\nu^2}$

steady NS
$$\begin{cases} \mu\left(\tilde{u}\cdot\nabla\tilde{u} - \nabla\tilde{p}\right) = \nabla\tilde{p} + \Delta\tilde{u} + f_\lambda \\ \nabla\cdot\tilde{u} = 0 \\ \displaystyle\int_{\mathbb{T}^2_\alpha} \tilde{u}\, dx dy = 0 \end{cases}$$

THEOREM: Let $\{u^\mu\}_{\mu>0}$ be a family of $H^2(\mathbb{T}^2_\alpha)$ steady solutions of Navier-Stokes. Then $\frac{\nu}{\gamma}u^\mu \to u^E$ strong in H^1 as $\mu \to \infty$ where $u^E \in H^2$ solves

$$u^E\cdot\nabla u^E = -\nabla p^E$$
$$\nabla\cdot u^E = 0$$
$$\int u^E\, dx dy = 0$$

with the additional constraint

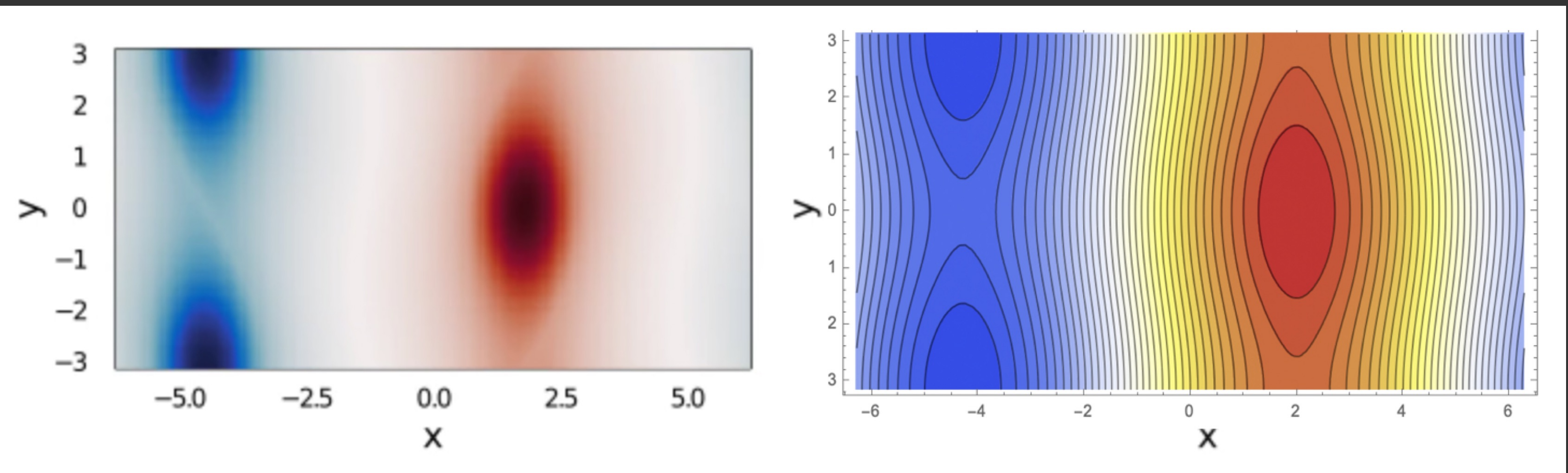
$$\lambda_1 \|\nabla u^E\|^2_{L^2} \le \|\Delta u^E\|^2_{L^2} \le \lambda \|\nabla u^E\|^2_{L^2}$$

Remark: • to normalize, Kolmogorov suggests $\gamma = \nu$

• nonempty: applies to laminar state (saturates)

$$\bar{u} = \frac{\gamma}{\nu\lambda} f_\lambda .$$

Numerical observation: By following on branch of the bifurcation diagram.



seems to converge to elliptical vortex pattern.

On right: assuming solution is 2 modes and saturates lower bound. Not solution except when $\alpha \to 1$.

Conjecture: $\exists$ a sequence of steady NS solutions with non-trivial but shrinking basin of attraction as $\mu \to \infty$. Thus strong Kolmogorov would be false,

A.N. Kolmogorov always emphasized that preservation of stability of a steady flow, even for the infinitely growing Reynolds number, would not contradict hydrodynamical experiments, under the assumption that the basin of the corresponding attractor shrinks fast enough.

Arnol'd - Khesin (1999)

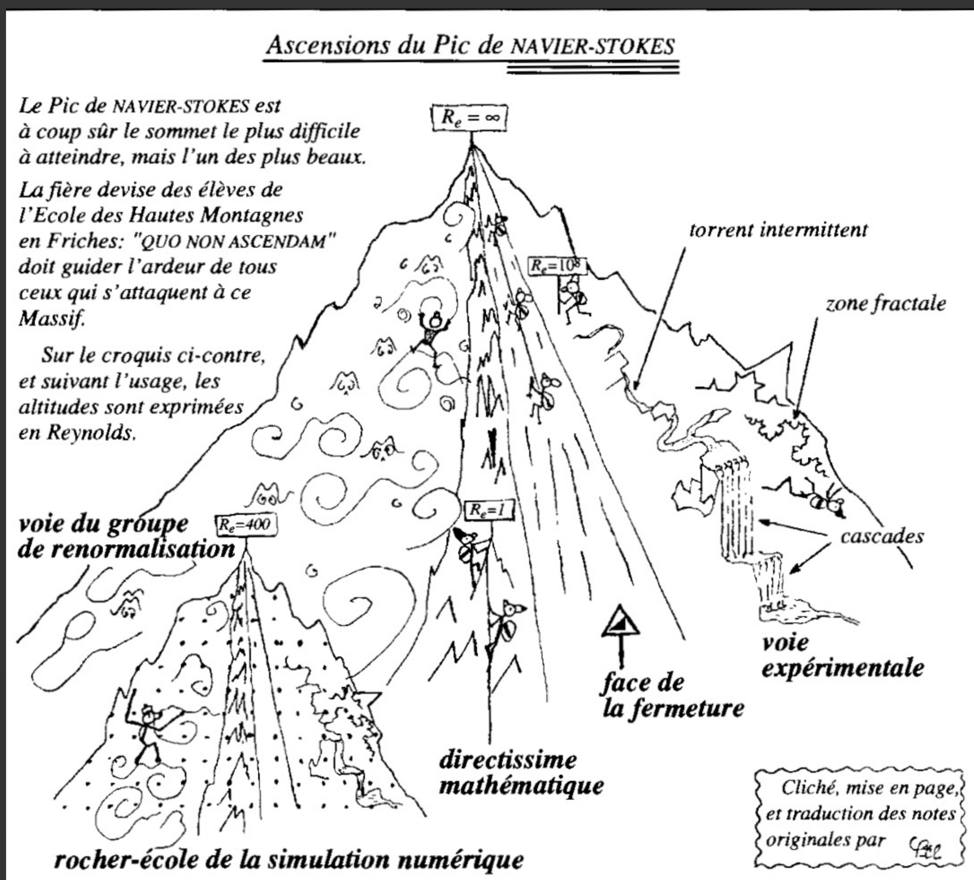
THANK-YOU!

Did not have time to cover

- Long time behavior of ideal fluids near laminar states.
- Mixing, instability and infinite time growth

  (Koch, Nadirashvili)
- Wandering solutions of Euler

  (Nadirashvili, Shnirelman,
  Khesin – Kuksin – Peralta-Salas)



LEMMA: Let $k = \alpha n \in (0,1)$. Then (**) admits a positive root $\mu_* = \mu_*(k)$ and it has only this one root. If $k > 1$, there is no root.

Lemma: let $\alpha \in (0,1)$. The problems

$$L[\phi] = \frac{1}{\mu}\phi \qquad \overset{*}{L}[\Phi] = \frac{1}{\mu}\Phi$$

have exactly $\lfloor \frac{1}{\alpha} \rfloor$ positive (as many negative) eigenvalues $1/\mu$. Each as multiplicity two.

THEOREM: There exist exactly $\lfloor \frac{1}{\alpha} \rfloor$ positive numbers

$$0 < \mu_*^{(1)} < \mu_*^{(2)} < \dots < \mu_*^{(n)} < \infty$$

which are points of bifurcation for NS. Each corresponds to a branch of eigenfunctions

$$-\infty > -\mu_*^{(m)} > \dots > -\mu_*^{(2)} > -\mu_*^{(1)} > 0.$$

are also points of bifurcation. The spectrum also contains the intervals:

$$(\mu_*^{(1)}, \mu_*^{(2)}), \quad (\mu_*^{(3)}, \mu_*^{(4)}), \quad \dots \quad (\mu_*^{(m-1)}, \mu_*^{(m)})$$

If m is odd, includes $(\mu_*^{(m)}, \infty)$. Also reflections.