Mathematical Aspects of Turbulence

Outline:

		Date
Lecture 1&2:	Phenomenology and Mathematics of fully developed turbulence	2/3, 2/5
Lecture 3:	Transistion to turbulence and a problem of kolmogorov	2/10, 2/11
Lecture 4:	Formation of small scales in ideal (inviscid) fluids	2/11

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Detailed Outline:

Lecture 1:

"$Re = \infty$"

- Equations of motion and basic properties
- High Reynolds number behavior, exp. & numerics
- Kolmogorov 1941 theory
- Landau's Kazan remark & intermittency

Lecture 2:

"$Re = \infty$"

- Onsager's Conjecture and dissipative weak Euler solutions.
- Examples of "anomalous dissipation"
  - convex integration constructions
  - Burgers
  - passive scalar constructions.

Lecture 3:

"$Re = 0\text{-}1000$"

- Transition to turbulence
- Kolmogorov's flow problem.
- Stability of laminar state at "low Re"
- Instability of laminar state at "high Re" (Meshalkin & Sinai)
- Bifurcation to stable secondary flows near onset (Yudovich).

Lecture 4:

"viscosity = 0"

laminar

- Long time behavior of ideal fluids near laminar states.
- Mixing, instability and infinite time growth (Koch, Nadirashvili)

Lecture 1:

The Euler equations (derived in 1757) for

velocity vector field $u(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$

internal pressure field $p(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$

- conservation of mass (incompressibility)

$$\nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \cdots + \partial_d u_d = 0$$

- conservation of momentum (Newton's second law)

$$\rho_0 (\partial_t u + u \cdot \nabla u) = -\nabla p$$

These are supplimented with

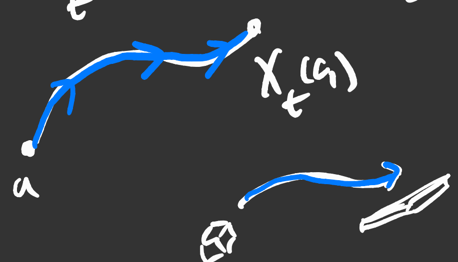
- initial datum u_0 for the Cauchy problem
- boundary conditions: e.g. $\mathbb{T}^d = [-\pi, \pi]^d$ periodic

  or $u \cdot \hat{n}\big|_{\partial\Omega} = 0$ non-penetration

Consider a particle

$$\frac{d}{dt} X_t(a) = u(X_t(a), t) \quad \Rightarrow \quad \rho_0 \ddot{X}_t(a) = -\nabla p (X_t(a), t)$$

$$m \, a = F$$



$$\det \nabla X_t = \exp\left(\int_0^t \nabla \cdot u (X_s, s) \, ds \right) = 1$$

Energy conservation on smooth solutions:

$$\frac{d}{dt} \int_\Omega \frac{\rho_0 |u|^2}{2} dx = -\rho_0 \int_\Omega u \cdot (u \cdot \nabla u + \nabla p) \, dx = -\rho_0 \int_\Omega u \cdot \nabla \left(\tfrac{1}{2} |u|^2 + p \right) dx$$

$$= -\rho_0 \int_{\partial\Omega} u \cdot \hat{n} \left(\tfrac{1}{2} |u|^2 + p \right) dS = 0 .$$

pressure:

- enforces incompressibility of the motion, recovered from the velocity field

$$\operatorname{div}(\partial_t u + u\cdot\nabla u = -\nabla p)$$

$$-\Delta p = \nabla\cdot(u\cdot\nabla u)$$

- defined up to a harmonic function. by the above. To determine the pressure, we require boundary conditions (e.g. periodicity & mean zero).

Vorticity:

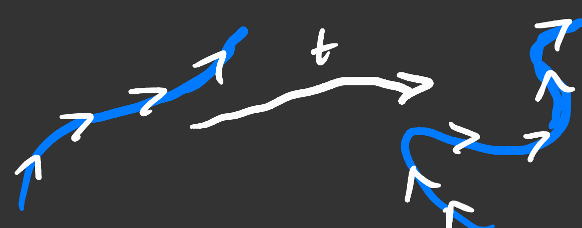
$$\omega = (\partial_2 u_3 - \partial_3 u_2,\ \partial_3 u_1 - \partial_1 u_3,\ \partial_1 u_2 - \partial_2 u_1) = \nabla\times u$$

geometric meaning: gives an axis and strength of local rotation

using the identities

- $(u\cdot\nabla)u = \omega\times u + \frac{1}{2}\nabla|u|^2$
- $\nabla\times(\omega\times u) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u$ (since $\nabla\cdot u = 0$, $\nabla\cdot\omega = 0$)

one derives the evolution

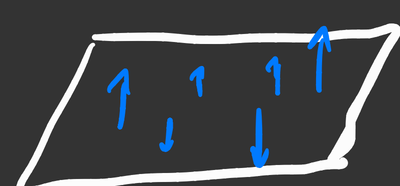


$\boxed{3D}$ $\quad \partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u.$

Note: if $u = (u_1(x,y), u_2(x,y), 0)$ then

$$\omega = (0, 0, \nabla^\perp\cdot(u_1,u_2)), \qquad \nabla^\perp := (-\partial_2, \partial_1)$$

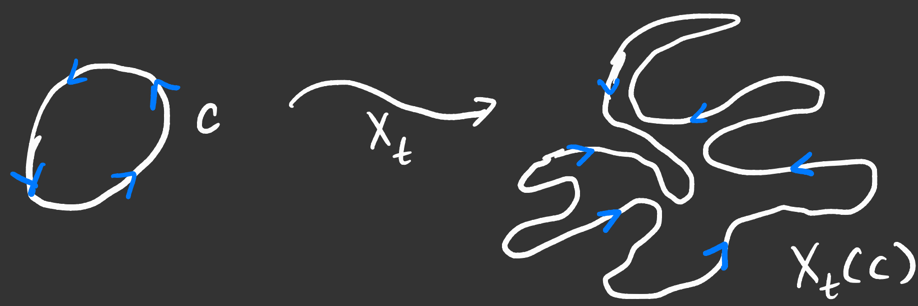
$\boxed{2D}$ $\quad \partial_t\omega + u\cdot\nabla\omega = 0.$



two-dimensional vorticity

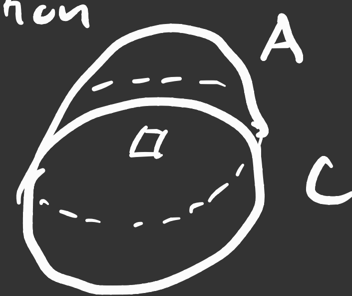
Kelvin theorem: The circulation around the curve $C(t) = X_t(C)$ is conserved by smooth 3D Euler flow, i.e.

$$\frac{d}{dt} \oint_{C(t)} u(t) \cdot d\ell = 0 \qquad \forall \ t > 0.$$



Remark: Equivalent to vorticity flux conservation

$$\frac{d}{dt} \oint_{A(t)} \omega(t) \cdot d\sigma = 0$$



Remark: In fact, the circulation theorem holding for every rectifiable loop characterizes smooth solutions of Euler.

Proof: parametrize $C(s) : [0,1] \to C$. Then

$$X_t(C) = \{ X_t(C(s)) \ : \ s \in [0,1] \}$$

$$\oint_{X_t(C)} u(t) \cdot d\ell = \int_0^1 u(X_t(C(s))) \cdot \frac{d}{ds} X_t(C(s))\, ds$$

$$(\partial_t u + u \cdot \nabla u)(X_t(C(s))) = -\nabla p(X_t(C(s)))$$

$$\frac{d}{dt} \oint_{X_t(C)} u(t) \cdot d\ell = \int_0^1 \frac{d}{dt} u(X_t(C(s))) \cdot \frac{d}{ds} X_t(C(s))\, ds$$

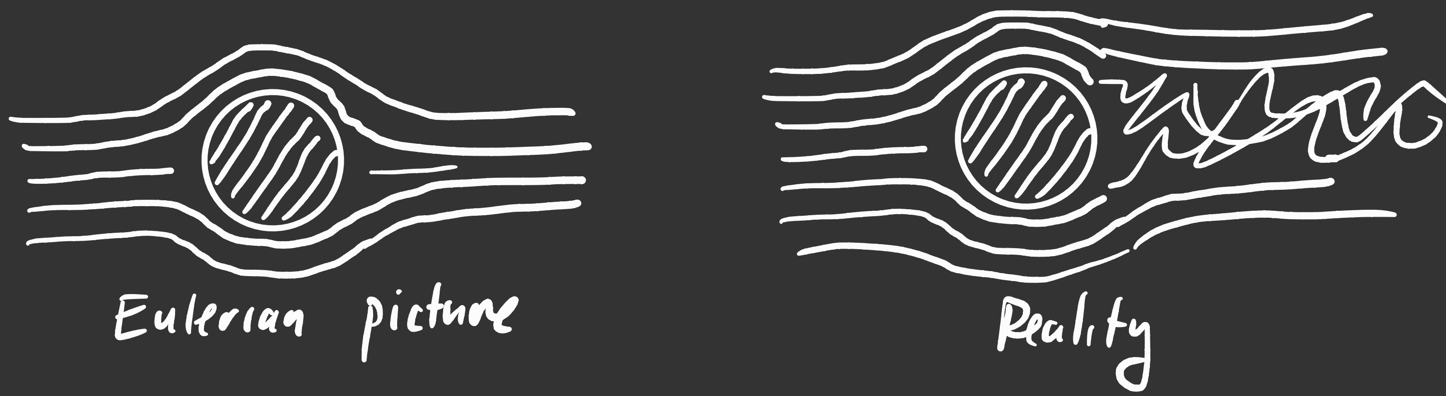
$$+ \int_0^1 u(X_t(C(s))) \cdot \frac{d}{ds} u(X_t(C(s)))\, ds$$

$$= \frac{1}{2} \frac{d}{ds} \left| u(X_t(C(s))) \right|^2$$

$$= 0.$$

D'Alembert's Paradox: in an irrotational Euler flow, the drag force on a body moving with constant velocity relative to the fluid is zero!

"Planes cannot fly in an irrotational Euler flow"



Solution:	One must take into account friction forces between adjacent molecules, i.e. viscosity

C.L. Navier (1822) and G. Stokes (1845) derived a model for this under the assumption that the shear stress is proportional to the symmetric part of the gradient:

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f$$
$$\nabla \cdot u = 0$$

- the parameter $\nu > 0$ is the kinematic viscosity of the fluid
- the function $f(x,t)$ is an external body force.
- widely accepted model of Newtonian fluid flow, arising in the joint limit of small Knudsen number and Mach numbers

Rigorous derivations:
- Quastle – Yau (1998) from Lattice gas model
- Saint-Raymond (2003) from Boltzmann
- Spohn (2012) from Molecular dynamics under assumptions

Non-dimensionalization (physical laws hold independent of units)

- U: characteristic velocity of the flow, e.g. rms $\left(\int |u|^2\right)^{1/2}$.
- L: characteristic length in the flow, e.g. domain size or period

Note, all the terms in NS have units of acceleration $\frac{L^3}{U^2}$.

Non-dimentionalizing $u \mapsto u/U$, $x \to x/L$, $t \to t/{L/U}$

$$\partial_t u + u\cdot\nabla u = -\nabla p + \frac{1}{Re}\Delta u$$
$$\nabla\cdot u = 0$$

The non-dimentional number

$$Re = \frac{UL}{\nu} \sim \frac{u\cdot\nabla u}{\nu\Delta u}$$

is the Reynolds number. It measures relative strength of inertial forces (nonlinearity) and viscous forces:

- bacteria $\quad Re \approx 10^{-5}$
- blood flow $\quad Re \approx 10^{2}$
- MLB pitch $\quad Re \approx 10^{5}$
- wake of blue whale $\quad Re \approx 10^{8}$
- wake of Boeing 747 $\quad Re \approx 10^{12}$

$\vdots$

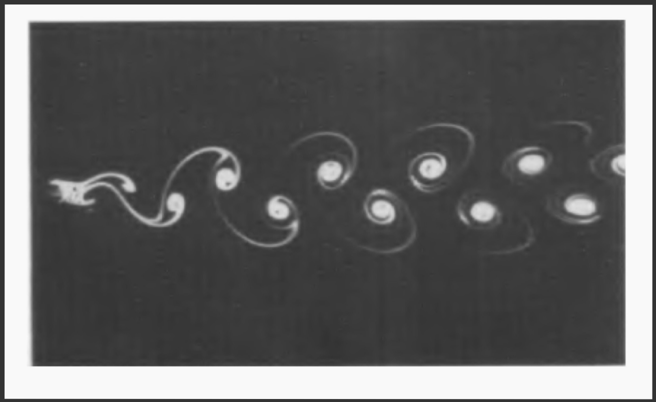
Note: in experiments, one often uses the Taylor-scale Re

$$Re_\lambda = \frac{U\lambda}{\nu}, \quad \lambda^2 = \frac{\int |u|^2}{\int |\nabla u|^2}$$

Typically $Re_\lambda \gg Re$.

Van Kármán vortex street behind cylinder.

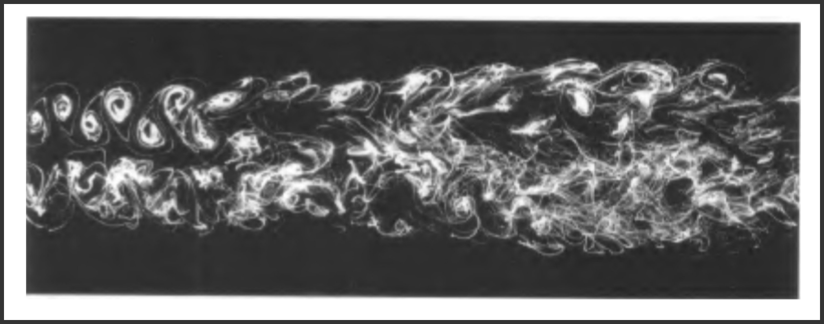
Van Dyke (1982)
Frisch (1995)



Re = 105

Wake behind two cylinders

Frisch (1995)



Re = 240

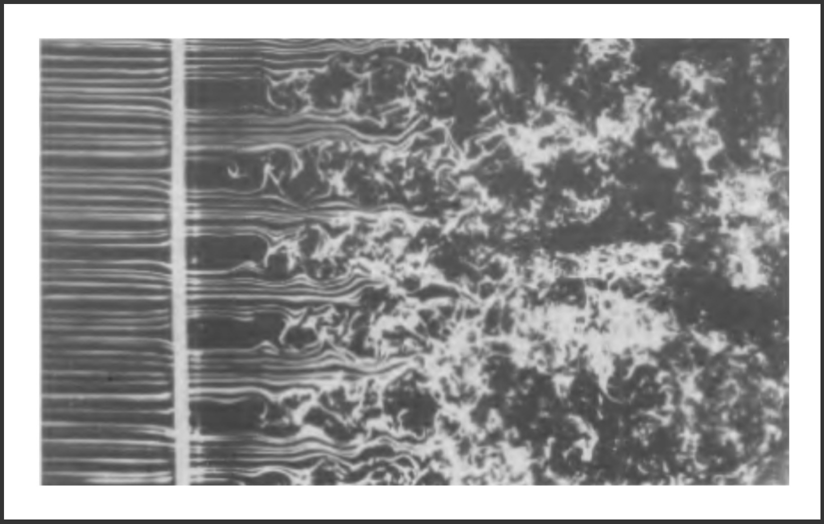
Wake behind two cylinders

Frisch (1995)

Re = 1800

Homogeneous turbulence behind grid.

Frisch (1995)



Re = 2300

Goals and Obstacles for Mathematics

- in real flows turbulence is generated at boundaries.

However, universal statistical features and small scale behavior are expected to hold away from walls.

- given the complexity of the flow, it may be unreasonable to hope to make pathwise predictions.

Instead, one may hope to predict averages / statistics

Physicist & Engineer Approach

- perform accurate experiments and simulations
- provide phenomenological theories based on heuristic principles and data fitting
- Often do not formalize precise statements/questions about turbulence since they may be hard to prove and one may give counterexamples

Mathematician Approach

- prove, from first principles (from NS or E equations) some experimental "facts". Due to the immense complexity, these are often of a conditional nature.
- Identify constraints on solutions of PDEs which make them "physical" or "observable".

Observables and Averages (Idealized setup)

- to achieve a nontrivial statistical steady state, one drives the fluid with a force acting on low modes (large scales).

  Typically $f = f(x)$ or $f = \sigma(x)\, dW_t$.

- example of observables F of solution u:
  Kinetic energy, dissipation rate, structure functions energy spectrum ...

In theory:

- attractors: long-time averages

$$\langle F(u) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(u(t))\, dt$$

observes solutions close to or on attractor

- statistical mechanics: ensemble averages is steady state

$$\langle F(u) \rangle = \int_{L^2} F(u)\, d\mu_{Re}(u)$$

ergodic invariant measure

- Ergodic hypothesis: these two concepts agree

note: Sometimes $\langle \cdot \rangle$ includes spatial ave. Follows if homogeneous.

Practice:
- (finite window) time average once reach equilibrium.
- (singular data) achieving equilibrium is hard, can result in rough data.
- (Taylors hypothesis) $\ell = U\tau$ relates time/space lags.

Anomalous Dissipation of Energy

The fundamental postulate of Kolmogorov's 1941 theory, the "zeroth law of turbulence" is a non-vanishing dissipation as $Re \to \infty$.

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \qquad \nabla \cdot u = 0$$

For dimensions $d \geq 3$, the only known a-priori controlled quantities which are controlled is from

$$\partial_t \left(\tfrac{1}{2}|u|^2\right) + \nabla \cdot \left(u \left(\tfrac{1}{2}|u|^2 + p\right) - \nu \nabla \tfrac{1}{2}|u|^2 \right) = -\nu |\nabla u|^2 + f \cdot u$$

provided the solution is smooth. Thus

$$(*) \quad \frac{d}{dt} \int_{\mathbb{T}^d} \tfrac{1}{2}|u|^2 dx = -\nu \int_{\mathbb{T}^d} |\nabla u|^2 dx + \int_{\mathbb{T}^d} f \cdot u \, dx$$

Gives apriori control of the solution in

$$u \in L^\infty_t L^2_x \cap L^2_t H^1_x .$$

Leray (1934) used this energy balance for a suitable approximation scheme combined with a compactness arguement to prove existence of a global-in-time weak solution of NS

- These satisfy $(*)$ with $\leq$.
- If $u \in L^4_{t,x}$ (Shinbrot 74) then interpolation gives $u \in L^3_t W^{1/3,3}_x$ $\Rightarrow \leq \; = \; =$.

what is known is the equality (Duchon-Robert, 2000)

$$\partial_t \left(\tfrac{1}{2}|u|^2\right) + \nabla\cdot\left(u\left(\frac{|u|^2}{2} + p\right) - \nu \nabla \frac{|u|^2}{2}\right) = -\nu|\nabla u|^2 - D[u] + u\cdot f$$

where the (x,t) - distribution $D[u]$ is defined by a weak form of the Kármán-Howarth-Monin relation:

$$D[u](x,t) = \lim_{\ell\to 0} \frac{1}{4}\int_{\mathbb{T}^d} \nabla\phi_\ell(r)\cdot \delta_r u(x,t)\, |\delta_r u(x,t)|^2\, dr = \mathcal{O}\left(\frac{(\delta_\ell u)^3}{\ell}\right)$$

where $\delta_r u(x,t) = u(x+r,t) - u(x,t)$ and $\phi_\ell(r) = \frac{1}{\ell^d}\phi\left(\frac{r}{\ell}\right)$.

D&R show the distributional limit of $L^1_{t,x}$ objects is independent of the choice of ϕ.

Moreover, it is nonnegative.

Proof: Let u_ℓ be a Leray regularized solution:

$$\partial_t u_\ell + (\phi_\ell * u_\ell)\cdot\nabla u_\ell = -\nabla p_\ell + \nu\Delta u_\ell .$$

Then, $u_\ell \to u$ weakly in $L^\infty_t L^2_x \cap L^2_t H^1_x$ and strongly in $L^3_t L^3_x$. Thus

$$D[u] = \lim_{\ell\to 0} \nu|\nabla u_\ell|^2 - \nu|\nabla u|^2$$

For any $\psi\in C_c^\infty$, $\psi \geq 0$, the map $u \mapsto \iint |\nabla u|^2 \psi\, dx\, dt$ is convex and lower semicontinuous w.r.t. the weak topology on $L^2_t H^1_x$. Thus $D[u] \geq 0$.

Understanding the defect distribution:

$$\partial_t u + u \cdot \nabla u = \dots$$

Let $u = u(x,t)$ and $u' = u(x+r,t)$. Note

$$\frac{1}{2}\int u \cdot \bar{u}_\ell = \frac{1}{2}\int G_\ell \, u u' \, dr \xrightarrow{\ell \to 0} \frac{1}{2}\int |u|^2$$

Derive equation for uu'

$$
\begin{aligned}
-\partial_t (uu') &= u \cdot (u' \cdot \nabla u') + u' \cdot [u \cdot \nabla u] \\
&= u \cdot (u' \cdot \nabla u') + u' \cdot \operatorname{div}(u \otimes u) \\
&= u \cdot (u' \cdot \nabla u') - u \otimes u : \nabla u' \\
&\quad + \operatorname{div}((u \cdot u') u).
\end{aligned}
$$

$$u \cdot (u' \cdot \nabla u') - u \otimes u : \nabla u' \qquad (\nabla^{1/3} u)^3$$

$$
\begin{aligned}
&= u \cdot ((u' - u) \cdot \nabla u') \\
&= u \cdot (\delta_r u \cdot \nabla u') \qquad \nabla_x u(x+r) = \nabla_r u(x+r) = \nabla_r \delta_r u \\
&= u \cdot (\delta_r u \cdot \nabla_r \delta_r u) = u \cdot \operatorname{div}_r (\delta_r u \otimes \delta_r u) \\
&= -\delta_r u \cdot \operatorname{div}_r (\delta_r u \otimes \delta_r u) - u' \cdot \operatorname{div}_r (\delta_r u \otimes \delta_r u) \\
&= \frac{1}{2} \operatorname{div}_r [\delta_r u |\delta_r u|^2] - \frac{1}{2} \operatorname{div}_x [|u'|^2 \delta_r u]
\end{aligned}
$$

$$\frac{1}{2}\partial_t (u \cdot u') = -\frac{1}{2} \nabla_r \cdot [\delta_r u |\delta_r u|^2] + \nabla_x \cdot J$$

$$\xrightarrow[\varphi_\ell \, \infty]{\ell \to 0} -D[u]$$

Thus, for weak solutions of Navier-Stokes, we have

$$\frac{d}{dt} E(t) = -\int_{\mathbb{T}^d} D[u]\,dx - \nu \int_{\mathbb{T}^d} |\nabla u|^2\,dx + \int_{\mathbb{T}^d} f \cdot u\,dx$$

We define the energy dissipation rate (per unit mass)

$$\boxed{\varepsilon^\nu[u] = \left\langle \fint_{\mathbb{T}^d} D[u]\,dx \right\rangle + \nu \left\langle \fint |\nabla u|^2 dx \right\rangle}$$

Experimentally, the zeroth law of turbulence

$$\boxed{\varepsilon = \liminf_{\nu \to 0} \varepsilon^\nu[u] > 0}$$

Non-zero energy dissipation in the inviscid limit!!

REMARK: 3D Phenomenon!
Theorems say AD cannot occur in 2D flows without boundary and smooth forcing.

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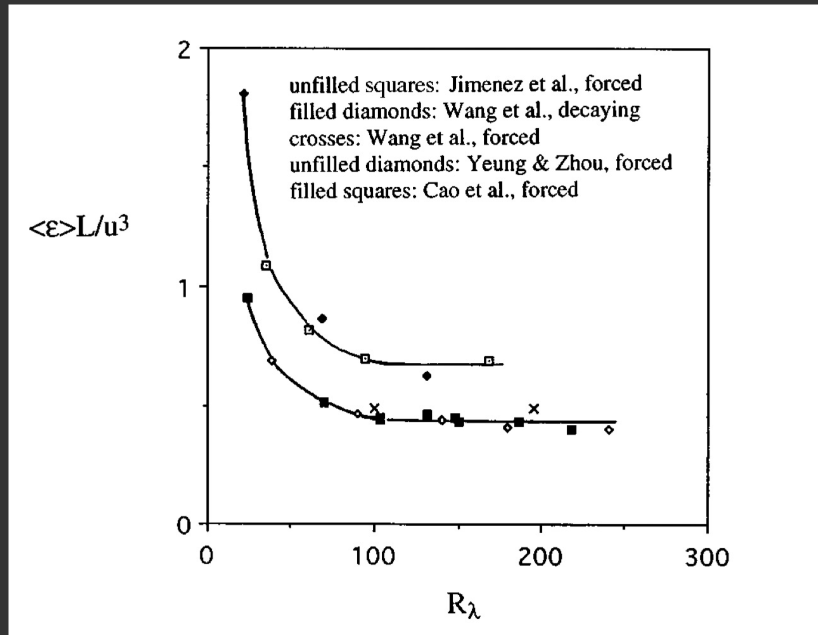
Sreenivasan, 1998



Pearson, Krogstad, de Water, 2001

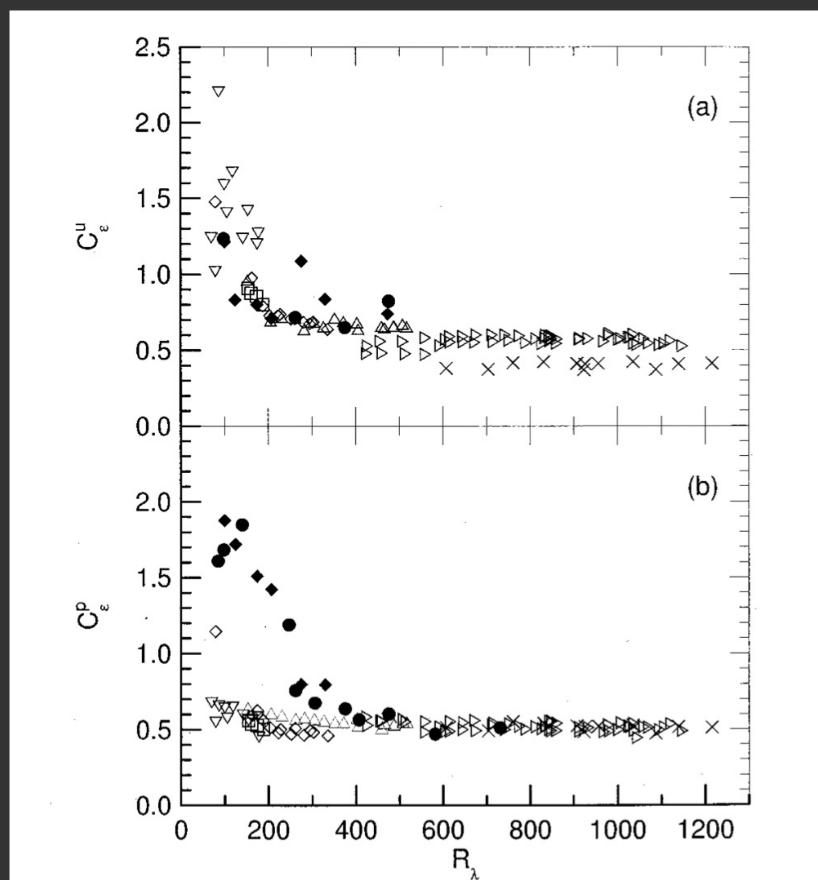


FIG. 1. Normalized dissipation rate for a number of shear flows. Details as found in this work and Refs. 14–16. (a) C_ϵ^u [Eq. (3)]; (b) C_ϵ^p [Eq. (4)]. □, circular disk, $154 \lesssim R_\lambda \lesssim 188$; ▽, pipe, $70 \lesssim R_\lambda \lesssim 178$; ◇, normal plate, $79 \lesssim R_\lambda \lesssim 335$; △, NORMAN grid, $174 \lesssim R_\lambda \lesssim 516$; × NORMAN grid (slight mean shear, $dU/dy \approx dU/dy|_{max}/2$), $607 \lesssim R_\lambda \lesssim 1217$; ▷, NORMAN grid (zero mean shear), $425 \lesssim R_\lambda \lesssim 1120$; ●, "active" grid Refs. 14, 15, $100 \lesssim R_\lambda \lesssim 731$; ◆, "active" grid, with L_u estimated by Ref. 16. For Ref. 14 data, we estimate $L_p \approx 0.1$ m and for Ref. 15 data we estimate $L_p \approx 0.225$ m.

Kolmogorov 1941 Theory

Assumptions

- $\varepsilon = \liminf_{Re \to \infty} \varepsilon^{Re} > 0$ zeroth law
- $\delta_{\ell \hat{z}} u(x) = u(x + \ell \hat{z}) - u(x)$ has same law for any $x \in \mathbb{T}^3$ and any $\hat{z} \in S^2$ for ℓ in the inertial range: $\ell_\nu \ll \ell \ll L$ homogeneity and isotropy
- there is a unique exponent $h > 0$ s.t. $\delta_{\lambda \ell \hat{z}} u$ has the same law as $\lambda^h \delta_{\ell \hat{z}} u$. self similarity
  Moreover $h = 1/3$ to be consistent with $\varepsilon > 0$.

Predictions:

- $\ell_\nu = \nu^{3/4} \varepsilon^{-1/4}$ (only length scale written as $\nu^\alpha \varepsilon^\beta$) molecular diffusion dominates
- For $p \geq 1$, define the longitudinal structure function

$$S_p^{\parallel}(\ell) = \left\langle \int_{\mathbb{T}^3} \int_{S^2} \left(\delta_\ell u(x) \cdot \hat{\ell} \right)^p d\hat{\ell}\, dx \right\rangle \sim (\varepsilon \ell)^{p/3}$$

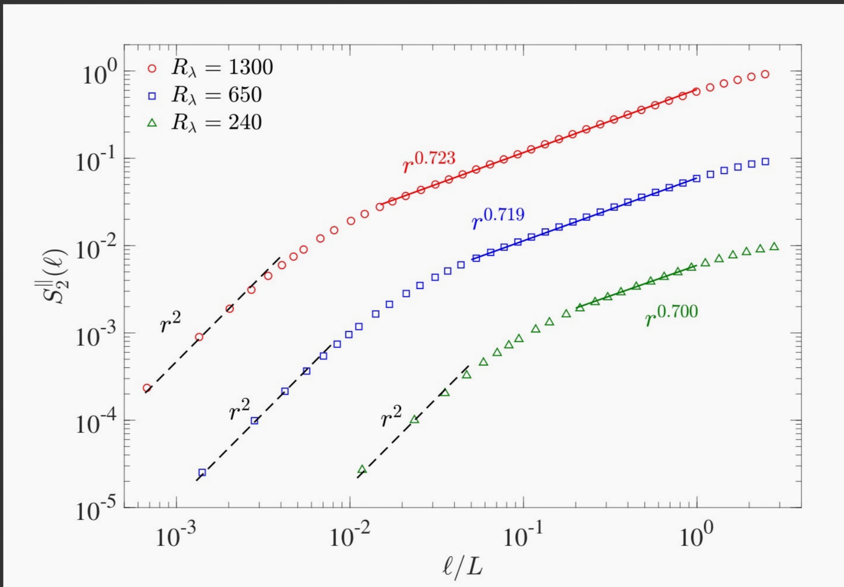
where $\langle \cdot \rangle$ is an ensemble/long-time ave.

- 4/5th law holds

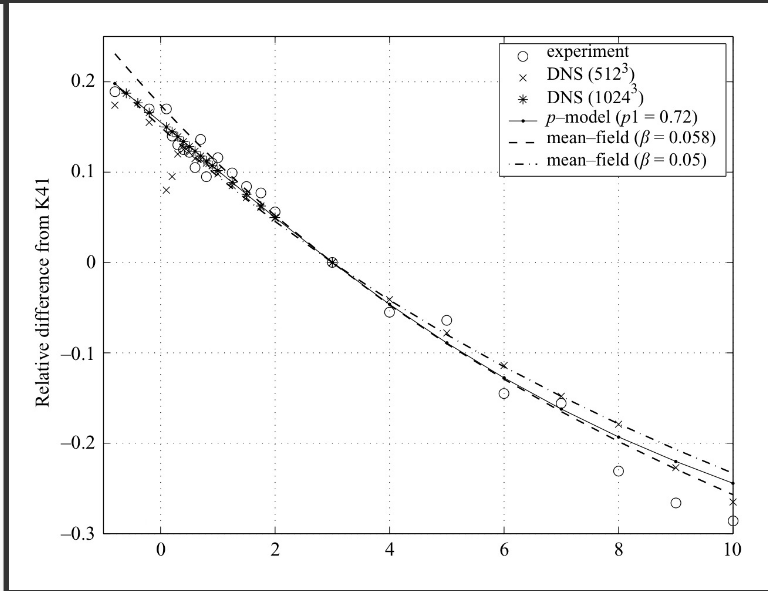
$$S_3^{\parallel}(\ell) = -\frac{4}{5}(\varepsilon \ell) \quad \text{as } Re \to \infty.$$

for ℓ in inertial range.

Numerical & Experimental Evidence



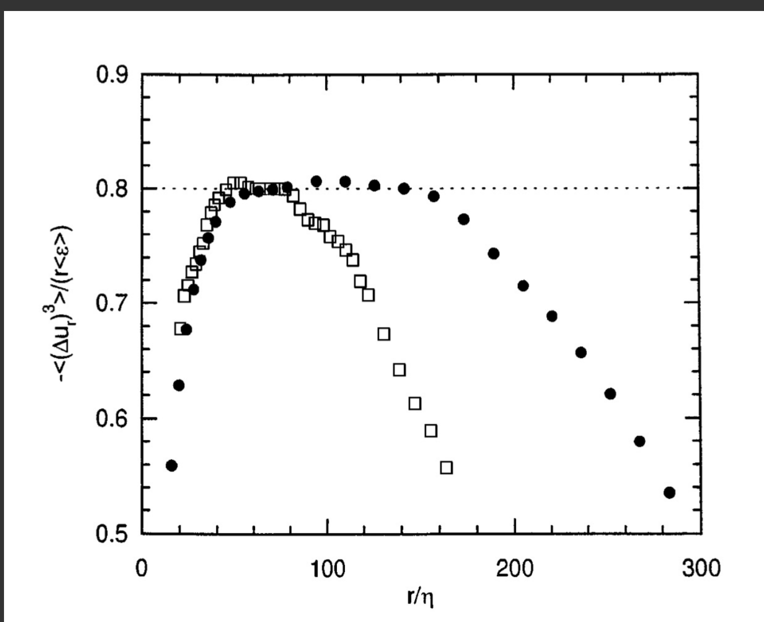
Iyer et al. 2020



Chen et al 2005

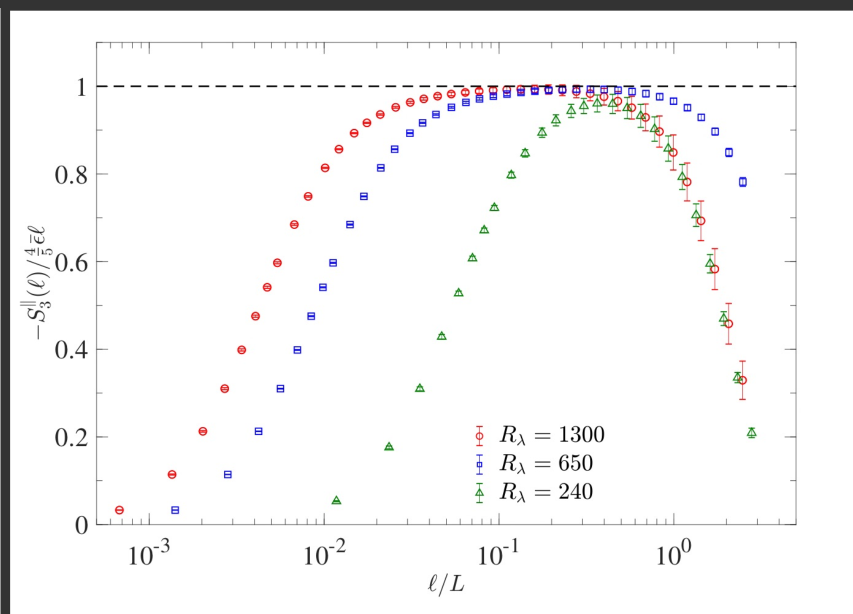
$$\zeta_p := \lim_{\ell \to 0} \lim_{\nu \to 0} \frac{\log(S_p^{\parallel}(\ell))}{\log(\varepsilon \ell)}$$

$$\left(\zeta_{|p|} - \frac{p}{3}\right)\frac{3}{p} \text{ v.s. } p$$



pipe flow Re = 230000

Sreenivasan et al. 1996



DNS Re = 1300

Iyer et al. 2020

$$S_3^{\parallel}(\ell) = -\frac{4}{5}(\varepsilon \ell)$$

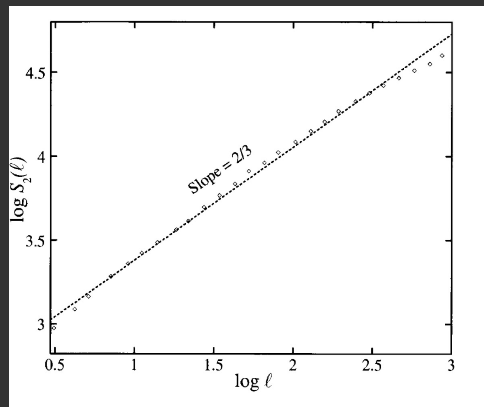
Landau's Remark and Intermittency

Famously, Kolmogorov predicted

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

$$\Longleftrightarrow$$

$$S_2(\ell) \sim (\varepsilon \ell)^{2/3}$$



Consequence of self-similar theory

$$S_p^{\parallel}(\ell) \sim (\varepsilon \ell)^{p/3}.$$

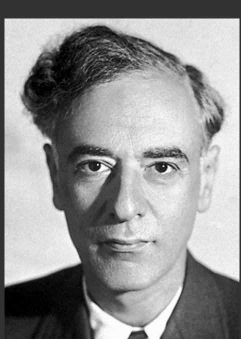
His exact 4/5th law (rigorously justified by DR 00)

$$S_3^{\parallel}(\ell) \sim (\varepsilon \ell).$$

Lets call ζ_p the scaling exponent at small scales

$$S_p^{\parallel}(r) \sim (\varepsilon \ell)^{\zeta_p}$$

Landau: 1942



the rate of energy dissipation is intermittent. I.e., it is spatially / temporally inhomogeneous. Thus ζ_p should not be a constant multiple of p.

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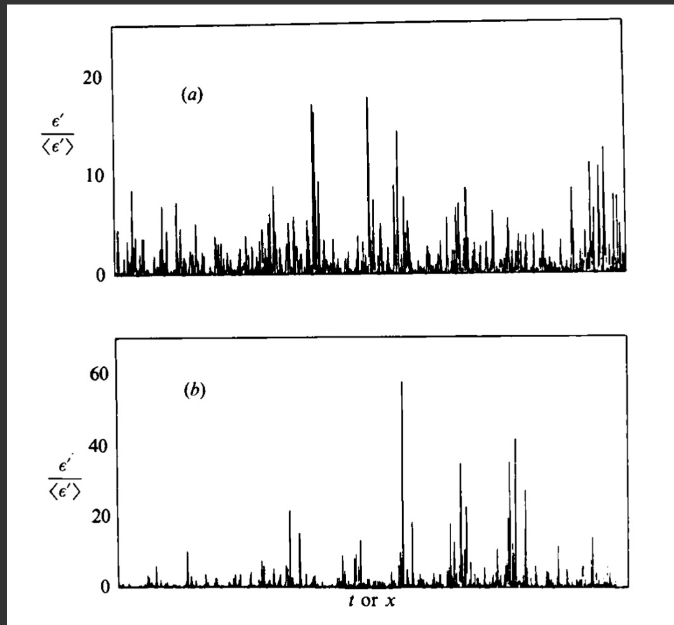
Meneveau & Sreenivasan 1991



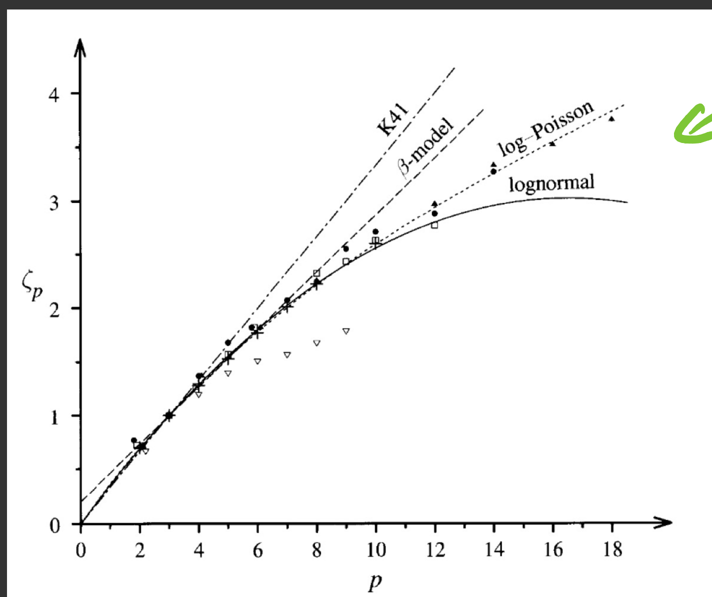
↖ laboratory boundary layer

↙ atmospheric boundary layer

surrogate:

$$\varepsilon' = \left(\frac{du_1}{dt}\right)^2$$

Frisch 1995



subject of many attempts to use renormalization group. success in Kraichnan model where small param. is either 1/d or α (Hölder index)

Falkovich, Chertkov, Kolokolov, Lebedev, Bernard, Gawedzki Kupiainen.

Models:

- log-normal : $\zeta_p = \frac{p}{3} - \frac{\mu}{18} p(p-3)$, $\mu = 0.25$ Kolmogorov 1962
- β-model : $\zeta_p = \frac{p}{3} + (3-D)\left(1-\frac{p}{3}\right)$, $D = 2.8$ Frisch et al 1978
- log-Poisson : $\zeta_p = \frac{p}{9} + 2\left(1-\left(\frac{2}{3}\right)^{p/3}\right)$ She-Leveque 1994
- mean-field : $\zeta_p = \frac{ap}{b-cp}$, $a = 0.185$, $b = 0.475$, $c = 0.0275$ Yakhot 2001

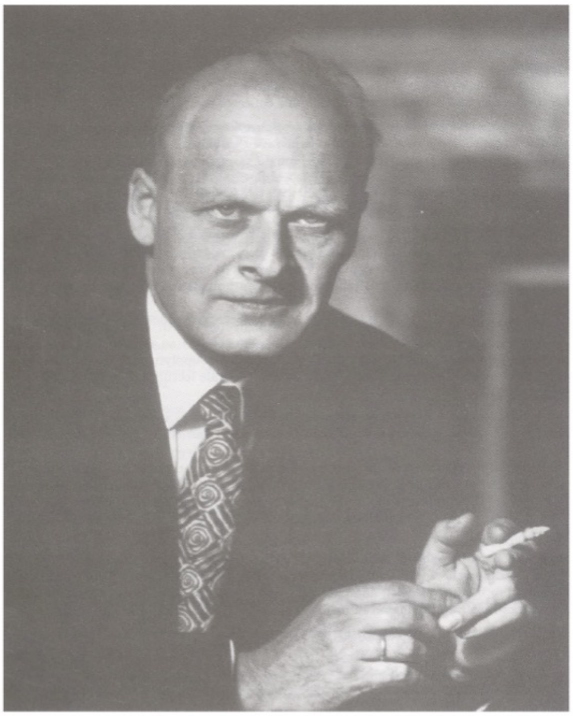
NEED: mathematical framework to impose constraints.

PUZZLE: As $\nu \to 0$ $(Re \to \infty)$

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f$$

$$\xrightarrow{\nu \to 0} \quad \partial_t u + u \cdot \nabla u = -\nabla p + f$$

which conserves energy. How can $\varepsilon > 0$?



Lars Onsager (1903-1976)

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

$$|\mathbf{v}(\mathbf{r}'+\mathbf{r}) - \mathbf{v}(\mathbf{r}')| < (\text{const.}) r^n \tag{26}$$

for any order n greater than $1/3$; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description...

"Statistical Hydrodynamics" (1949)

Generalized description: weak solutions

Idea: replace PDEs with integrated balances:

cons momentum: $$\int_{\mathbb{T}^d}\int_{\mathbb{R}} \left(u \cdot \partial_t \phi + (u \otimes u) : \nabla \phi\right) dt\, dx = 0 \qquad \forall \phi \in C_0^\infty,\ \nabla \cdot \phi = 0$$

cons mass: $$\int_{\mathbb{T}^d}\int_{\mathbb{R}} \left(u \cdot \nabla \psi\right) dt\, dx = 0 \qquad \forall \psi \in C_0^\infty$$

NEED ONLY: $u \in L^2_{t,x}$

Onsager's Conjecture:

Weak: If u is a weak solution of Euler in the class $C_t C_x^{1/3+}$, then u conserves energy

Strong: there exists an Euler solution with $u \in C^{1/3-}$ such that energy is dissipated

Strongest: Euler solutions as in (Strong) should arise as vanishing viscosity limits of Navier-Stokes, ultimate goal: prove

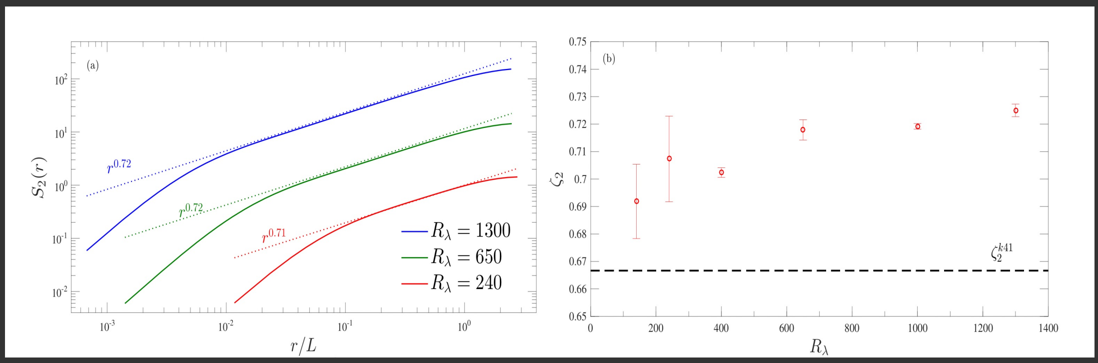
$$\nu \int_0^T \int |\nabla u|^2 \, dx \, dt \geq \varepsilon_* > 0 \quad \forall \, \nu > 0$$

Foundations: why weak solutions? Constantin-Vicol (2018), Drivas-Nguyen (2019)

Theorem: If $S_2(\ell) \leq C|\ell|^s$, $s \in (0,2)$ for $\forall \, |\ell| \geq \nu^{\frac{1}{2-s}}$ Then every weak limit $u^\nu \to u$ in $L^2_{t,x}$ is a weak soln.

DNS data

Iyer et al 2020



Weak: energy conservation for $C^{1/3+}$

Recall, for any C^1 smooth incompressible v

$$\int_{\mathbb{T}^3} v\otimes v : \nabla v \, dx = \int_{\mathbb{T}^3} v_i v_j \partial_i v^j \, dx$$

$$= \int_{\mathbb{T}^3} v_i \, \partial_i \frac{|v|^2}{2} \, dx$$

$$= -\int_{\mathbb{T}^3} \partial_i v_i \, \frac{|v|^2}{2} \, dx = 0.$$

Thus, if $u \in C^0_t C^1_x$ is a strong solution of Euler, then kinetic energy is conserved:

$$\frac{d}{dt} \int \frac{|u|^2}{2} dx = \int (u\otimes u) : \nabla u \, dx = 0.$$

What about weak solutions? Formally

$$\int_{\mathbb{T}^3} v\otimes v : \nabla v \, dx \approx \int (\nabla^{1/3} v)^3 \, dx$$

IBP justified if u is "1/3-differentiable".

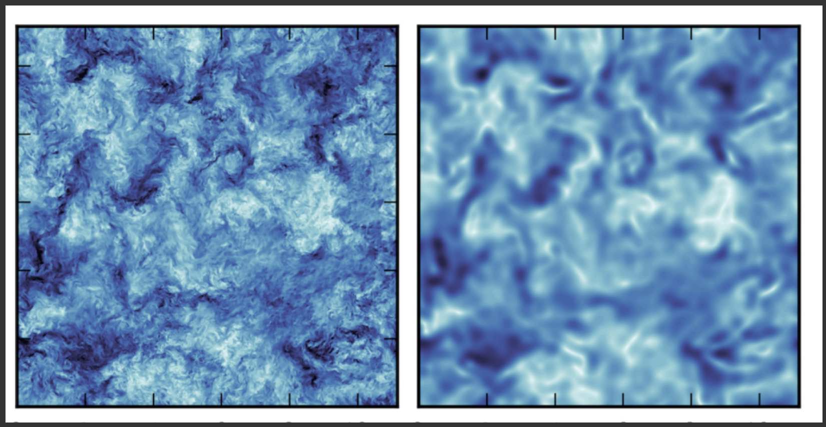
Energy transfer through scale

Coarse-graining

$$\bar{u}_\ell(x) = \int_{\mathbb{T}^d} G_\ell(r)\, u(x+r)\, dr, \quad G_\ell(r) = \ell^{-d} G\left(\frac{r}{\ell}\right)$$

or

$$u_k(x) = \mathbb{P}_{\le k}[u] \qquad \text{projection onto low freq.}$$



If $u \in L^\infty_t L^2_x$, then

$$E(t) = \lim_{\ell \to 0} E_\ell(t) = \lim_{\ell \to 0} \int_{\mathbb{T}^d} |\bar{u}_\ell|^2\, dx$$

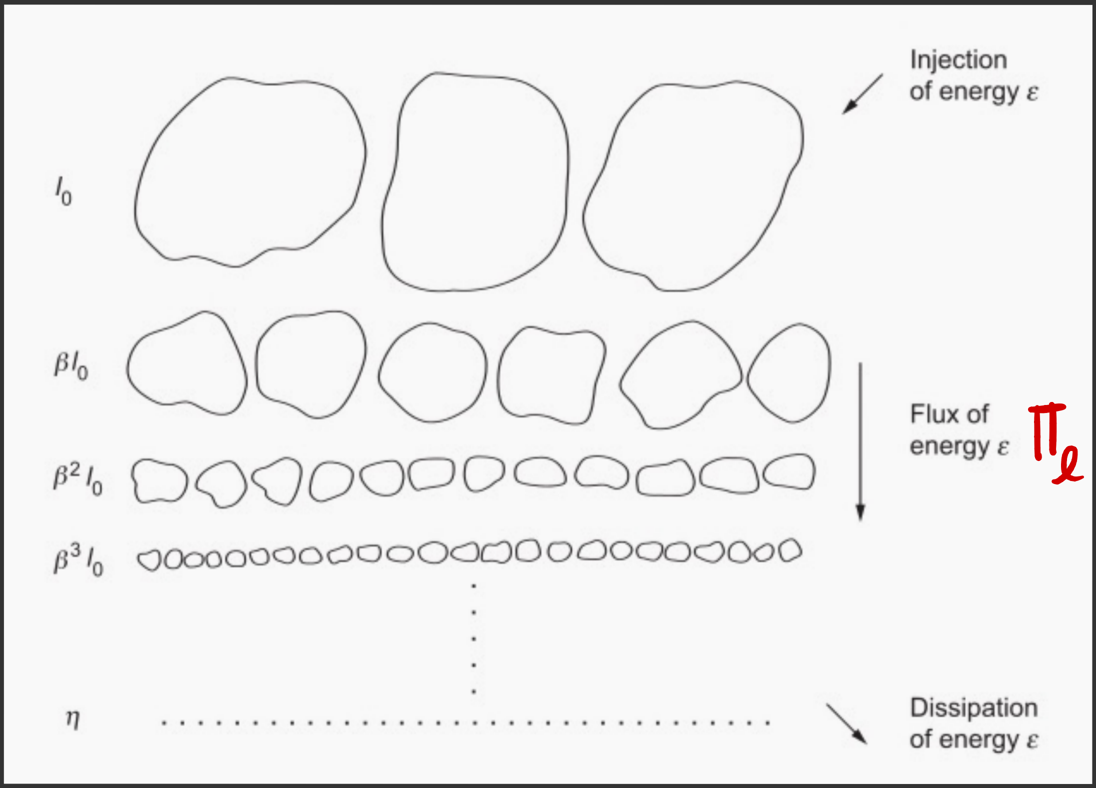
$$= \lim_{k \to \infty} E_k(t) = \lim_{k \to \infty} \int_{\mathbb{T}^d} |u_k|^2\, dx.$$

Dynamics of large-scale energy

$$\partial_t \bar{u}_\ell + \nabla\cdot(\bar{u}_\ell \otimes \bar{u}_\ell) = -\nabla \bar{p}_\ell - \nabla\cdot\tau_\ell(u,u)$$

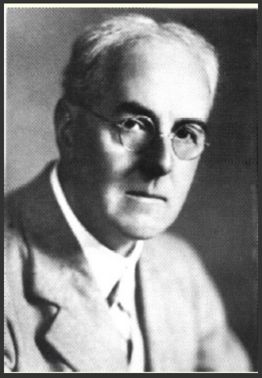
$$\nabla\cdot\tau_\ell(u,u) := \overline{(u\otimes u)}_\ell - \bar{u}_\ell\otimes\bar{u}_\ell$$ closure problem!

$$\boxed{\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int|\bar{u}_\ell(x,t)|^2dx = -\int \Pi_\ell[u]\,dx.\\ &\Pi_\ell[u] = -\nabla\bar{u}_\ell : \tau_\ell(u,u).\end{aligned}}$$



Big whirls have little whirls, That feed on their velocity; And little whirls have lesser whirls, And so on to viscosity

L. F. Richardson



We just proved: energy is constant on $[0,T]$ iff

$$\lim_{\ell\to 0}\int_0^T\int_{\mathbb{T}^d}\Pi_\ell[u]\,dx\,dt.$$

Thus, for a weak Euler solution arising as a zero-viscosity limit (Duchon-Robert 2000)

In spirit of RG invariance.

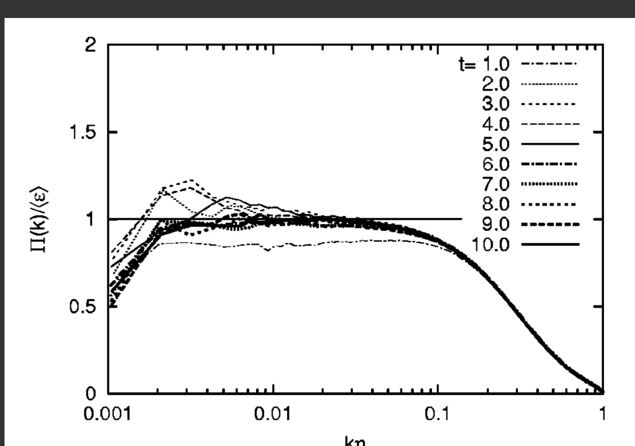
$$\boxed{\lim_{\ell\to 0}\Pi_\ell[u] = \lim_{\nu\to 0}\varepsilon^{\nu}[u^{\nu}]}$$

nonlinear flux
turbulent cascade

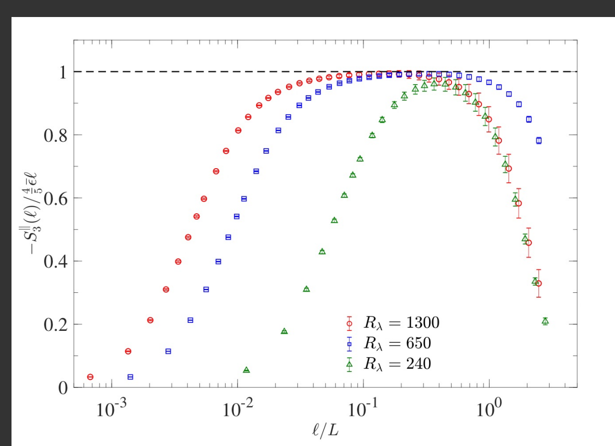
viscous dissipation

Rigorous form of Kolmogorov 4/5th law.

$$\frac{S_3^{\|}(\ell)}{|\ell|} := \frac{1}{|\ell|}\int_{S^{d-1}}(\delta_\ell u\cdot\hat{\ell})^3 \xrightarrow{\ell\to 0} -\frac{4}{5}\Pi[u]$$



Kaneda et al 2003



Iyer et al, 2020.

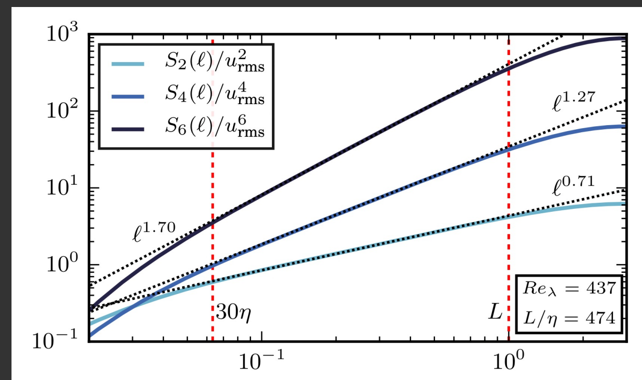
Whenever we show $\Pi_\ell[u] \to 0$, energy is conserved.

How should we measure regularity? Besov spaces

$u \in B_p^{\sigma,\infty}$ for $p \geq 1$

iff $\begin{cases} u \in L^p \\ \|\delta_\ell u\|_{L^p} \lesssim |\ell|^\sigma \quad \forall |\ell| < 1 \end{cases}$

Note: $S_p(\ell) = \|\delta_\ell u\|_{L^p}^p$.



THEOREM: (Eyink 1992, Constantin-E-Titi 1994)

Let $u \in L^3(0,T; B_3^{\sigma,\infty}(\mathbb{T}^d)) \cap C(0,T; L^2(\mathbb{T}^d))$ with $\sigma > 1/3$, then

$$\frac{1}{2}\int_{\mathbb{T}^d} |u(x,t)|^2 \, dx = \frac{1}{2}\int_{\mathbb{T}^d} |u_0(x)|^2 \, dx \qquad \forall\, t \in [0,T].$$

Idea: Follows from commutator identities

$$\tau_\ell(u,u) = \int G_\ell(r)\, \delta_r u \otimes \delta_\ell u - \int G_\ell(r)\, \delta_r u \otimes \int G_\ell(r)\, \delta_r u$$

$$\nabla \bar{u}_\ell = -\frac{1}{\ell}\int (\nabla G)_\ell(r)\, \delta_r u$$

$$\Pi_\ell[u] = \nabla \bar{u}_\ell : \tau_\ell(u,u).$$

$$\Rightarrow \quad \|\Pi_\ell\|_{L^1_{x,t}} \lesssim \ell^{3\sigma - 1}\int_0^T |u|^3_{B_p^{\sigma,\infty}}$$

This proof can be connected to NS and made quantitative:

THEOREM: (Drivas - Eyink 2018) if $\{u^\nu\}_{\nu>0}$ and $\sigma \in (0,1]$,

$$\sup_{\nu>0} \|u^\nu\|_{L^3_t B^{\sigma,\infty}_3} < \infty \implies \int_0^T \int_{\mathbb{T}^d} \nu |\nabla u^\nu|^2 \, dx\, dt \lesssim \nu^{\frac{3\sigma-1}{\sigma+1}}.$$

Thus, if dissipation decays slowly, there can be no uniform boundedness of Navier-Stokes in $B^{\sigma,\infty}_p$.
Experimental evidence robustly points to Euler singularity.

MATHEMATICAL GOAL

dissipative anomaly (1) There is data $u_0 \in L^2$, $T>0$ and $\varepsilon>0$ s.t.

$$\int_0^T\int_{\mathbb{T}^3} \nu |\nabla u|^2 \, dx\, dt + \int_0^T \int_{\mathbb{T}^3} D[u] \, dx\, dt > \varepsilon$$

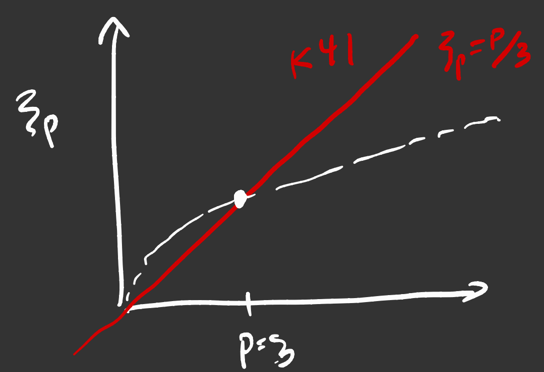
weak compactness (2) The family $\{\bar{u}^\nu\}_{\nu>0}$ is compact and along subsequences converge to weak Euler

Onsager Conjecture (3) These Euler solutions exhibit constant mean flux $\langle \Pi[\bar{u}] \rangle = \varepsilon$ and live in $L^3_t B^{1/3,\infty}_3$

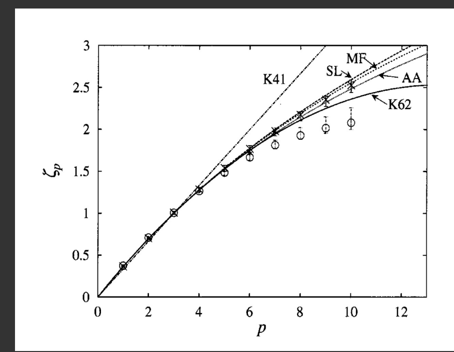
Moreover, this behavior should be generic (statistically stationary regime)

Some Consequences of Onsager Theory

- 4/5th law is rigorously justified if $\{u^\nu\}$ compact in L^3
- Intermittency constrained:



- $\zeta_p \le p/3$
- ζ_p concave in $p \in [0,\infty)$



Multifractal formalism: (Parisi – Frisch)

Assume local Hölder exponents $h(x) \in [h_{min}, h_{max}]$.
$h_{min} > 0$, $h_{max} < 1$

Set $S(h) = \{ x : h(x) = h \}$
$D(h) = \dim(S(h))$

By definition, for x within distance r of $S(h)$,

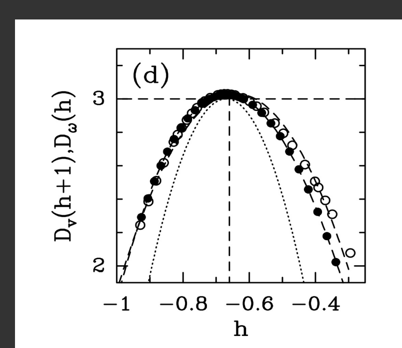
$$|u(x+r) - u(x)| \sim r^h$$

$$\text{Fraction}(x : \text{dist}(x, S(h)) \sim r) \sim r^{K(h)}, \quad K(h) = d - D(h)$$

$$S_p(\ell) = \langle |\delta_\ell u|^p \rangle \sim \int_{h_{min}}^{h_{max}} d\mu(h)\, \ell^{hp + K(h)} \sim \ell^{\zeta_p}$$

where

$$\zeta_p = \lim_{\ell \to 0} \frac{\ln(S_p(\ell))}{\ln(\ell)} = \inf_h \{ ph + K(h) \}$$



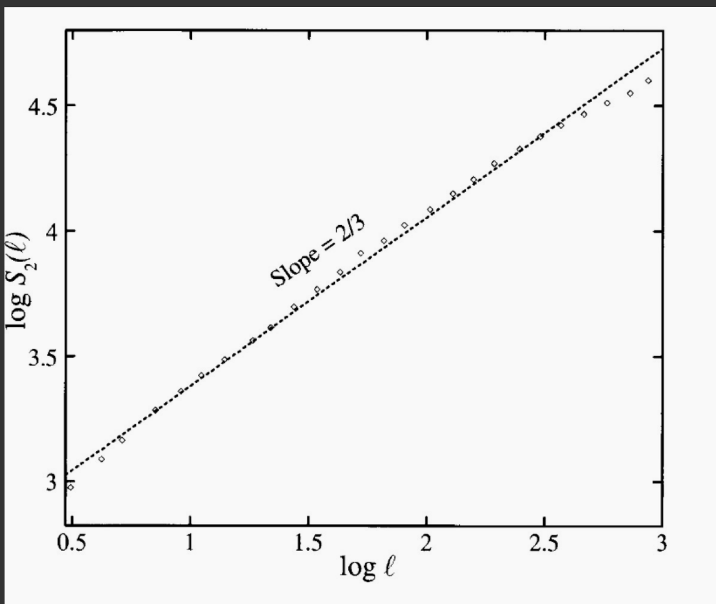
Legendre transform

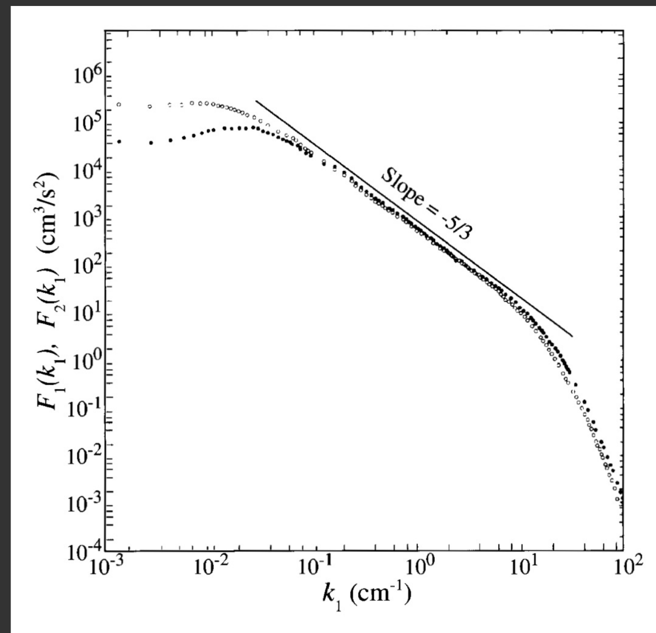
$$D(h) = \inf_p \{ ph + (d - \zeta_p) \}.$$

Used to show "most probable" exponent $h_0 \approx 0.34$ (Kestener et al 2004)

Connection with Kolmogorov spectra

2/3rds law $\Longleftrightarrow$ $-5/3$ law





$$S_2(\ell) = \int_{S^{d-1}}\int_{\mathbb{T}^d} |u(x+\ell\xi) - u(x)|^2 \, dx \, d\omega(\xi)$$

$$E(k) = \sum_{p \in \mathbb{Z}^d} \delta(k - |p|) \, |\hat{u}(k)|^2$$

By Wiener-Khinchin theorem

$$\boxed{S_2(\ell) \sim \ell^{2s} \Longleftrightarrow E(k) \sim k^{-(2s+1)}}$$

$s = 1/3$ $\quad E(k) \sim \varepsilon^{2/3} k^{-5/3} \Longleftrightarrow S_2(\ell) \sim (\varepsilon \ell)^{2/3}$

Recall 1/3 is the Onsager exponent, which is maximal regularity consistent with anomalous dissipation

In K41, all L^p have 1/3 derivative, so these are connected!

- <u>Locality of cascade:</u> one can show that

energy flux $\Pi_\ell[u]$

has contributions primarily from a band of scales $[\ell-\delta, \ell+\Delta]$ using Littlewood-Paley

Eyink, 2005

Constantin - Cheskidov - Friedlander - Shvydkoy, 2008

- <u>Link between</u> Lagrangian reversibility:

THEOREM: (Drivas, 2019) Let $u \in L^3(0,T;L^3)$ be a weak Euler solution. Then

$$\Pi[u] = \lim_{\substack{\tau\to 0\\ \ell\to 0\\ R\to 0}} \left[\frac{\langle |\delta X^\ell_{t,t-\tau}(r;x)-r|^2\rangle_{R,x,t} - \langle |\delta X^\ell_{t,t+\tau}(r;x)-r|^2\rangle_{R,x,t}}{4\tau^3} \right]$$

where $\dot{X}^\ell_{t,s} = \bar{u}_\ell(X^\ell_{t,s}, s)$.

Note: $\Pi[u] = \varepsilon[u] > 0$ in 3d

$\Pi[u] = -I[u] < 0$ in 2d, small-scale forced

Uses rigorous version of Ott-Mann-Gawedzki relation, noted by Jucha et al (2014) to link with irreversibility.

Flexible Side of Onsager's Conjecture

THEOREM (Isett, 2018, Buckmaster - De Lellis - Szekelyhidi - Vicol 19)

Let $e : [0,T] \to \mathbb{R}$ be a strictly positive smooth function. For any $\alpha \in (0, 1/3)$, there exists a weak solution $u \in C^\alpha([0,T] \times \mathbb{T}^3)$ of the Euler equations with

$$\int \frac{1}{2} |u(x,t)|^2 \, dx = e(t) \quad \forall \; t \in [0,T]$$

Long history: Scheffer 1993, Shnirelman 1997, 2000, De Lellis - Szekelyhidi 2009-2011, 2012, Buckmaster - De Lellis - Szekelyhidi 2013, 2014

Built off ideas of Nash-Kuiper Theorem and Gromov's h-principle. (Buckmaster - Vicol 2020, De Lellis - Szekelyhidi, 2011 great reviews!)

Ideas: Inverse Renormalization group (Frisch)

- Stages $S_0, S_1, \ldots, S_q, \ldots$ adding ever smaller motions ($2\pi \to 2\pi/2 \to \ldots \to 2\pi/2^q \to \ldots$)

$$\partial_t \bar{u}_{\ell_q} + \nabla \cdot (\bar{u}_{\ell_q} \otimes \bar{u}_{\ell_q}) = -\nabla \bar{P}_{\ell_q} + \nabla \cdot \tau_{\ell_q}$$

$$\tau_\ell = \overline{(u \otimes u)}_\ell - \bar{u}_\ell \otimes \bar{u}_\ell \geq 0$$

where $\bar{u}_\ell = \sum_{|k| < \ell^{-1}} e^{ik\cdot x} \hat{u}(t)$ and $\ell := \lambda^{-q}$

Must show $\tau_{\ell_q} \to 0$ as $q \to \infty$.

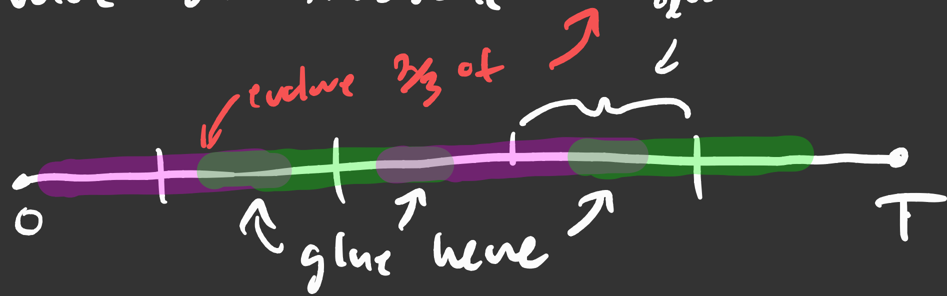
STEP 1: (Coarse-graining) Take output of stage S_{q-1} and apply $\overline{(\cdot)}_{\ell_q}$. Filters out scales $< \ell_q$.

STEP 2: (Euler dynamics)

- from previous stage, have approx solution to Euler which is filtered in STEP 1 to kill all frequencies $> 2^q$.

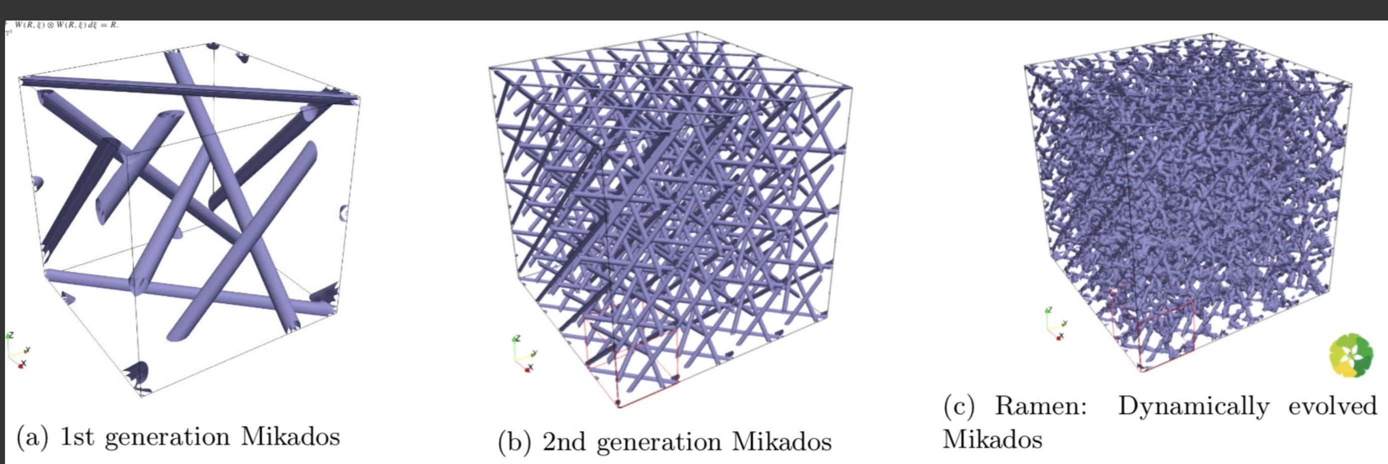
GOAL: Improve error from being soln by letting it evolve in time & develop smaller scales

- evolve on timescale $\ell/\delta_{q,n}$ where $\ell = 2^{-q}$, $\delta_\ell \sim \ell^h$.



STEP 3: (Small-scale perturbation)

To compensate the "stress" from previous stage by adding small scale perturbations at S_q at scale 2^{-q}. Amplitudes are used to reduce stress. Analogous to Nash's isometric embedding const.



(a) 1st generation Mikados (b) 2nd generation Mikados (c) Ramen: Dynamically evolved Mikados

Remarks:

subsolution

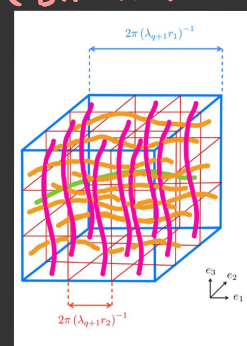
$\downarrow$

weak solution

high freq oscillations introduce stresses in passing to weak limits

- Because the equation for subsolutions is highly underconstrained, it is easy to construct them.
- An iteration process reintroduces high-wavenumber oscillations, perturbations designed to cancel low frequencies of old stress.
- Difficulty lies in controlling error terms. Accomplished by judicious choice of building blocks.

(B.M.N.V 21')



- oscillations introduced in highly non-unique way $\Rightarrow$ infinitely many solutions.

- Solutions are "monofractal" in that the velocity has just one exponent h, which can be $1/3^-$. They have "Kolmogorov-like" spectra (not quite).

- Notable recent exception: Buckmaster-Masmoudi-Novak-Vicol, 21 Solutions with $> 1/3$ derivative in L^2 but $< 1/3$ derivative in L^3 constructed. Towards more realistic flows!

- High degree of non-uniqueness! Dissipative Euler solutions do not provide a predictive theory alone. Must consider viscosity!

Examples in related models

1. Burgers equation: $u : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$

$$\partial_t u + u\,\partial_x u = \nu\,\partial_x^2 u \qquad \tfrac{1}{2}\partial_x u^2$$

Remarks:
- for $\nu > 0$, model is globally wellposed
- for $\nu = 0$, model shocks in finite time.

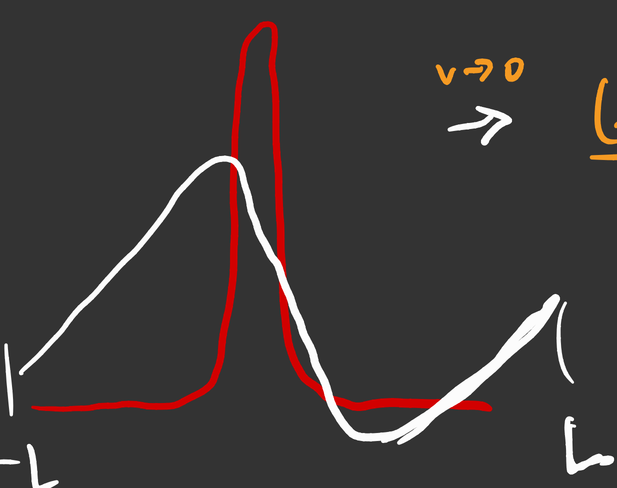
What happens to the dissipation after shock forms?

Exact 1d solution: Khokhlov sawtooth

$$\overset{\nu}{u}(x,t) = \frac{1}{t}\left[x - L\tanh\left(\frac{Lx}{2\nu t}\right)\right]$$

$$\xrightarrow{\nu \to 0} \begin{cases} \dfrac{x+L}{t} & -L \le x \le 0 \\ \dfrac{x-L}{t} & 0 < x \le L \end{cases}$$

$$\varepsilon^\nu(x,t) = \nu|\partial_x u^\nu|^2 \simeq \frac{L^4}{4\nu t^4}\,\mathrm{sech}^2\left(\frac{L\nu}{2\nu t}\right)$$



$$\xrightarrow{\nu \to 0} \frac{(\Delta u)^3}{12}\,\delta(x), \qquad \Delta u = u^-(0) - u^+(0) = \frac{2L}{t}$$

This behavior is general.
- convergence is toward global entropy weak solution
- such solns. necessarily dissipate.

Moreover shock solutions (entropy weak solns) with countably many shocks live in:

$$u \in L^\infty_t (L^\infty_x \cap BV_x)$$

Since $$L^\infty \cap BV \subseteq B^{1/p,\infty}_p, \qquad p \geq 1.$$

shocks live at the Onsager-critical threshold

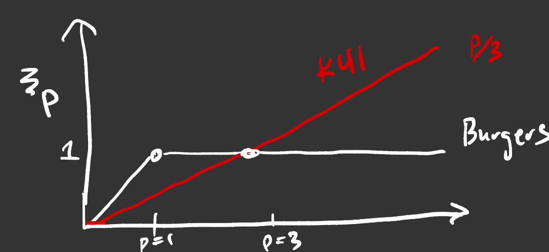
They are also <u>intermittent</u>.

$$u = \begin{cases} \frac{x+L}{t} & -L \leq x \leq 0 \\ \frac{x-L}{t} & 0 < x \leq L \end{cases}$$

$$\langle |\delta_\ell u|^p \rangle = \frac{1}{2L} \int_{-L}^{L} |u(x+\ell) - u(x)|^p \, dx$$

$$= \left(1 - \frac{\ell}{2L}\right) \left(\frac{\ell}{t}\right)^p + \frac{\ell}{2L} \left(\frac{2L + \ell}{t}\right)^p$$

$$\sim (\Delta u)^p \begin{cases} (\ell/L)^p & 0 < p < 1 \\ \frac{\ell}{2L} & p \geq 1 \end{cases}$$



$$\partial_t u + u \partial_x u = \nu \partial_x^2 u + \sum_k \sigma(x) \, dW^k_t$$

<u>Mathematical Dream</u>: E-Khanin-Mazel-Sinai (1997, 2000)

Unique invariant measure μ_0 supported on entropic shocks, realized $\mu_\nu \to \mu_0$. displays AD & Intermittency.

Review: Bec-Khanin "Burgers Turbulence" 2007

Passive Scalars: Let $\theta \in \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{R}$ satisfy

$$\partial_t \theta^\kappa + u \cdot \nabla \theta^\kappa = \kappa \Delta \theta^\kappa$$
$$\nabla \cdot u = 0$$

$$\theta^\kappa\big|_{t=0} = \theta_0^\kappa, \quad \fint \theta_0^\kappa \, dx = 0$$

Here, θ^κ represents temperature or dye being stirred by the velocity u. Scalar 'energy' is dissipated:

$$\frac{1}{2}\frac{d}{dt}\int |\theta^\kappa|^2 dx = -\kappa \int |\nabla \theta^\kappa|^2 dx$$

Even though the the velocity field does not feature in this balance, it is crucially important to the process

velocity acts to filament the scalar, causing $\nabla\theta^\kappa$ to grow and contribute more to dissipation.

$$\boxed{\text{Anomalous Dissipation:} \quad \kappa \int_0^T \int |\nabla \theta^\kappa|^2 dx\, dt \geq \chi > 0}$$

Donzis-Sreeni (2005)

Obukhov (1949) & Corrsin (1951) Theory


'turbulent' velocity "$u \in C^\alpha$" $\alpha \in (0,1]$
gives rise to "$\theta \in C^\beta$" with $\beta = \frac{1-\alpha}{2}$.

THEOREM: Suppose $u \in L^1(0,T; C^\alpha)$, $\alpha \in (0,1]$, incompressible.
Suppose $\{\theta^\kappa\}_{\kappa>0}$ is uniformly bdd in $L^\infty(0,T; C^\beta)$, then

$$\kappa \int_0^T \int |\nabla \theta^\kappa|^2 \, dx \, dt \lesssim C \, \kappa^{\frac{\alpha+2\beta-1}{1+\alpha}}.$$

In particular, if $\beta > \frac{1-\alpha}{2}$ there is no anomalous diss.

This can be refined under stronger assumptions

Suppose $u \in L^1_{loc}(0,T; W^{1,\infty}) \cap L^1([0,T]; C^\alpha)$, then

$$\lim_{\kappa \to 0} \int_0^T \int |\nabla \theta^\kappa|^2 \, dx \, dt \to 0$$

if $\beta = \frac{1-\alpha}{2}$.

No (deterministic) rigorous examples of anomaly!

Kraichnan model: Bernard, Gawedzki, Kupiainen, Falkovich, Lebedev ...

Stochastic fluids: Bedrossian - Blumenthal - Punshon-Smith (2018-)

(Drivas, Elgindi, Iyer, Jeong, 2020)

THEOREM (Anomalous Diss): Fix $T>0$, $d\geq 2$, $\alpha\in[0,1)$ & $\theta_0\in H^2$.
There exist a divergence-free velocity field

$$u\in C^\infty([0,T)\times\mathbb{T}^d)\cap L^1([0,T];C^\alpha)$$

such that we have

$$\kappa\int_0^T\int|\nabla\theta^\kappa|^2\,dx\,dt\geq\chi>0$$

where $\chi:=\chi(\theta_0,\alpha)$.

<u>Remarks</u>:

- In our construction, θ^κ retains no Hölder regularity.
  Thus, in the endpoint case $\beta=0$, $\alpha<1$, this demonstrates the sharpness of the <u>Obukhov-Corrsin theory</u>

- gives an example of non-uniqueness for weak solutions of the transport equation.
  simple consequence of time irreversibility.

Proof of the Theorem

balanced growth implies anomalous dissipation.

LEMMA: If $u \in L^\infty_{loc}(0,T; W^{1,\infty})$, $\theta_0 \in H^2$ and

$$\lim_{t \to T} \int_0^t \int |\nabla\theta|^2 \, dx\, dt = +\infty \quad \& \quad |\nabla\theta|^2_{L^2} \geq c\left(|D^2\theta|_{L^2} + |D^2\theta^\kappa|_{L^2}\right), \quad c \in (0,1)$$

then

$$\kappa \int_0^T \int |\nabla\theta^\kappa|^2 \, dx\, dt \geq \left(\frac{c}{2}\right)^4 |\theta_0|^2_{L^2} = \chi.$$

Proof: Take $|\theta_0|_{L^2} = 1$, $T = 1$.

$$\frac{1}{2}\frac{d}{dt}|\theta^\kappa - \theta|^2_{L^2} = \kappa \int \Delta\theta^\kappa (\theta^\kappa - \theta) \implies |\theta^\kappa - \theta|^2_{L^2} \leq \overbrace{\sqrt{\kappa \int_0^1 |\theta^\kappa|^2_{\dot H^1}}}^{\delta_\kappa} \sqrt{\kappa \int_0^1 |\theta|^2_{\dot H^1}}$$

For contradiction, assume $\exists\ \kappa_n \to 0$ s.t. $\delta_{\kappa_n} < \chi$. Let $T_\kappa < 1$

$$\boxed{\kappa \int_0^{T_\kappa} |\theta|^2_{\dot H^1}\, dt = 1.}$$

Note $T_\kappa \to 1$ as $\kappa \to \infty$.

Interpolation:

$$|\theta - \theta^\kappa|^2_{\dot H^1} \leq |\theta - \theta^\kappa|_{L^2} |\theta - \theta^\kappa|_{H^2} \leq \frac{\sqrt{2}\,\chi^{1/4}}{c} |\theta|^2_{\dot H^1}$$

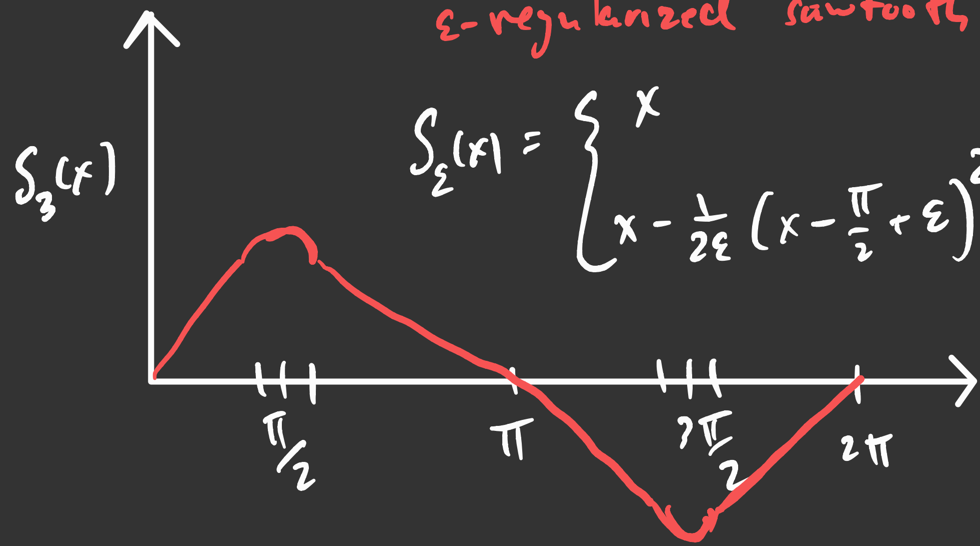
On other hand

$$|\theta^\kappa|_{\dot H^1} \geq \left| |\theta|_{\dot H^1} - |\theta^\kappa - \theta|_{\dot H^1} \right| > \boxed{\left(1 - \frac{2^{1/4}\chi^{1/8}}{\sqrt{c}}\right)|\theta|_{\dot H^1}}$$

The boxed terms give lower bound on dissipation contradicting assumption.

With this in hand, we proceed with the construction. Our building block is

ε-regularized sawtooth



$$S_\varepsilon(x) = \begin{cases} x & x \in [0, \frac{\pi}{2} - \varepsilon] \\ x - \frac{1}{2\varepsilon}\left(x - \frac{\pi}{2} + \varepsilon\right)^2 & x \in [\frac{\pi}{2} - \varepsilon, \pi/2] \end{cases}$$

even w.r.t. $\pi/2$

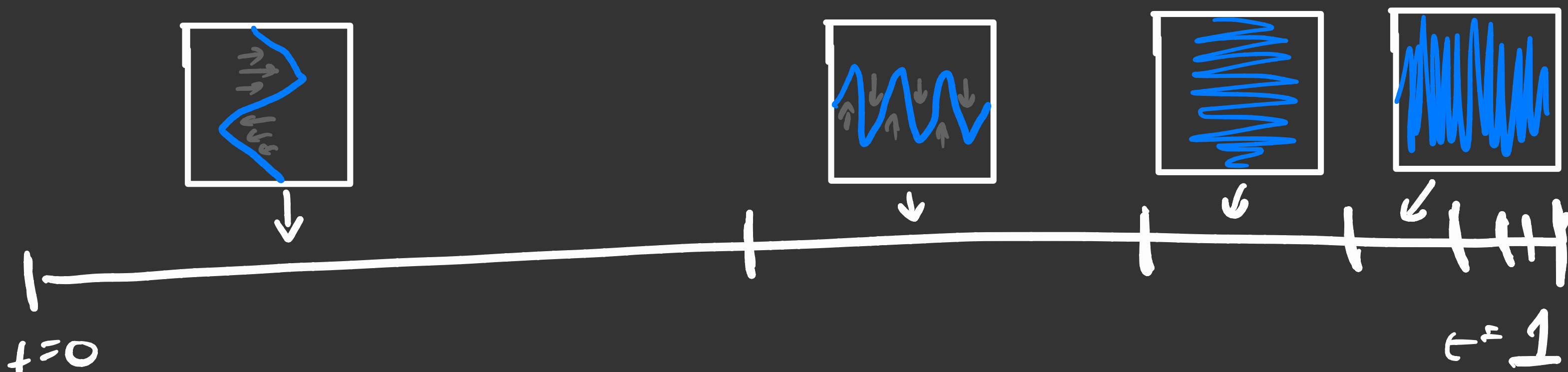
odd w.r.t. 0

Enjoys $|S_\varepsilon'|_{L^\infty} \leq 1$ but $|S_\varepsilon''|_{L^\infty} \sim 1/\varepsilon$.

Out of it, we make shear flows

$$u^N(x,y) = \begin{cases} \pm \begin{pmatrix} S_\varepsilon(Ny) \\ 0 \end{pmatrix} & \text{horizontal shear} \\ \pm \begin{pmatrix} 0 \\ S_\varepsilon(Nx) \end{pmatrix} & \text{vertical shear} \end{cases}$$

Fix dyadic sequences $t_j = 2^{-j}$, $N_j = 2^{(1+\alpha)j}$, $\varepsilon_j = C(\alpha) 2^{-2j}$.



Transport by shearing is solvable:

$$\partial_t \theta + S(Ny)\,\partial_x \theta = 0, \qquad \theta|_{t=0} = \theta_0$$

$$\Rightarrow \quad \theta(t) = \theta_0(x - t\,S(Ny), y).$$

Thus, at $t_j = 2^{-j}$

$$\theta_j(x,y) = \theta_{j-1}(x - t_j S(Ny), y).$$

One can then show

- $|\theta_j|_{\dot H^1} \geq C_\alpha 2^{\frac{\alpha(j+1)j}{2}} |\theta_0|_{\dot H^1}$
- $|\theta_j|_{W^{1,\infty}} \leq C_\alpha |\theta_j|_{\dot H^1}$
- $|\theta|_{\dot H^2}, |\theta|_{\dot H^2} \leq C_\alpha |\theta_j|^2_{\dot H^1}$

Direction of velocity field depends on θ_0

Lemma applies. Note $|\nabla u|_{L^\infty} \sim \frac{1}{(1-t)^2} \sim |\theta|^2_{\dot H^1} \sim |\theta|_{\dot H^2}$.

Moreover, for any $s \in (0,1)$:

$$|u|_{L^1(0,T;C^s)} \lesssim \sum_j t_j N_j^s \leq \sum 2^{(1+\alpha)sj - j} < \infty$$

once we choose

$$0 < \alpha < \frac{1}{s} - 1.$$

Some open issues

① Construction of u that works for <u>all</u> data.
Randomize?

② Sharpness of Obukhov-Corrsin theory

QUESTION: Does there exist div-free

$$u \in L^1(0,T; C^\alpha(\mathbb{T}^d)), \qquad \alpha \in (0,1)$$

and smooth initial data θ_0

such that $\{\theta^\kappa\}_{\kappa>0}$ is bounded in

$$L^\infty([0,T]; C^\beta), \qquad \beta < \frac{1-\alpha}{2}$$

and

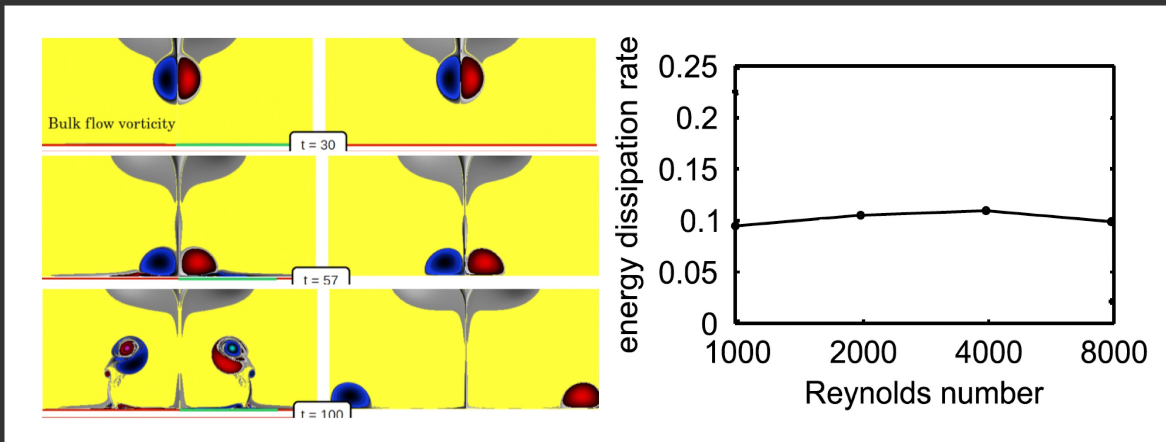
$$\liminf_{\kappa \to 0} \kappa \int_0^T \int |\nabla \theta^\kappa|^2 \, dx \, dt > 0 ?$$

Note, as velocity gets rougher, the scalar is smoother.

③ QUESTION: Conditions on an Euler blowup which would ensure anomalous dissipation for Navier-Stokes.

Recent Euler Blowups: Elgindi (2018-)

Extensions and Different Directions



Nguyen, Farge, Schneider, 2011

- bounded domains: Bardos - Titi (2018), Drivas - Nguyen (2018)

$\sigma \in [1/3, 1]$ $u \in B^{\sigma,\infty}_p$ (interior)

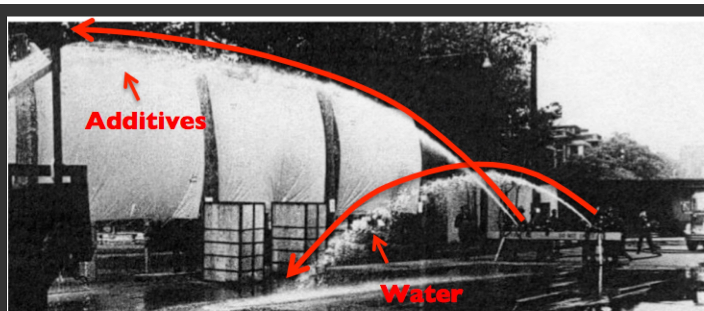
← $\mathrm{dist}(\partial\Omega) \sim \nu^{\frac{1}{2(1-\sigma)}}$

velocity equicontinuous

kolmogorov length

$\sigma = 1/3 \implies \nu^{3/4}$ B.L. Theory makes contact Blasius.

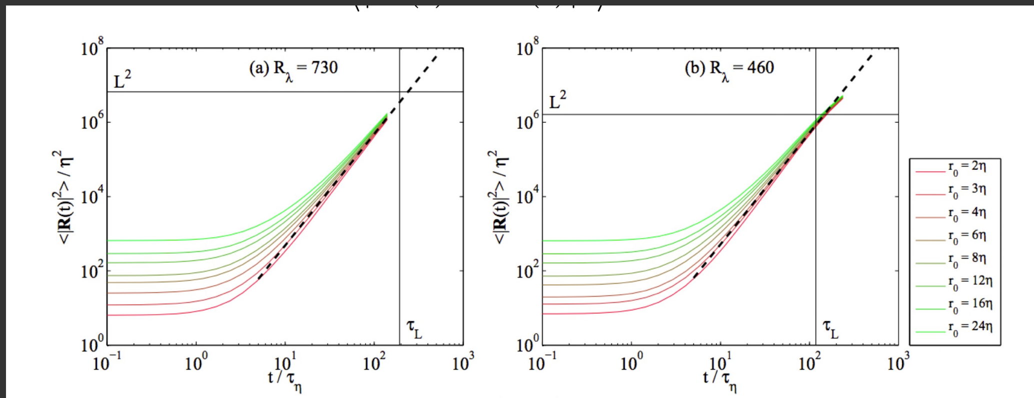
QUESTION: Identify physical mechanism that suppress this behavior. Drag reduction! Engineering



Polymer: Drivas - La, 2019 ← polymer included only at walls

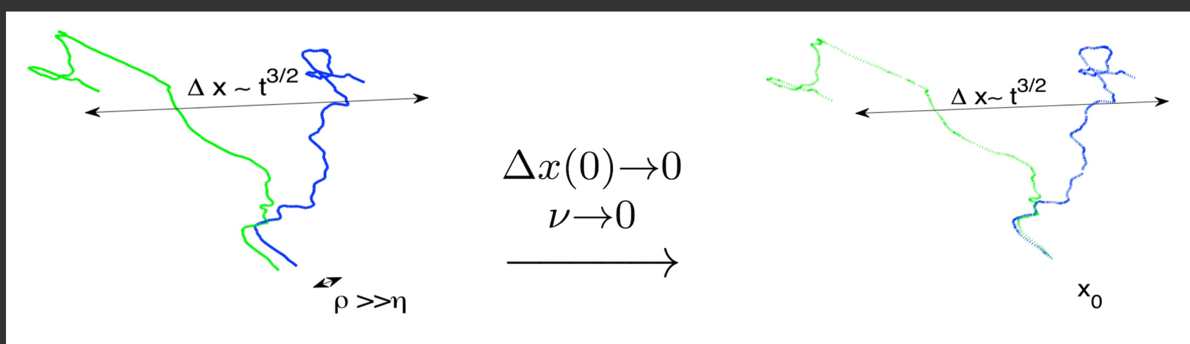
Rough walls: Mikelić, Jäger ← only near very special flows.

Richardson (1926): $\langle |x_1(t) - x_2(t)|^2 \rangle \sim \varepsilon t^3$



Toy understanding: K41 velocity $\delta_\ell u \sim (\varepsilon \ell)^{1/3}$.

$$\dot{\delta x} = \delta u \approx (\varepsilon\, \delta x)^{1/3} \Rightarrow |\delta x|^2 \sim \varepsilon t^3$$



Spontaneous Stochasticity:

Bernard, Gawedzki-Kupiainen (1998) ...

Gawedzki - Vergassola (2000) ...

Eyink (2006 ...)

Drivas - Mailybaev, Drivas - Mailybaev - Raibekas (2018, 2020)

ALSO IN SPACE OF VELOCITIES! Kraichnan-Leith

Kolmogorov 1958-1959 seminar

> 8. *Consideration (at least on models) of the conjecture that, in the situation at the end of 5 above, in the limit the dynamical system turns into a random process (the conjecture of the practical impossibility of a long-term weather forecast).*

Arnol'd - Khesin

NEXT TIME

Lecture 3:

- Transition to turbulence
- Kolmogorov's flow problem.
- Stability of laminar state at "low Re"
- Instability of laminar state at "high Re" (Meshalkin & Sinai)

Lecture 4:

- Bifurcation to stable secondary flows near onset (Yudovich).
- Long time behavior of ideal fluids near laminar states.
- Mixing, instability and infinite time growth (Koch, Nadirashvili)