Remarks on Anomalous Dissipation and Intermittency

Einstein Chair Lecture, Feb 2021

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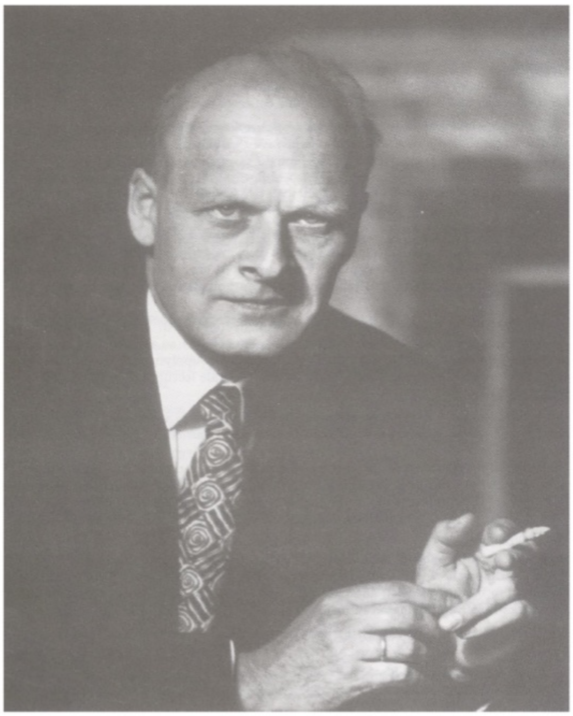
We are interested in the turbulent dynamics of slightly viscous flow governed by incompressible NS:

$$\partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u$$
$$\nabla\cdot u = 0$$

exhibiting anomalous dissipation

$$E(0) - E(T) = \nu\int_0^T\int|\nabla u|^2\,dx\,dt \geq \varepsilon > 0.$$



Lars Onsager (1903-1976)

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

(26) $|\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')| < (\text{const.})r^n$

for any order n greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description...

"Statistical Hydrodynamics" (1949)

<u>Conjecture:</u> Turbulence at small viscosity exhibiting anomalous dissipation is described by singular weak solutions of Euler

$$\partial_t u + \nabla\cdot(u\otimes u) = -\nabla p, \qquad \nabla\cdot u = 0$$

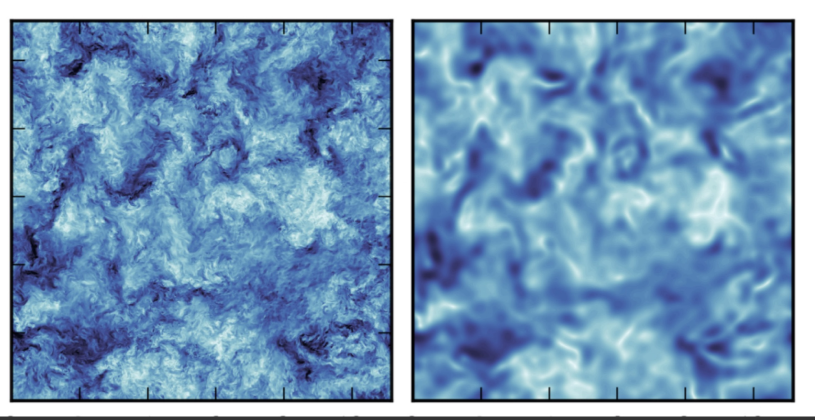
possessing at most 1/3 of a derivative.

To study energy cascade in weak solutions,

$$\bar{u}_\ell(x) = \int_{\mathbb{T}^d} G_\ell(r)\, u(x+r)\, dr, \quad G_\ell(r) = \ell^{-d} G\left(\frac{r}{\ell}\right)$$

or

$$u_K(x) = \mathbb{P}_{\leq K}[u] \quad \text{projection onto low freq.}$$



If $u \in L^\infty_t L^2_x$, then

$$\partial_t \tfrac{1}{2}|\bar{u}_\ell|^2 \longrightarrow \partial_t \tfrac{1}{2}|u|^2 \quad \text{as} \quad \ell \to 0$$

since
$$\left|\int |\bar{u}_\ell|^2 \partial_t \phi \, dx\,dt - \int |u|^2 \partial_t \phi\, dx\,dt\right| \quad \phi \in C_0^\infty$$

$$= \left|\int \bar{u}_\ell \cdot (\bar{u}_\ell - u)\partial_t \phi + \int u \cdot (\bar{u}_\ell - u)\partial_t \phi\right|$$

$$\lesssim |\partial_t \phi|_{L^\infty} |u|_{L^2} |\bar{u}_\ell - u|_{L^2} \to 0 \quad \text{since } \bar{u}_\ell \xrightarrow{L^2} u.$$

Coarse-grained dynamics:

$$\partial_t \bar{u}_\ell + \bar{u}_\ell \cdot \nabla \bar{u}_\ell = -\nabla \bar{p}_\ell - \nabla \cdot \tau_\ell(u,u)$$

$u(u\cdot\nabla u)$

$$\tau_\ell(u,u) := \overline{u\otimes u} - \bar{u}_\ell \otimes \bar{u}_\ell$$

$\sum_i A_{ik} B_{ik}$

Then, the resolved kinetic energy satisfies "$A:B$"

$$\frac{1}{2}\partial_t |u|_\ell^2 + \nabla\cdot\left(\left(\frac{1}{2}|\bar{u}_\ell|^2 + p_\ell\right)\bar{u}_\ell + \tau_\ell(u,u)\bar{u}\right) = -\Pi_\ell[u]$$

$$\Pi_\ell[u] = -\nabla \bar{u}_\ell : \tau_\ell(u,u)$$

Recall, by Calderon-Zygmund theory $\Delta^{-1}\nabla\otimes\nabla$ is a bounded operator $L^p \to L^p$ $p\in(1,\infty)$. Thus $u\in L^3_{t,x} \Rightarrow p = \Delta^{-1}\nabla\otimes\nabla(u\otimes u) \in L^{3/2}_{t,x}$

<u>Theorem</u> (Duchon-Robert 2000): If $u\in L^3(0,T;L^3)$, then

$$-\Pi[u] = -\lim_{\ell\to 0}\Pi_\ell[u]$$

$$= \frac{1}{2}\partial_t |u|^2 + \nabla\cdot\left(\left(\frac{1}{2}|u|^2 + p\right)u\right)$$

in the sense of distributions

<u>Remark</u>: One can show also that with

$$D_\ell[u] = \frac{1}{4\ell}\int_{\mathbb{T}^d} (\nabla G)_\ell(r)\cdot \delta_r u\, |\delta_r u|^2$$

that

$$\Pi[u] = \lim_{\ell\to 0} D_\ell[u]$$

in the sense of distributions.

Theorem: (Constantin - E - Titi, 1994) If $u \in L^3(0,T;L^3)$ and

$$\int_0^T \|\delta_\ell u\|_{L^3}^3 \, dt \leq C|\ell|^{1+}, \quad \text{"third}^+ \text{ of a derivative in } L^3\text{"},$$

then $\Pi[u] \equiv 0$. No anomalous dissipation.

Proof: $D_\ell[u] = \frac{1}{4|\ell|}\int_{\mathbb{T}^d} (\nabla G)_\ell(r) \cdot \delta_r u \, |\delta_r u|^2$

Thus $\int_0^T |D_\ell[u]|_{L^1} dt \leq |\nabla G|_{L^1} \frac{|\delta_\ell u|_{L^3}^3}{|\ell|} \xrightarrow{\ell \to 0} 0.$

Remark: $u \in B_p^{s,\infty}$ means $u \in L^p$ and $\sup_{|\ell|>0} \frac{1}{|\ell|^s} |\delta_\ell u|_{L^p} < \infty$.

$$u \in L^3(0,T; B_p^{s,\infty}(\mathbb{T}^d)) \qquad s > 1/3.$$

Remark: what is required for the theorem is

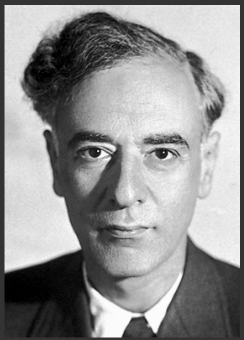
$$u \in L^3_{t,x} \quad \text{and} \quad \lim_{|\ell| \to 0} \frac{\int_0^T \|\delta_\ell u\|_{L^3}^3 dt}{|\ell|} \to 0$$

This is the sharpest criterion known for conservation.

$$u \in L^3(0,T; B_3^{1/3, c_0(\mathbb{N})}(\mathbb{T}^d))$$ critical Onsager space.

$u \in L^p,\ \|\delta_\ell u\|_{L^p}^p \lesssim |\ell|^{\zeta_p} \iff u \in L^p(0,T; B_p^{\zeta_p/p,\infty})$

Landau: 1942



the rate of energy dissipation is intermittent.

I.e., it is spatially / temporally inhomogeneous.

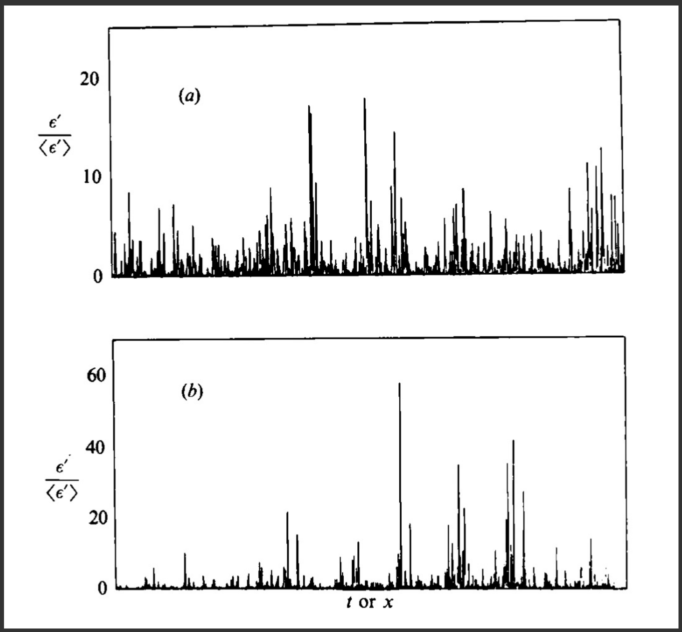
Thus ζ_p should not be a constant multiple of p.

i.e. $u \notin L^p(0,T; B_p^{1/3,\infty})$.

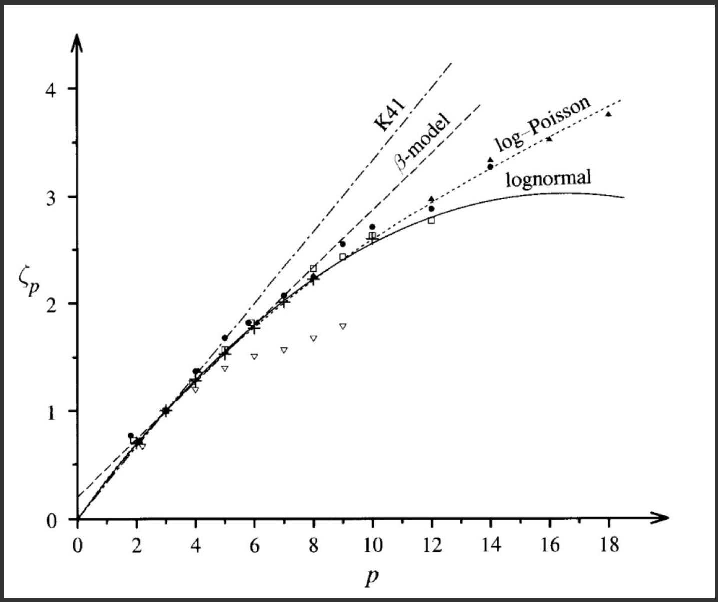
Meneveau & Sreenivasan 1991

surrogate:

$\varepsilon' = \left(\frac{du_1}{dt}\right)^2$



Frisch 1995



Dennis (yesterday)

THEOREM: (Isett, 2018): Let u be a weak solution of Euler of class $u \in L^p(0,T; B_p^{1/3,\infty})$ for some $p \geq 3$.
i.e., u has a $1/3$ derivative in L^p, then the distribution $\Pi[u]$

$$-\Pi[u] = \frac{1}{2}\partial_t |u|^2 + \nabla\cdot\left(\left(\tfrac{1}{2}|u|^2 + p\right)u\right)$$

is a (signed) measure. If, furthermore $p > 3$, then this measure is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is of class $\Pi[u] \in L^{p/3}_{t,x}$.

Thus, if u has $1/3$ derivative in L^p $p>3$, then the energy dissipation would need to take place on a full measure set.

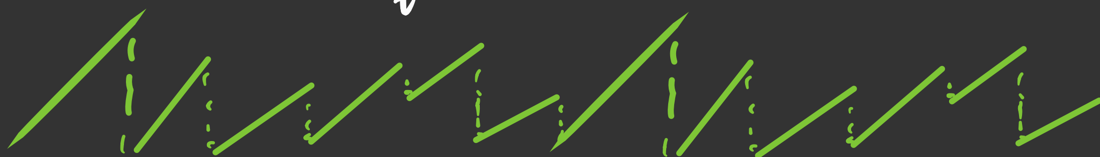
This, in accord with Landau's remark, would contradict experimental evidence of intermittent dissipation. Rigorous version of Landau's objection!

Best experimental measurements suggest dissipation takes place on of set of Hausdorff dimension $D < 3$
≈ 2.8

Meneveau & Sreenivasan 1991

Remark: Proof does not give absolute continuity in the case of $p=3$. Indeed, inviscid Burgers

$$u \in L^\infty_t(L^\infty_x \cap BV_x)$$



$$L^\infty \cap BV \subseteq B^{1/p,\infty}_p, \qquad p \geq 1.$$

and the dissipation takes place at just the shock locations (not absolutely continuous).

Proof: Recall that

$$-\Pi[u] = \frac{1}{2}\partial_t |u|^2 + \nabla\cdot\left(\left(\tfrac{1}{2}|u|^2 + p\right)u\right)$$

$$= \lim_{\ell\to 0} \nabla \bar{u}_\ell : T_\ell(u,u)$$

where $\quad T_\ell(u,u) = \overline{(u\otimes u)}_\ell - \bar{u}_\ell \otimes \bar{u}_\ell$

and $\quad \bar{f}_\ell = G_\ell * f \qquad G_\ell(r) = \frac{1}{\ell^d} G(r/\ell)$

Thus, by Hölder's inequality with $\frac{3}{p} = \frac{1}{p} + \frac{2}{p}$

$$\|\Pi_\ell\|_{L^{p/3}_{t,x}} \leq \|\nabla \bar{u}_\ell\|_{L^p_{t,x}} \|\bar{\tau}\|_{L^{p/3}_{t,x}}$$

$$\lesssim \left(\ell^{1/3-1} \|u\|_{L^p_t B^{1/3,\infty}_3}\right)\left(\ell^{2/3} \|u\|^2_{L^p_t B^{1/3,\infty}_3}\right)$$

$$\lesssim \|u\|^3_{L^p_t B^{1/3,\infty}_3}.$$

Consequently, the sequence $\{\Pi_\ell[u]\}_{\ell>0}$ is uniformly bounded in $L^{p/3}_{t,x}$ independent of $\ell > 0$.

Thus, using $p \geq 3$, the weak limit $\Pi[u] = \lim_{\ell \to 0} \Pi_\ell[u]$ is a Radon measure. More explicitly by (*) and Hölder (with the characteristic function of K as a factor), for any compact K and any test $\phi \in C_0^\infty(K)$:

$$|\langle \phi, \Pi[u]\rangle_{D'([0,T]\times\mathbb{T}^d)}| \leq C_K \|\phi\|_{C^0} \|u\|^3_{L^p_t B^{1/3,\infty}_3}.$$

Moreover, when $p>3$, the measure $\Pi[u]$ is absolutely continuous with density function in $L^{p/3}_{t,x}$.

This follows from the duality characterization of L^p. Namely, if $q \in (1,\infty)$ is the dual exponent

$$\frac{1}{q} + \frac{3}{p} = 1,$$

we have

$$|\langle \phi, \Pi[u] \rangle_{\mathcal{D}'((0,T)\times \mathbb{T}^d)}| \lesssim C \|\phi\|_{L^q_{t,x}} \|u\|^3_{L^p_t B^{1/3,\infty}_3}.$$

By density of test functions in L^q, we have that $\Pi[u]$ is in dual of $L^q_{t,s}$, which is $L^{p/3}_{t,x}$.

$\blacksquare$

Question: Can this argument be quantified:

If the dissipation is supported on a set of Hausdorff dimension D, must $u \notin L^p B^{\sigma_p,\infty}_p$ satisfy for some $\bar{\sigma} = \bar{\sigma}(D)$ that

$$\sigma_p \lesssim \bar{\sigma}(D) < 1/3 \quad \text{for} \quad p > 3$$

In fact, the information on the support of the dissipation can be used to get information on the singular support of u.

THEOREM (IScH, 2018) Let u be any weak solution of Euler in the class $u \in L^p_t B^{1/3,\infty}_p$ for some $p>3$ that does not conserve energy. Then u must be singular on a subset of spacetime with strictly positive $(d+1)$-dimensional Lebesgue measure.

This shows the necessary complexity of singularities in the Onsager endpoint class.

Corollary: Any solution with lower-dimensional singularity which does not conserve cannot be $L^p_t B^{1/3,\infty}_p$. $p>3$

Consistent with all available evidence!

LEMMA: Let u be a weak solution of Euler in the class $L^3_{t,x}$. Then the distribution $\Pi[u]$

$$-\Pi[u] = \frac{1}{2}\partial_t |u|^2 + \nabla\cdot\left(\left(\frac{1}{2}|u|^2 + p\right)u\right)$$

has support contained in the singular support of u relative to the critical space $L^3_t B^{1/3,c_0(\mathbb{N})}_3$.

DEF: The singular support of u relative to class of dist. X is the complement of those points $q=(t,x)$ for which $\exists$ an open neighborhood $\mathcal{O}_q$ of q on which u is represented by a generalized function of the class X.

Recall:

$$|u|_{B_p^{1/3,\infty}} := |u|_{L^p} + \sup_{|r|<1} \frac{|u(\cdot+r)-u(\cdot)|_{L^p}}{|r|^{1/3}}$$

and $B_p^{1/3,\, c_0(\mathbb{N})}$ is the class of all distributions s.t.

$$\lim_{|r|\to 0} \frac{|u(\cdot+r)-u(\cdot)|_{L^p}}{|r|^{1/3}} = 0 .$$

- The singular support of u relative to $L_+^p B_p^{1/3,\, c_0(\mathbb{N})}$ is a subset of the usual singular support of u as a distribution.

Proof of Theorem assuming Lemma:

Let $u \in L^3_{t,x}$ a weak solution of Euler such that

$e(t) = \frac{1}{2} \int |u(x,t)|^2 \, dx$ is not constant.

Then the distribution $\Pi[u]$:

$$-\Pi[u] = \frac{1}{2} \partial_t |u|^2 + \nabla \cdot \left(\left(\tfrac{1}{2}|u|^2 + p \right) u \right)$$

is well defined, and is nontrivial (not zero dist).

For a solution of class $u \in B^p_t B^{1/3,\infty}_p$ with $p > 3$, we have that $\Pi[u]$ is of class $L^{p/3}_{t,x}$.

Thus, for $\Pi[u]$ to be non-zero, the support of $\Pi[u]$ as a distribution must occupy a closed set with positive Lebesgue measure.

Since the non-trivial support of $\Pi[u]$ gives a lower bound for singular support of u as a distribution, this concludes the proof.

∎

Proof of Lemma:

Let $u \in L^3_{t,x}$ be a weak Euler solution, so that $p \in L^{3/2}_{t,x}$. Let q be a point in the complement of the s.s. relative to $L^3_t B_3^{1/3,c_0}$.

That is, $\exists$ an open neighborhood of $\mathcal{O}$ s.t. $u \in L^3_t B_3^{1/3,c_0}(\mathcal{O})$. Let $\phi \in C_c^\infty(\mathcal{O})$

Then by $-\Pi[u] = \lim_{\ell \to 0} \nabla \bar{u}_\ell : T_\ell(u,u)$

$$\langle \phi, -\Pi[u] \rangle_{\mathcal{D}'} = \lim_{\ell \to 0} \int \phi \, \nabla \bar{u}_\ell : T_\ell(u,u) \, dx\, dt$$

where, by assumption, $L^3_t B_3^{1/3,c_0}(\mathcal{O})$. Note:

$$(*) \quad |\langle \phi, -\Pi[\bar{u}] \rangle_{\mathcal{D}'}| \leq \limsup_{\ell \to 0} |\phi|_{C^0} \int_0^T |\nabla \bar{u}_\ell|_{L^3} \, |T_\ell(u,u)|_{L^{3/2}} dt$$

Note, integrand is bounded by $|u|^3_{B_3^{1/3,\infty}}$, which is integrable

Moreover, a.e. t one has that $u(t,\cdot) \in B_3^{1/3,c_0}$ belongs to closure of C^∞ in $B_3^{1/3,\infty}$ norm.

For each such t, $\limsup_{\ell \to 0} \ell^{1-1/3} |\nabla \bar{u}_\ell|_{L^3} \to 0$.

Combined with $|T_\ell(u,u)|_{L^{3/2}} \leq C |\ell|^{2/3}$, we have

$(*) = 0$ by dominated convergence theorem