Intermittency and Dissipation in turbulent flows

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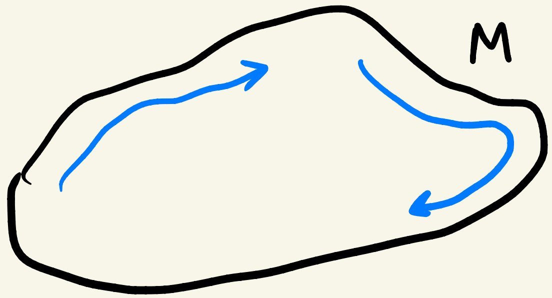
How do fluids move in the Platonic ideal?

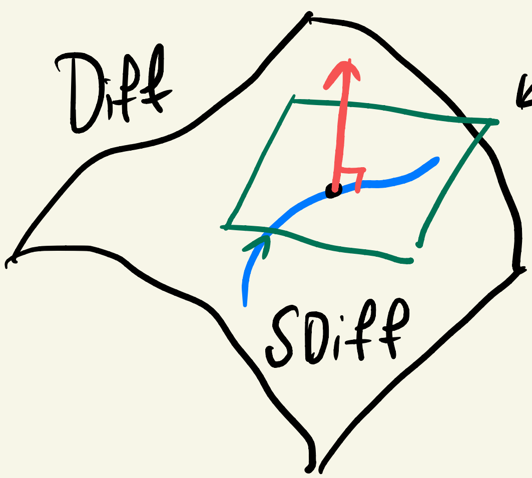


$$\dot{\phi}_t = u_t \circ \phi_t$$

- State of the fluid is a diffeomorphism, preserving volume.

$$\phi_t \in \{\phi : M \circlearrowleft \ : \ \det \nabla\phi = 1\} = SDiff(M)$$

- The motion obeys this constraint, but is otherwise free



$$T_\phi SDiff = \{u \circ \phi : u : M \to \mathbb{R}^d \quad \nabla\cdot u = 0, \ u\cdot\hat{n}|_{\partial M} = 0\}$$

$$N_\phi SDiff = \{\nabla p \circ \phi \ : \ p : M \to \mathbb{R}\}$$

D'Alembert's principle		Euler equation
$\ddot{\phi}_t \in N_{\phi_t} SDiff$ $\phi_t \in SDiff$	$\Leftrightarrow$	$\partial_t u + u\cdot\nabla u = -\nabla p$ $\nabla\cdot u = 0$

The Euler equations: "flow of dry water" - Feynman

$$\partial_t u + u\cdot\nabla u = -\nabla p$$
$$\nabla\cdot u = 0$$

① Local well posedness for $u \in C^{1,\alpha}$. Global in dim 2.
(Hölder, Wolibner)

Conservation of Energy

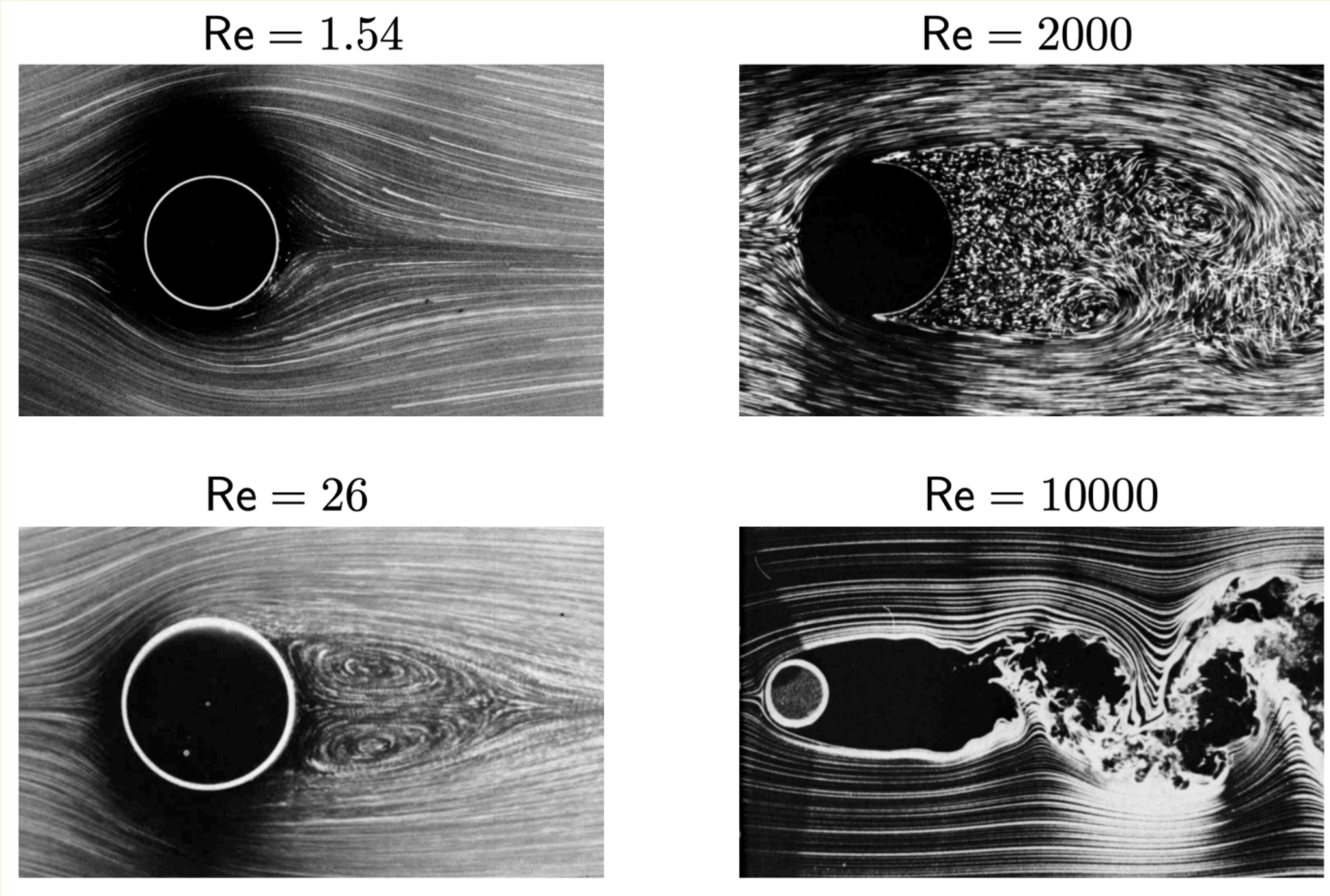
$$\boxed{\frac{d}{dt}\int_M \frac{1}{2}|u|^2\,dx = 0}$$

We call this good situation "not turbulent"

② Finite time singularity in dimension 3 (Elgindi)

Moral: Cannot stay in the non-turbulent regime forever.

What does turbulence look like?



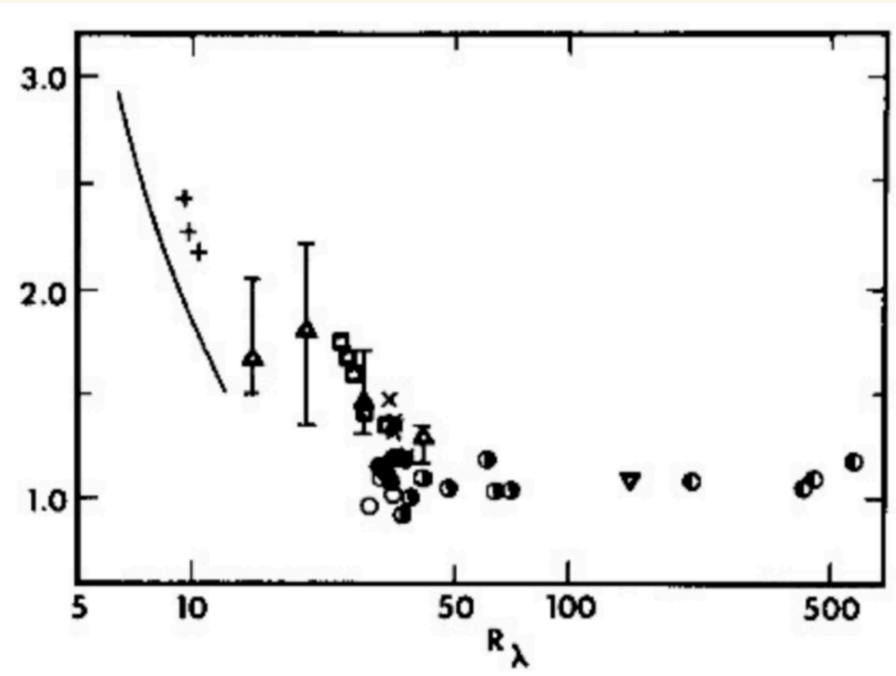
Wet water governed, to some approximation, by the Navier Stokes equations

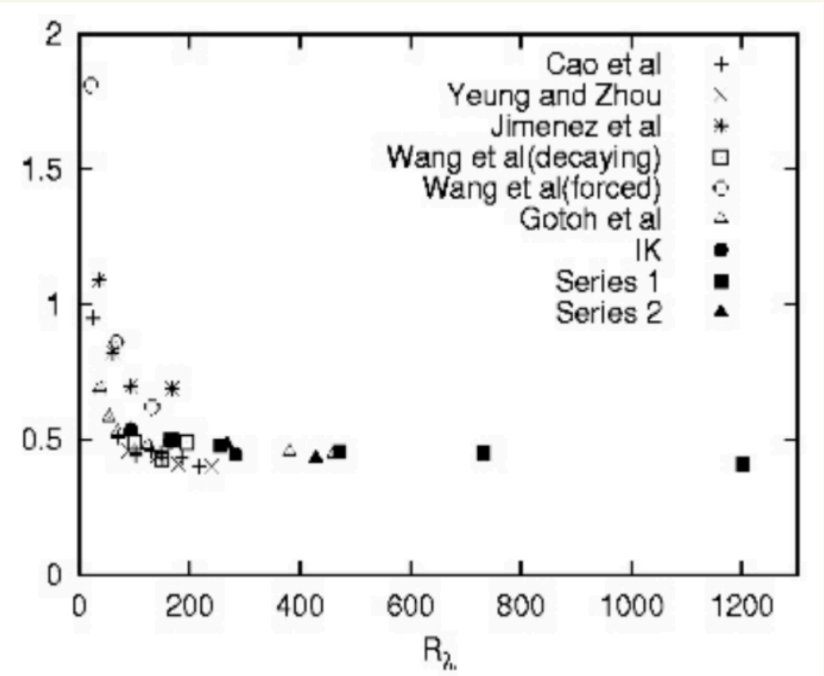
$$\boxed{\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p + \nu \Delta u \qquad \nu = \frac{1}{Re} \\ \nabla \cdot u &= 0 \end{aligned}}$$

How does turbulence behave?

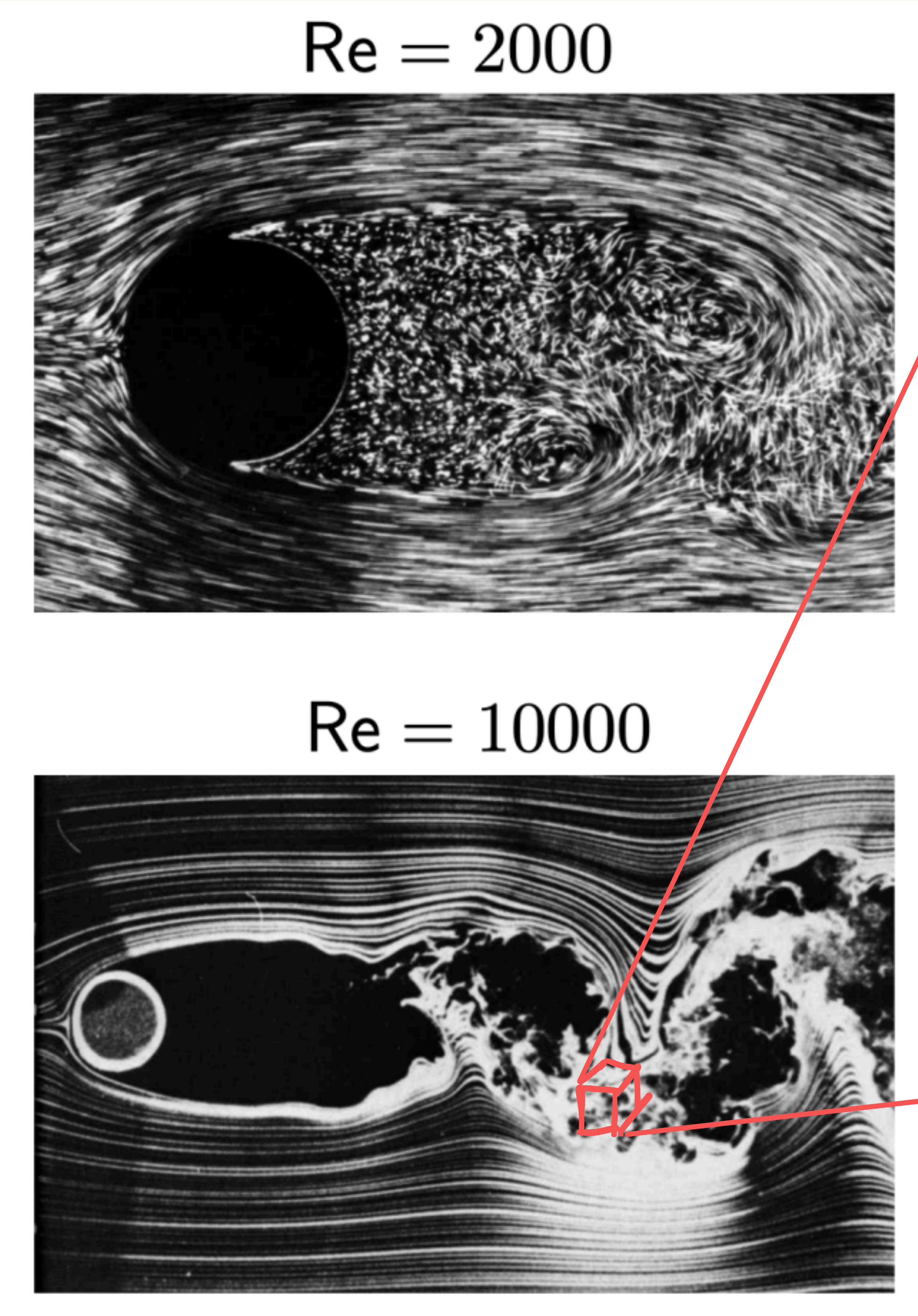
$$\frac{d}{dt}\int_M \frac{1}{2}|u^\nu|^2\,dx = \boxed{\nu\int_M |\nabla u^\nu|^2\,dx \longrightarrow \varepsilon > 0}$$ "Zeroth Law"

Termed anomalous dissipation. Necessitates roughness $\|\nabla u\|_{L^2_{t,x}} \approx \frac{1}{\sqrt{\nu}}$.

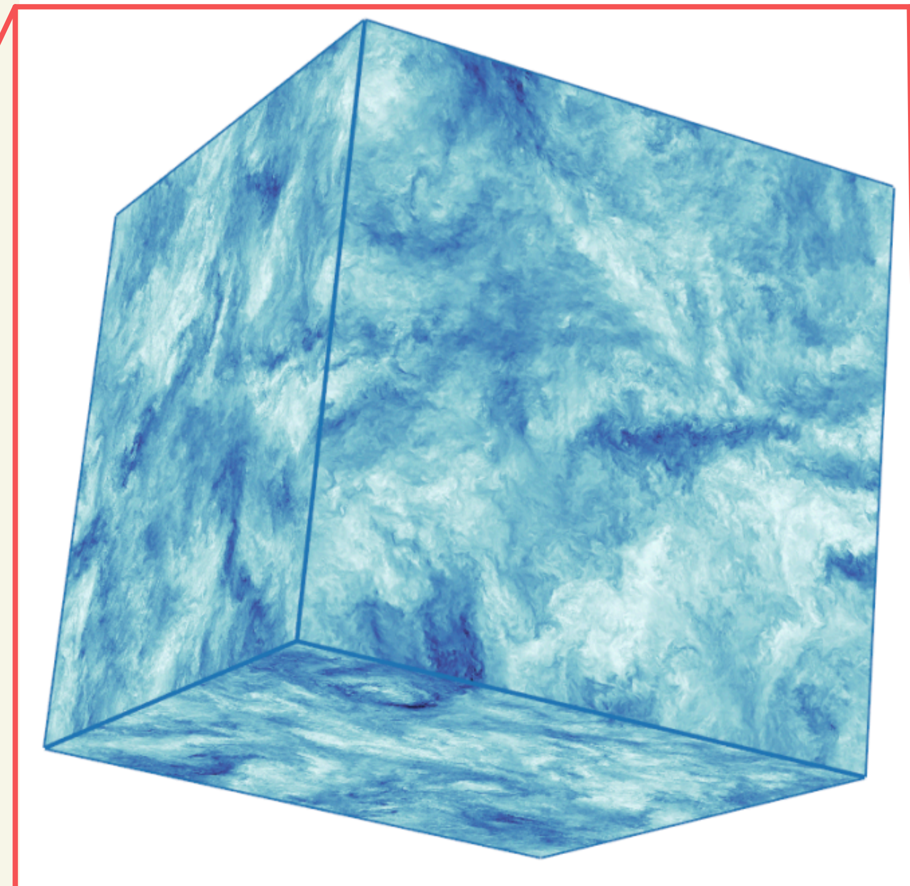




"Practical Ideal Turbulence"



$$\boxed{M = \mathbb{T}^3}$$



Numerical simulation of spatially periodic turbulence

Observations on pth order Structure Functions:

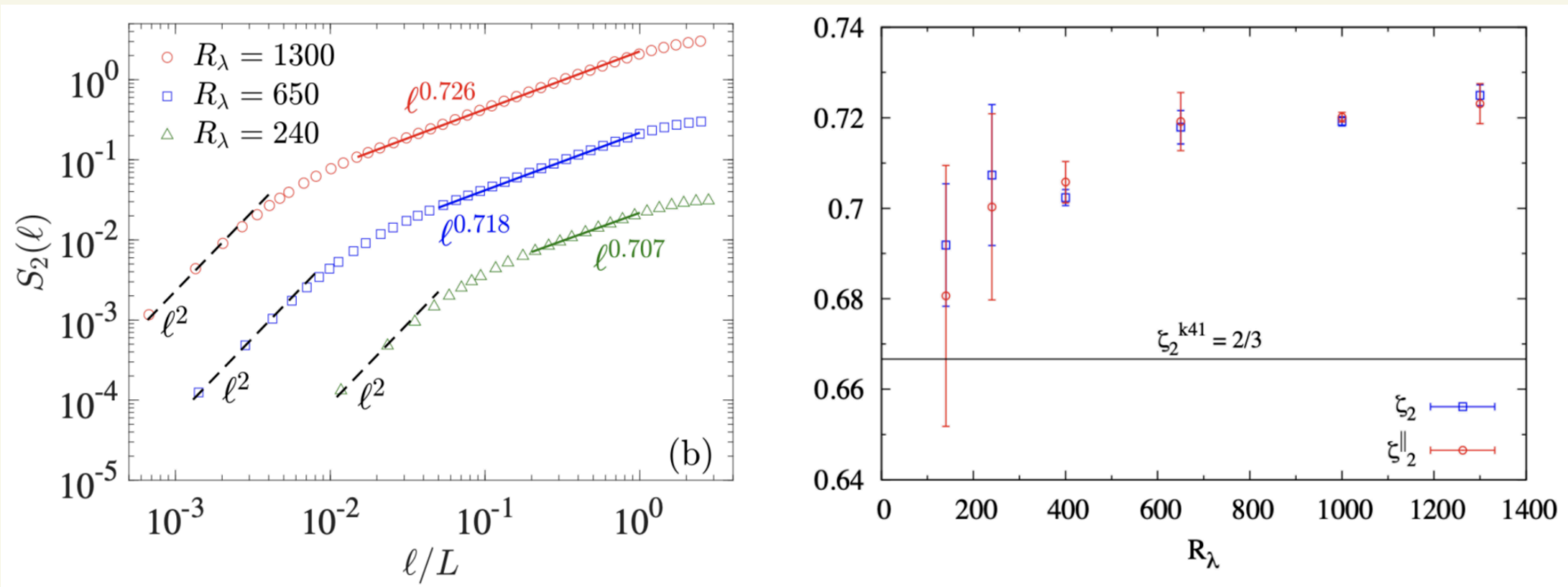
$$\boxed{S_p^\nu(\ell) := \fint_0^T \fint_M |u^\nu(x+\ell) - u^\nu(x)|^p \, dx \, dt}$$

Besov space

$u \in B^{\sigma}_{p,\infty}$ iff

$u \in L^p$ & $S_p(\ell) \lesssim \ell^{\sigma p}$!

that $S_2^\nu(\ell) \lesssim \ell^{\zeta_2}$ for some $\zeta_2 > 0$ gives L^2 precompactness. As such $u^\nu \xrightarrow{L^2} u$, where u is a weak solution of Euler.

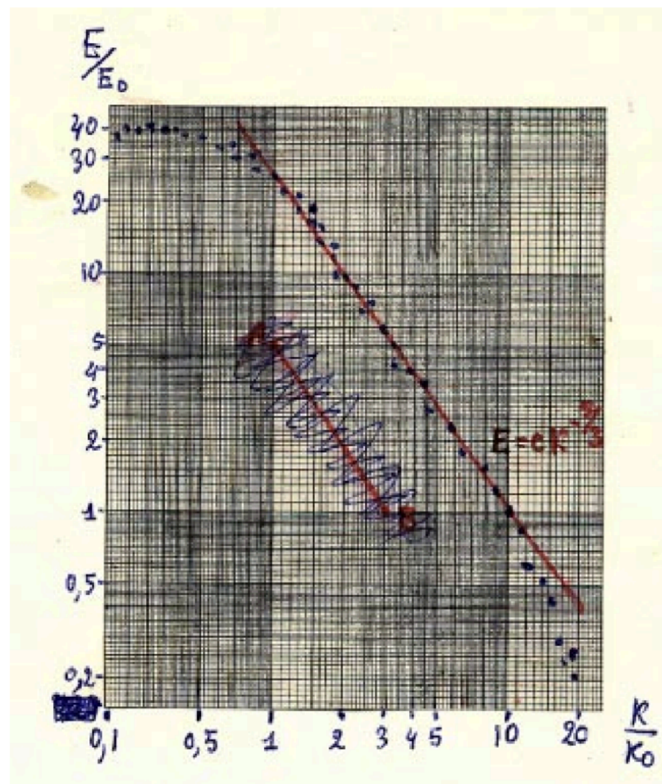


Kolmogorov's 1941 theory (homogeneity, isotropy, self similarity)

$$\boxed{S_p^v(\ell) \sim (\varepsilon \ell)^{p/3}} \quad \text{for } \ell \in [\ell_D, \ell_I] \quad \text{"inertial range"}$$

$$\ell_D \sim \left(\frac{v^3}{\varepsilon}\right)^{1/4} \sim v^{3/4}$$

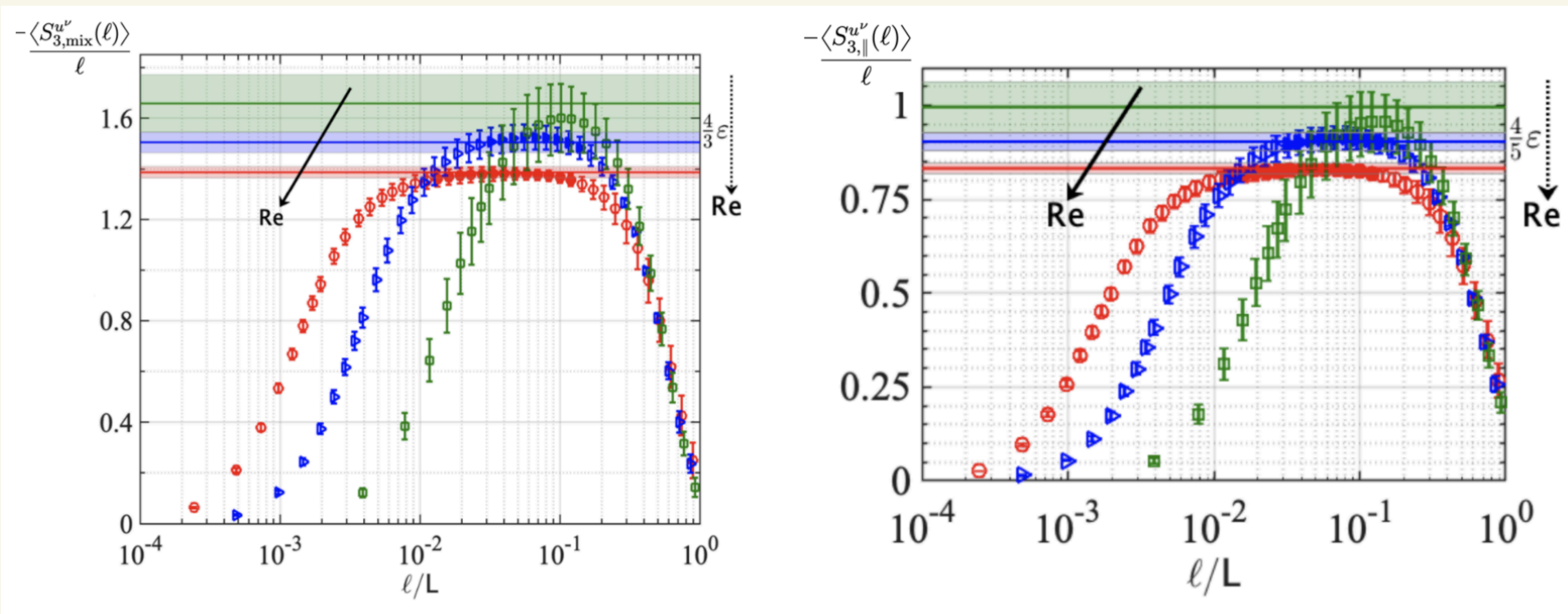
For instance, $S_2(\ell) \sim \ell^{2/3} \iff E(k) = \sum_{|p|=k} |\hat{u}(p)|^2 \sim k^{-5/3}$



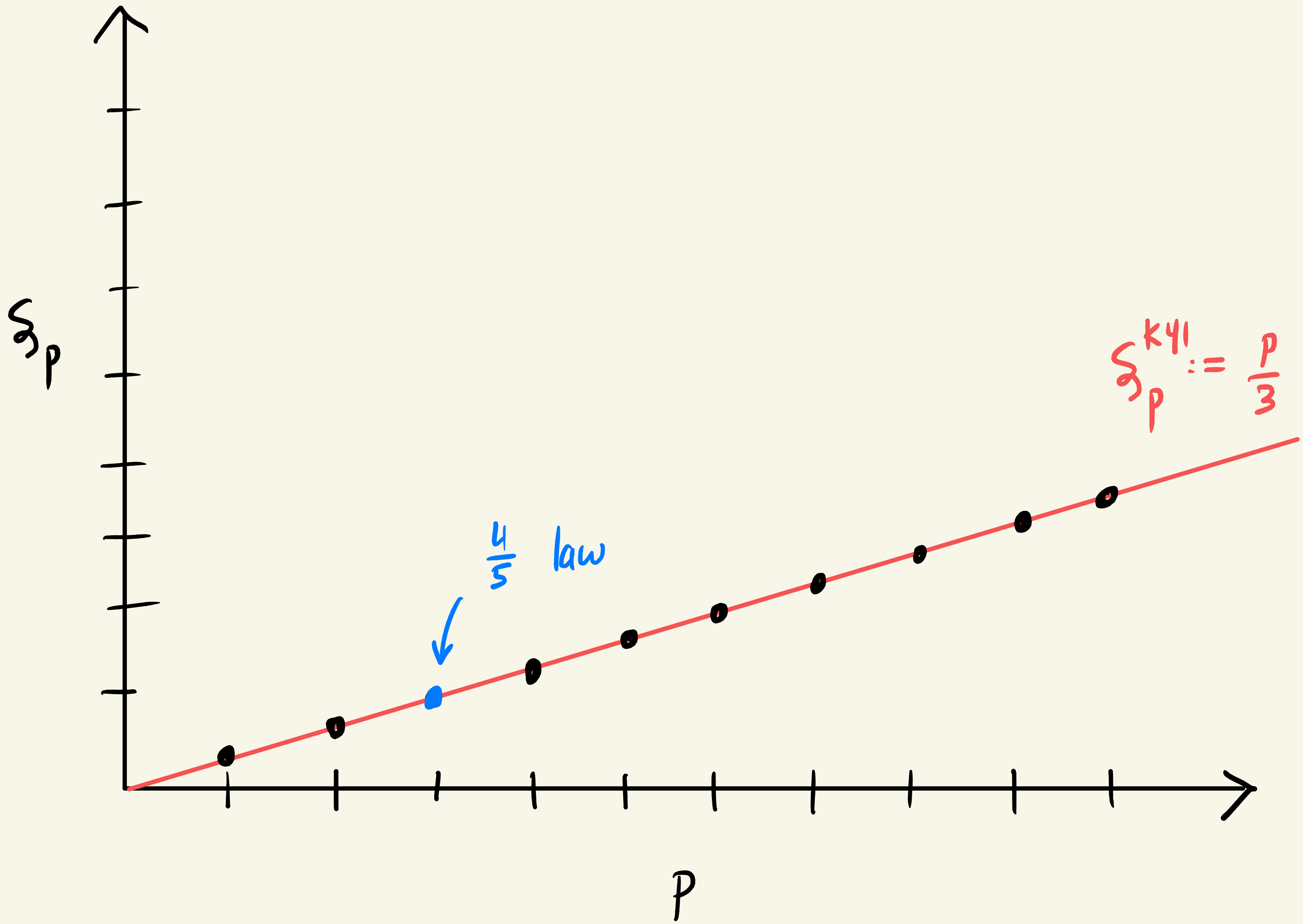
Kolmogorov's 4/5 and 4/3 law (modern interpretation)

Theorem: Let u^ν be sequence of Navier-Stokes solutions.
Suppose $\{u^\nu\}_{\nu>0}$ has a uniform L^3 modulus $\phi_u(\ell) = \sup_{|h|\le\ell}\sup_{\nu>0} \|u^\nu(\cdot+\ell)-u(\cdot)\|_{L^3}$.
If $u^\nu \to u$ in L^3, there exists a monotone sequence $\ell_\nu \searrow 0$ s.t.

$$\limsup_{\nu\to 0}\ \sup_{\ell\in[\ell_\nu,\ell_I]} \left| \left(\frac{\langle (\hat{z}\cdot\delta_{\ell\hat{z}}u^\nu)^3\rangle_{ang}}{\ell} + \frac{4}{5}\varepsilon^\nu[u^\nu],\ \varphi \right)_{L^2} \right| \lesssim \phi_u(\ell_I)$$



$$S_p^{v}(\ell) := \fint_0^T \fint_M |u^{v}(x+\ell) - u^{v}(x)|^p \, dx\, dt \sim \ell^{\zeta_p}$$

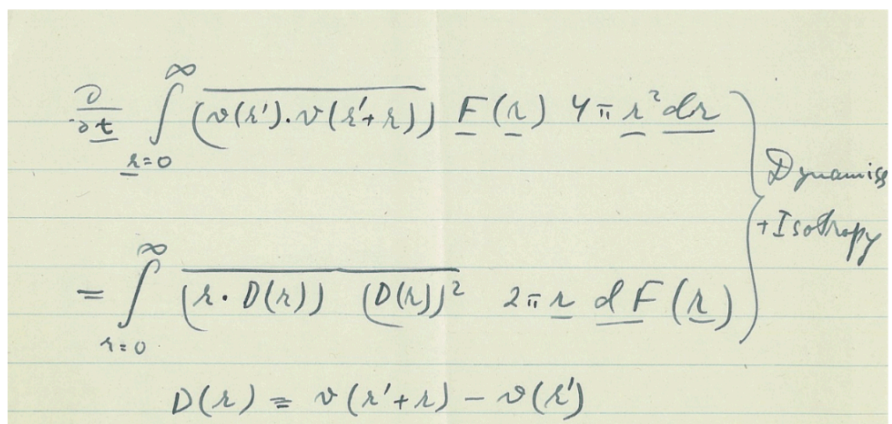


$\langle (\hat{\ell}\cdot\delta_\ell u)^3\rangle \sim \ell \Rightarrow$ "$1/3$ derivative in L^3": Onsager '49 understood this as a deterministic threshold for dissipation. Duchon-Robert: $u^\nu \xrightarrow{L^3} u$

$$\partial_t\left(\tfrac{1}{2}|u|^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}|u|^2+p\right)u\right) = -D[u], \qquad D[u] = \lim_{\nu\to 0} \nu|\nabla u^\nu|^2$$

Theorem: (Eyink '94, CET '94) Let u be a weak sln' in $u\in L^3_t B^{1/3+}_{p,\infty}$. Then $D[u]=0$.

Thus, $\zeta_p \le \frac{p}{3}$ for all $p \geq 3$ for anomalous dissipation to exist.



$$\frac{\partial}{\partial t}\int_{r=0}^{\infty} \overline{(v(r')\cdot v(r'+r))}\, F(r)\, 4\pi r^2 dr$$

$$= \int_{r=0}^{\infty} \overline{(r\cdot D(r))\, (D(r))^2}\, 2\pi r\, dF(r)$$

Dynamics + Isotropy

$$D(r) = v(r'+r) - v(r')$$

Proof: Let $G(\cdot)$ be a mollifier, and $G_\ell(\cdot) = \frac{1}{\ell^d} G(\frac{\cdot}{\ell})$. Let $u_\ell = u * G_\ell$

mollified Euler equation
$$\begin{cases} \partial_t u_\ell + u_\ell \cdot \nabla u_\ell = -\nabla p_\ell - \nabla \cdot \tau_\ell \\ \tau_\ell = (u \otimes u)_\ell - u_\ell \otimes u_\ell \end{cases}$$

Resolved energy
$$\partial_t \left(\tfrac{1}{2}|u_\ell|^2\right) + \nabla \cdot \left(\left(\tfrac{1}{2}|u_\ell|^2 + p_\ell\right) u_\ell + u_\ell \tau_\ell \right) = \boxed{\nabla u_\ell : \tau_\ell}$$

$$\lesssim \ell^{3\sigma - 1}$$

Commutator estimates $p \geq 3$
$$\|\nabla u_\ell\|_{L^3_{x,t}} \lesssim |u|_{L^3_t B^\sigma_{p,\infty}} \ell^{\sigma-1} \qquad \|\tau_\ell\|_{L^{3/2}_{x,t}} \lesssim |u|^2_{L^3_t B^\sigma_{p,\infty}} \ell^{2\sigma}$$

Compactness
$$\tfrac{1}{2}|u_\ell|^2 \xrightarrow{\ell \to 0} \tfrac{1}{2}|u|^2$$
$$\left(\tfrac{1}{2}|u_\ell|^2 + p_\ell\right) u_\ell + u_\ell \tau_\ell \xrightarrow{\ell \to 0} \left(\tfrac{1}{2}|u|^2 + p\right) u \qquad \text{as} \quad u_\ell \xrightarrow{L^3} u, \quad p_\ell \xrightarrow{L^{3/2}} p$$
$$\|\nabla u_\ell : \tau_\ell\|_{L^1} \lesssim \ell^{3\sigma - 1} \xrightarrow{\ell \to 0} 0$$

Negative side of Onsager's "Conjecture"

- Convex integration has produced $C^{1/3-}$ solutions of $d\geq 3$ Euler which do not conserve energy (Isett, De Lellis, Szekelyhidi, ...)
  This holds also in $d=2$ (Giri - Radu)

- Critical case $C^{1/3}$ is completely open!!

- Solutions are quite pathological, Energy is generically non-monotone

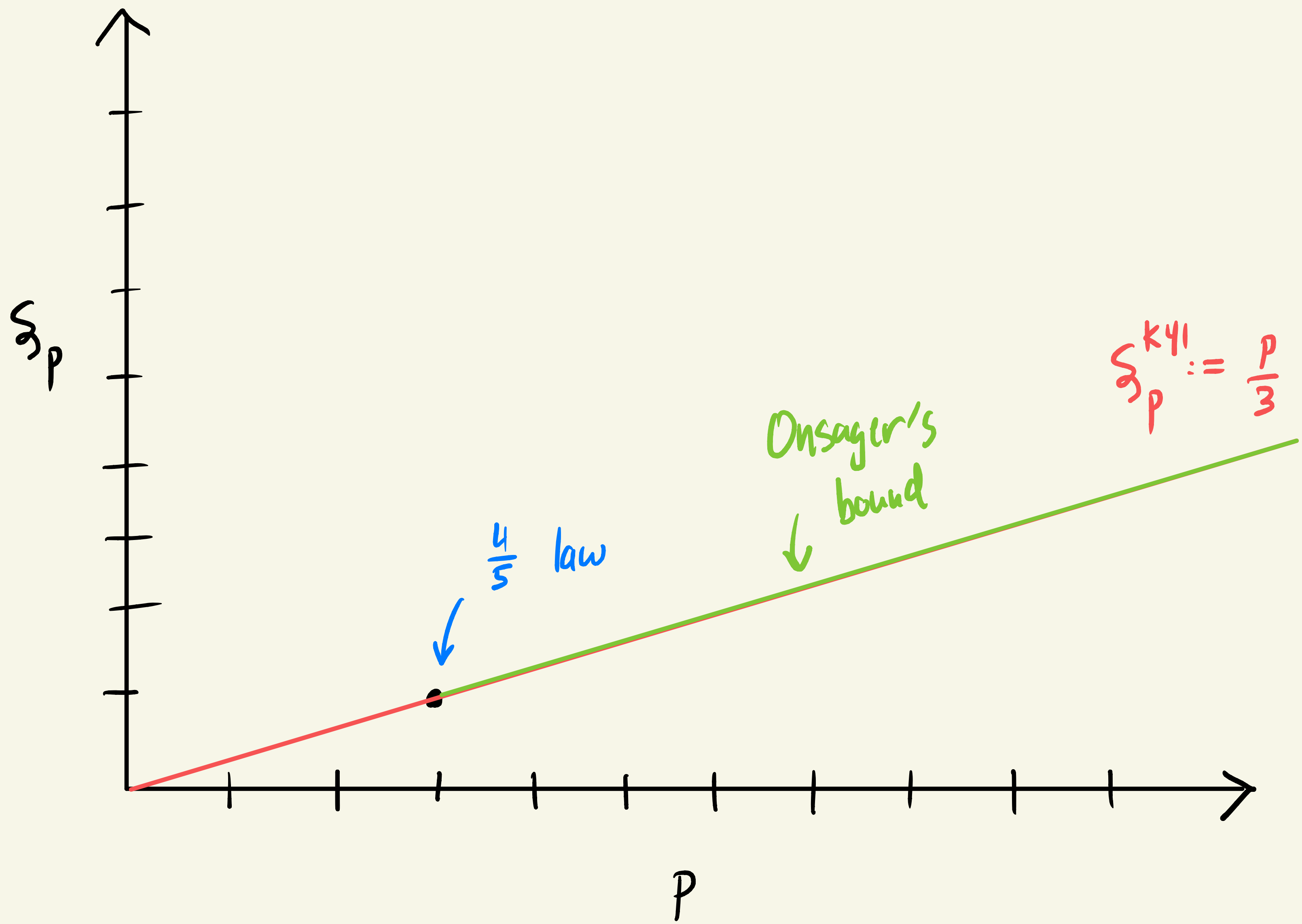
Theorem (De Rosa - Tione) Fix $\alpha \in [0, 1/3)$. There is a complete metric space $X := \{u : u \text{ solves Euler}\} \subseteq L^\infty_t C^\alpha_x$ in which the set

$$Y = \{u \in X : \tfrac{1}{2}\int |u|^2 dx \notin BV(I) \text{ for any open } I \subseteq [0,T]\}$$

is residual.

Anomalous dissipation unstable below $C^{1/3}$.

$$S_p^\nu(\ell) := \fint_0^T \fint_M |u^\nu(x+\ell) - u^\nu(x)|^p \, dx\, dt \sim \ell^{\zeta_p}$$



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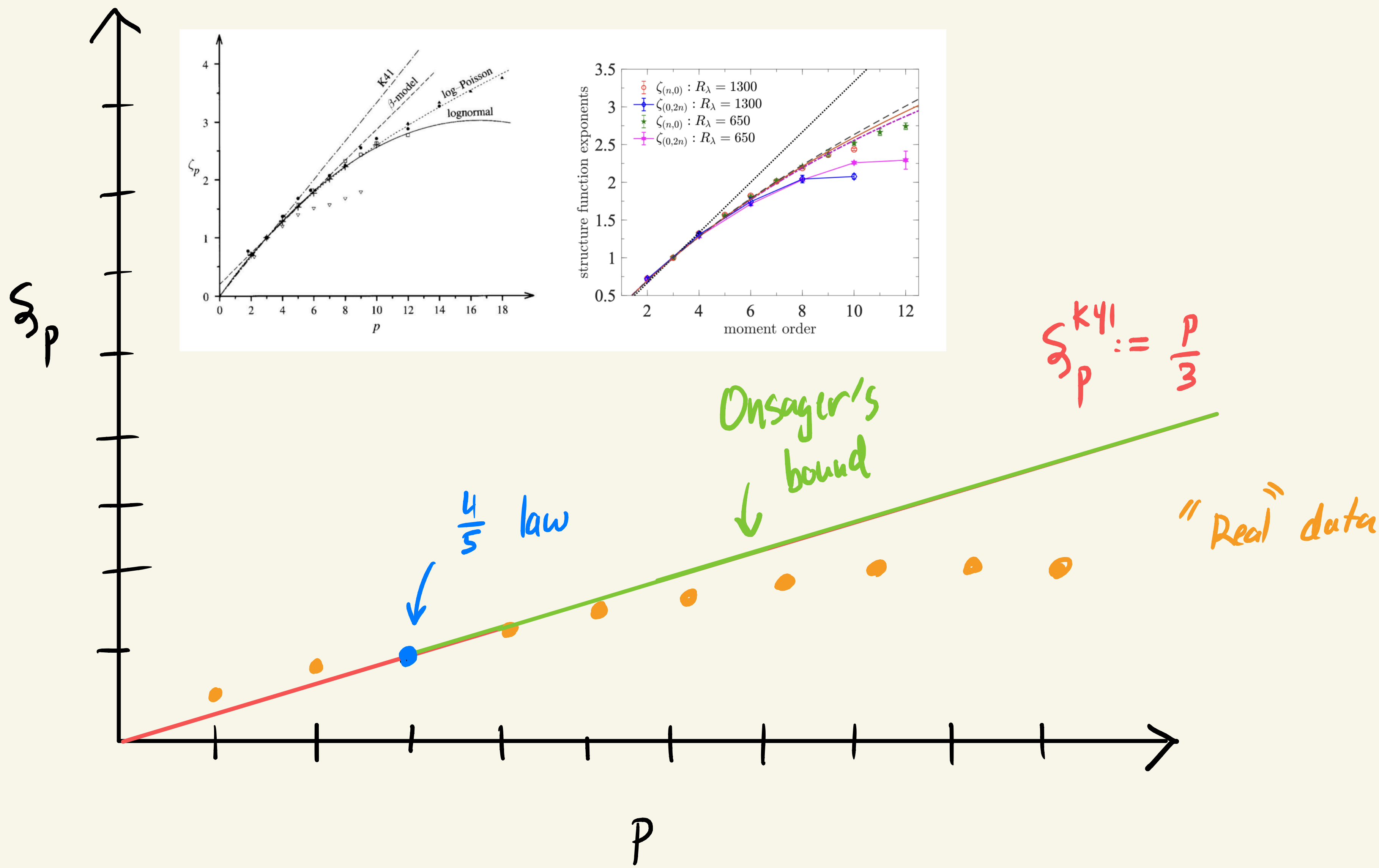


Intermittency

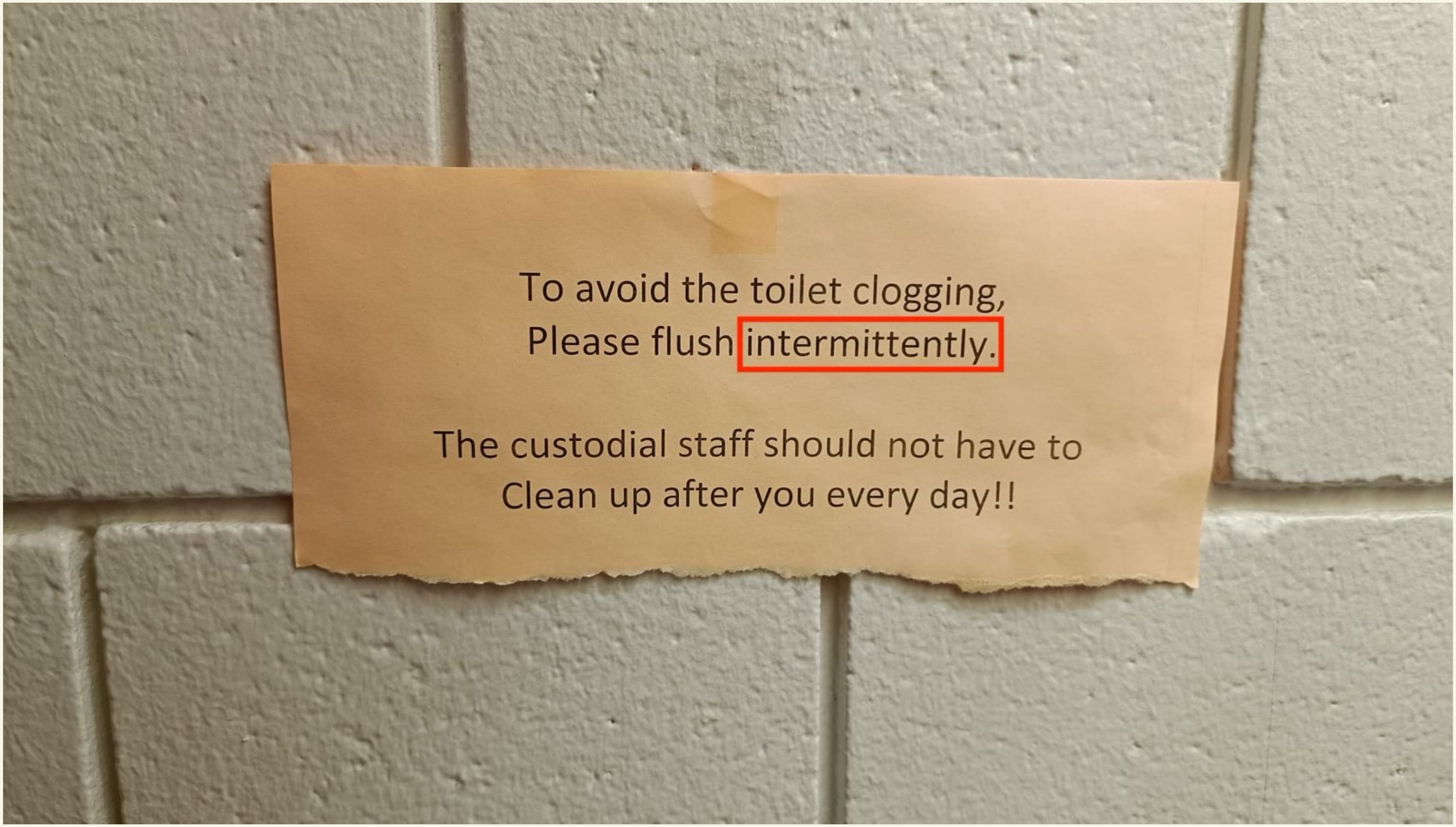
$$S^v_p(\ell) := \fint_0^T \fint_M |u^v(x+\ell) - u^v(x)|^p \, dx \, dt \lesssim \ell^{\zeta_p}$$

largest such

> **Definition:** We say that the fluid is intermittent if $\zeta_p = cp$ for all p.

- K41 theory is not intermittent.
- Real fluids are intermittent!
- Many guesses:
  - log-normal: $\zeta_p = \frac{p}{3} - \frac{\mu}{18} p(p-3)$, $\mu = 0.25$ — Kolmogorov 1962
  - β-model: $\zeta_p = \frac{p}{3} + (3-D)\left(1 - \frac{p}{3}\right)$, $D = 2.8$ — Frisch et al 1978
  - log-Poisson: $\zeta_p = \frac{p}{9} + 2\left(1 - \left(\frac{2}{3}\right)^{\frac{p}{3}}\right)$ — She–Leveque 1994
  - mean-field: $\zeta_p = \frac{ap}{b - cp}$, $a = 0.185$, $b = 0.475$, $c = 0.0275$ — Yakhot 2001

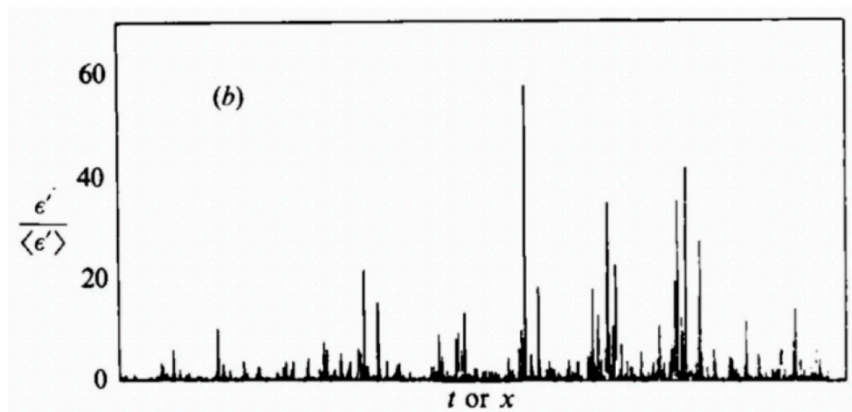
  ⋮



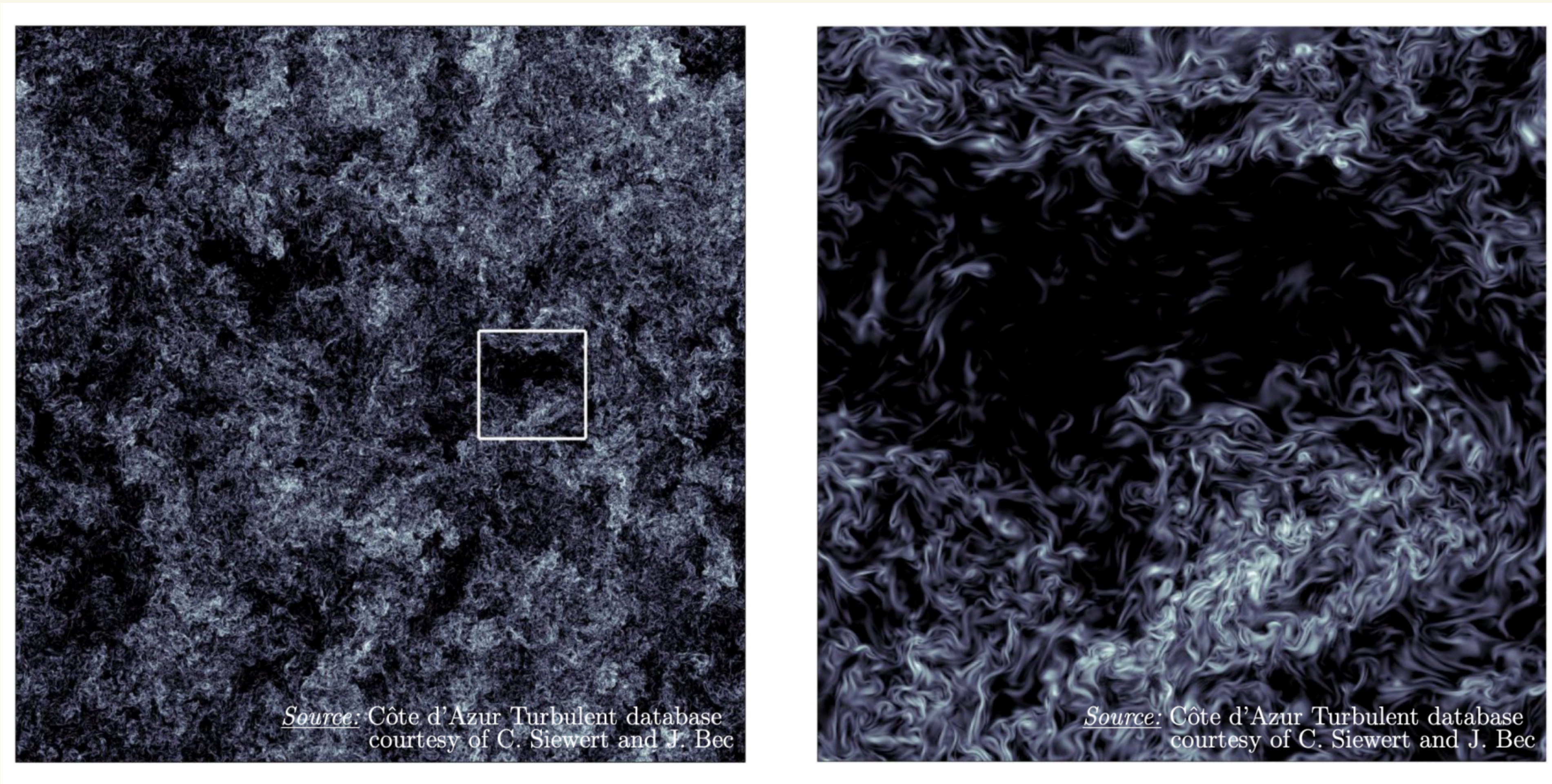
Stony Brook Math Department
circa 2026

Landau objected to Kolmogorov's non-intermittent theory based on the spottiness of turbulent energy dissipation.

$$\left\langle \left(\hat{z}\cdot \delta_{\ell\hat{z}}\mathbf{u}\right)^3\right\rangle \approx -\frac{4}{5}\langle\varepsilon\rangle \ell \quad \not\Longrightarrow \quad \left\langle \left(\hat{z}\cdot \delta_{\ell\hat{z}}\mathbf{u}\right)^p\right\rangle \approx \left(\langle\varepsilon\rangle \ell\right)^{p/3}$$



Meneveau & Sreenivasan (1991) estimated dissipation to take place on a fractal set of space-time dimension ≈ 3.87



Why is spottiness of energy dissipation linked to $\zeta_p \neq p/3$?

$$\partial_t\left(\tfrac{1}{2}|u|^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}|u|^2+p\right)u\right) = -D[u], \qquad D[u] = \lim_{\nu\to 0} \nu|\nabla u^\nu|^2$$

Radan measure

Theorem: (De Rosa - D - Inversi - Isett) Suppose $u \in L^3_{x,t}$ weak solution with $\dim(\operatorname{supp} D[u]) = \gamma \in (0, d+1)$. Then, for all $p \geq 3$ s.t. $u \in L^p_t B^{\sigma_p}_{p,\infty}$

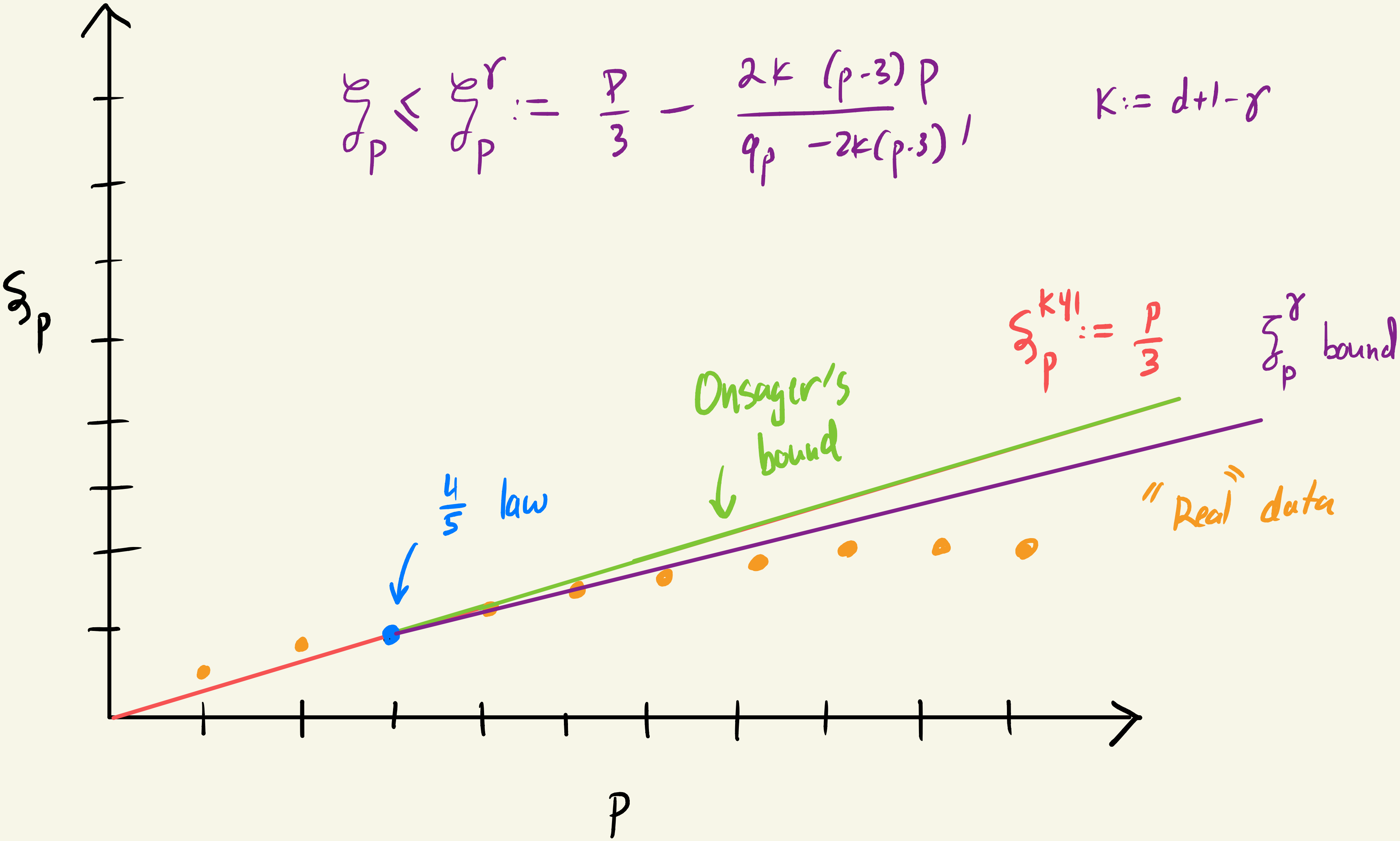
$$\frac{2\sigma_p}{1-\sigma_p} \leq 1 - \frac{p-3}{p}(d-1-\gamma)$$

- $$u \in L^p_t B^{\sigma_p}_{p,\infty} \iff S_p(\ell) \lesssim \ell^{\zeta_p} \quad \text{with} \quad \zeta_p := \sigma_p \cdot p$$

  since $\frac{2\sigma}{1-\sigma} = 1 \iff \sigma = \frac{1}{3}$. Thus, if $\gamma < d+1$, then we have a quantitative deviation from K41 prediction $\zeta_p = p/3$.
- For $p = 3$, there is no deviation
- If $u \in L^p_{t,x}$, then $\gamma \geq d+1-\frac{p}{p-1}$, which is sharp (De Rosa, D, Inversi)

$$S_p^\nu(\ell) := \fint_0^T \fint_M |u^\nu(x+\ell) - u^\nu(x)|^p \, dx \, dt \sim \ell^{\zeta_p}$$

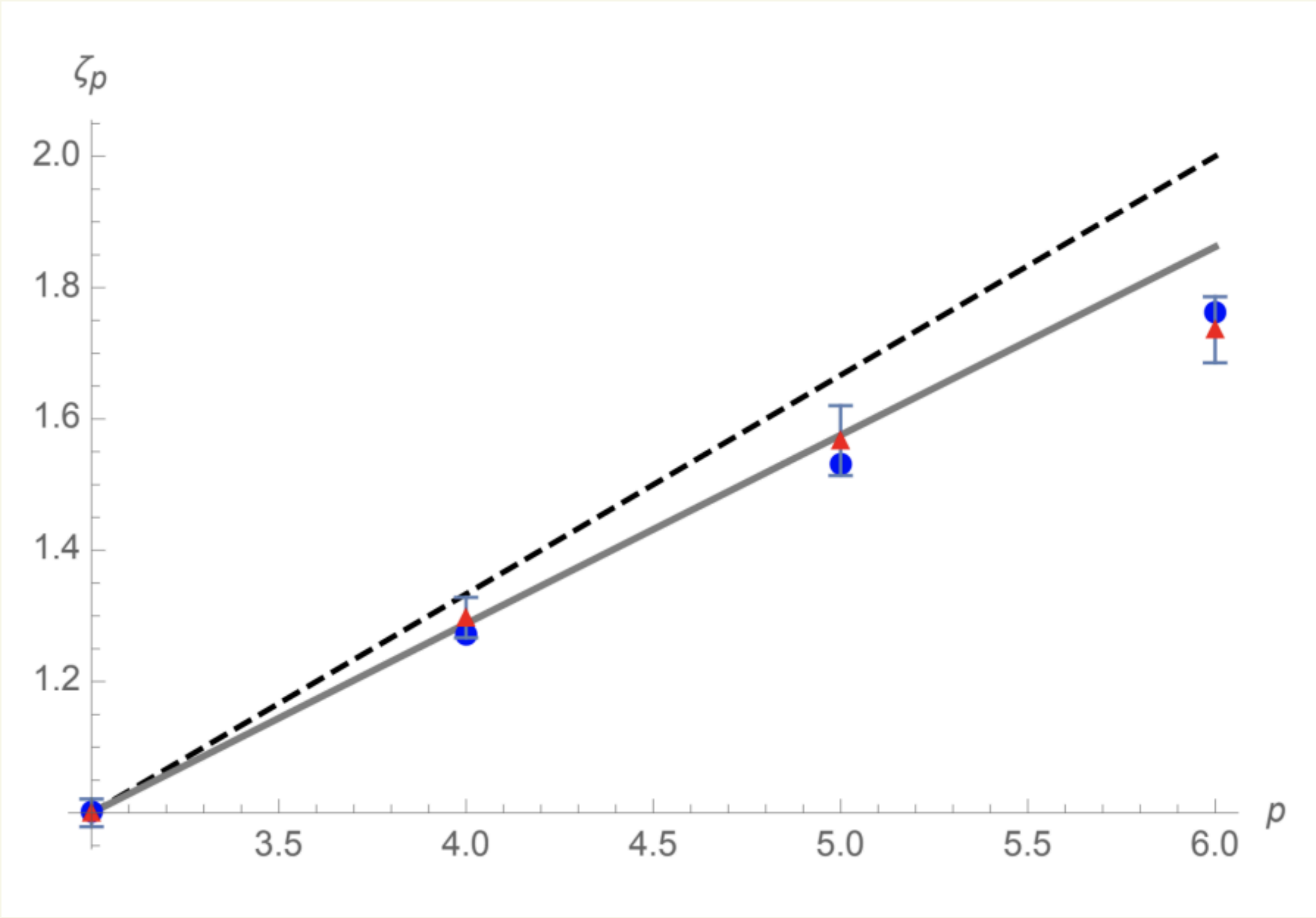
$$\zeta_p \le \zeta_p^\gamma := \frac{p}{3} - \frac{2\kappa(p-3)p}{9p - 2\kappa(p-3)}, \qquad \kappa := d+1-\gamma$$



Estimated $\gamma = \dim(\operatorname{supp} D)$ from JHU database gives

$$\gamma \approx 3.85$$

Meneveau & Sreenivasan value was 3.87.



The intermittency result follows from

> **Theorem:** (De Rosa - D - Inversi - Isett) Let $u \in L^p_t B^\sigma_{p,\infty}$ be a weak solution of Euler, for some $p \in [3,\infty]$, $\sigma \in (0, 1/3)$. Let D be its dissipation measure. Then
>
> - $D \in B^{\frac{2\sigma}{1-\sigma} - 1}_{p/3, \infty}$ in spacetime
> - D is absolutely cont. w.r.t. $\mathcal{H}^\gamma$ for any $\gamma \geq 0$ s.t.
>
> $$\frac{2\sigma}{1-\sigma} > 1 - \frac{p-3}{p}(d+1-\gamma)$$

- The number $\frac{2\sigma}{1-\sigma}$ appears due to the anisotropy of

$$\nabla_{t,x} \cdot V = -D[u]$$

where $V = \left(\frac{1}{2}|u|^2, \left(\frac{1}{2}|u|^2 + p\right)u\right)$

Linear Toy problem:

Proposition: Let $\alpha \in (0,1)$, $p \in (1,\infty)$. If $V : M \to \mathbb{R}^N$ be $V \in B^{\alpha}_{p,\infty}$ and $\mu = \operatorname{div} V$ is a nonnegative measure. Then

(i) $\mu \in B^{\alpha-1}_{p,\infty}$

(ii) $\mu \ll \mathcal{H}^{\gamma}$ for all γ s.t. $\alpha > 1 - \frac{p-1}{p}(N-\gamma)$

- This is optimal, i.e. $\exists$ measure μ and set S s.t.

$$\mu \in B^{\alpha-1}_{p,\infty}, \qquad S \subseteq \mathbb{R}^N$$

and

$$\mu(S) > 0 \quad \text{and} \quad \dim_{\mathcal{H}} S = \gamma$$

where

$$\alpha = 1 - \frac{p-1}{p}(N-\gamma)$$

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(i) $\mu \in B^{\alpha-1}_{p,\infty}$ ← trivial, by definition

(ii) $\mu \ll \mathcal{H}^{\gamma}$ for all γ s.t. $\alpha > 1 - \frac{p-1}{p}(N-\gamma)$

Proof of (ii). Assume $\mathcal{H}^{\gamma}(A) = 0$ for some γ to be found.

For any $\varepsilon > 0$, we find a covering $A \subset \bigcup_i B_{r_i}$ with $\sum_i r_i^{\gamma} < \varepsilon$.

On each ball B_{r_i}, put a smooth cutoff χ_i and set $\chi(x) = \max_i \chi_i(x)$.

Then $\chi \in W^{1,\infty}$. Up to small loss

$$B^{\alpha-1}_{p,\infty} = \left(B^{1-\alpha}_{p/(p-1),1}\right)^* \sim \left(W^{1-\alpha,\frac{p}{p-1}}\right)^*$$

Then

$$\mu(A) \le \int \chi \, d\mu \le \|\chi\|_{W^{1-\alpha,\frac{p}{p-1}}} \lesssim \left(\sum_i \|\chi_i\|^{\frac{p}{p-1}}_{W^{1-\alpha,\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} \lesssim \left(\sum r_i^{(\alpha-1)\frac{p}{p-1}+N}\right)^{\frac{p-1}{p}} \lesssim \varepsilon^{\frac{p-1}{p}}$$

for any $\gamma < (\alpha-1)\frac{p}{p-1} + N$. This completes the proof.

In the case of Euler..

$$D = -\partial_t\left(\tfrac{1}{2}|u|^2\right) - \operatorname{div}\left(\left(\frac{|u|^2}{2}+p\right)u\right) := \operatorname{div}_{t,x} V$$

Guided by the previous observation, want to prove

$$(*) \qquad \left(\frac{|u|^2}{2}, \left(\frac{|u|^2}{2}+p\right)u\right) \in B^{\frac{2\sigma}{1-\sigma}}_{p/3,\infty} \quad \text{on} \quad \mathbb{R}^{d+1}$$

Then absolute continuity follows as in toy problem.

For $u \in L^p_t B^{\sigma}_{p,\infty}$, $(*)$ looks false! However by abstract interpolation, one can still prove $D \in B^{\frac{2\sigma}{1-\sigma}-1}_{p/3,\infty}$ without using $(*)$

Note: The $\operatorname{div}_{t,x}$ operator is not elliptic. Thus $\operatorname{div}^{-1}_{t,x}$ does not map $B^{\frac{2\sigma}{1-\sigma}-1}_{\frac{p}{3},\infty}$ into $B^{\frac{2\sigma}{1-\sigma}}_{p/3,\infty}$

Lemma: Fix $p \geq 3$, $\sigma \in (0,1)$. Let $u \in L^p_t B^{\sigma}_{p,\infty}$ be a weak solution of Euler with energy dissipation measure $D[u]$. Then $D[u] \in B^{\frac{2\sigma}{1-\sigma}-1}_{p/3,\infty}$ in spacetime.

Remark: (De Rosa-Tione) Shows this is sharp. Namely, $\exists$ uncountably many weak Euler solutions $u \in L^p_t B^{\sigma}_{p,\infty}$ s.t.

$$D[u] \in B^{\frac{2\sigma}{1-\sigma}-1}_{p/3,\infty} \setminus \bigcup_{\varepsilon>0} B^{\frac{2\sigma}{1-\sigma}-1-\varepsilon}_{p/3,\infty}.$$

Lemma: Fix $p > 3$, $\sigma \in (0,1)$. Let $u \in L^p_t B^\sigma_{p,\infty}$ be a weak solution of Euler with energy dissipation measure $D[u]$. Then $D[u] \in B^{\frac{2\sigma}{1-\sigma}-1}_{p/3,\infty}$ in spacetime.

Remark: (De Rosa–Tione) Shows this is sharp. Namely, $\exists$ uncountably many weak Euler solutions $u \in L^p_t B^\sigma_{p,\infty}$ s.t.

$$D[u] \in B^{\frac{2\sigma}{1-\sigma}-1}_{p/3,\infty} \setminus \bigcup_{\varepsilon > 0} B^{\frac{2\sigma}{1-\sigma}-1-\varepsilon}_{p/3,\infty}.$$

To establish this improved regularity, we aim to find a splitting

$$D[u] = D_1^\ell[u] + D_2^\ell[u]$$

where

$$\|D_1^\ell[u]\|_{W^{-1,p/3}_{x,t}} \lesssim \ell^\alpha \quad \text{and} \quad \|D_2^\ell[u]\|_{L^{p/3}_{x,t}} \lesssim \ell^{\alpha-1}$$

Then, abstract interpolation of Banach couples gives

$$D \in \left(W^{-1,p/3}_{x,t}, L^{p/3}_{x,t}\right)_{\alpha,\infty} = B^{\alpha-1}_{p/3,\infty}$$

To obtain these, we use a new (distributional) identity:

$$-D[u] = (\partial_t + u_\ell \cdot \nabla) E^\ell + \operatorname{div} Q^\ell + C^\ell$$

where

$$E^\ell = \tfrac{1}{2} |u - u_\ell|^2 \qquad \|E^\ell\|_{L^{p/2}} \lesssim \ell^{2\sigma}$$

$$Q^\ell = \left(\frac{|u-u_\ell|^2}{2} + p - p_\ell \right)(u - u_\ell) \qquad \|Q^\ell\|_{L^{p/3}} \lesssim \ell^{3\sigma}$$

$$C^\ell = (u - u_\ell) \cdot \operatorname{div} T_\ell + (u-u_\ell) \otimes (u - u_\ell) : \nabla u_\ell \qquad \|C^\ell\|_{L^{p/3}} \lesssim \ell^{3\sigma - 1}$$

Thus, if $D_1[u] = (\partial_t + u_\ell \cdot \nabla) E^\ell + \operatorname{div} Q^\ell$ and $D_2[u] = C^\ell$,

$$\|D_1[u]\|_{W^{-1,p/3}} \lesssim \ell^{2\sigma} \quad \text{and} \quad \|D_2[u]\|_{L^{p/3}} \lesssim \ell^{3\sigma - 1}$$

Not quite what we want, as 2σ and 3σ are unbalanced, but $\frac{2\sigma}{1-\sigma}$ arises by optimally balancing.

Questions & Directions

- Exhibit a weak Euler solution with lower dimensional dissipation, saturating our bound. Note recent work (Giri, Kwon, Novack) can prescribe D as an arbitrary smooth function.

- Are there conditions that guarantee lower dimensionality?

- What can be said about $\zeta_p > p/3$ for $p \in (1,3)$.

Thank you for your attention!