Remarks on Anomalous Dissipation and Intermittency

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We are interested in the turbulent dynamics of slightly viscous flow governed by incompressible NS:
\[ \partial_t u + u \cdot \nabla u = - \nabla p + \nu \Delta u \]
\[ \nabla \cdot u = 0 \]

exhibiting anomalous dissipation

\[ E(0) - E(T) = \nu \int_0^T \int |\nabla u|^2 \, dx \, dt \geq 70. \]

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such “ideal” turbulence cannot obey any LIPSCHITZ condition of the form

\[ |v(r+r') - v(r')| < (\text{const.})r^n \]

for any order \( n \) greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description...

"Statistical Hydrodynamics" (1949)

\[ \text{Turbulence at small viscosity exhibiting anomalous dissipation is described by singular weak solution of Euler} \]
\[ \partial_t u + \nabla \cdot (u \otimes u) = - \nabla p, \quad \nabla \cdot u = 0 \]

possessing at most \( \gamma \) of a derivative.
To study energy cascade in weak solutions,

\[ \overline{u}_k(x) = \int_{\mathbb{R}^d} g_\varepsilon(r) u(x+r) \, dr, \quad G_\varepsilon(r) = \varepsilon^d G\left(\frac{r}{\varepsilon}\right) \]

or

\[ u_k(x) = P_k[u] \quad \text{projection onto low freq.} \]

If \( u \in L^\infty_t H^1_x \), then

\[ \partial_t \frac{1}{2} \overline{u}_k^2 \rightarrow \partial_t \frac{1}{2} u^2 \quad \text{as} \quad \varepsilon \rightarrow 0 \]

since

\[
\begin{align*}
\left| \int \overline{u}_k^2 \phi_x \, dx \, dt - \int u^2 \phi_x \, dx \, dt \right| &= \left| \int \overline{u}_k \cdot (\overline{u}_k - u) \phi \, dx \, dt + \int u \cdot (\overline{u}_k - u) \phi \, dx \, dt \right| \\
&\leq \|\phi\|_{L^\infty_t H^1_x} \|u\|_{L^2}^2 \|\overline{u}_k - u\|_{L^2} \\
&\rightarrow 0 \quad \text{since} \quad \overline{u}_k \rightharpoonup u.
\end{align*}
\]
course-grained dynamics:
\[
\frac{\partial}{\partial t} \tilde{u}_e + \nabla \cdot \tilde{u}_e \cdot \nabla \tilde{u}_e = -\nabla \tilde{p}_e - \nabla \cdot \tau_e(\eta,\nu)
\]
\[
\tau_e(\eta,\nu) := \nabla \tilde{u}_e - \tilde{u}_e \otimes \tilde{u}_e
\]

Then, the resolved kinetic energy satisfies:
\[
\frac{1}{2} \frac{\partial}{\partial t} |\tilde{u}_e|^2 + \nabla \cdot \left( \left( \frac{1}{2} |\tilde{u}_e|^2 + \tilde{p}_e \right) \tilde{u}_e + \tilde{\tau}_e(\eta,\nu) \tilde{u}_e \right) = -\Pi_{e} \tilde{u}_e
\]
\[
\Pi_{e} \tilde{u}_e = -\nabla \tilde{u}_e : \tilde{\tau}_e(\eta,\nu)
\]

Recall, by Calderon-Zygmund theory, \( \hat{\Delta} \hat{\nabla} \hat{\nabla} \) is a bounded operator \( L^p \rightarrow L^p \), \( p \in (1,\infty) \). Thus, \( u \in L^3 \Rightarrow \hat{p} = \Delta^{1/2} \nabla \nabla (\eta \nu) \in L^{3/2}
\]

**Theorem (Duchow-Robert 2000):** If \( u \in L^3(\omega_0; L^3) \), then

\[
-\Pi[u] = \lim_{\varepsilon \to 0} \Pi_{\varepsilon} [u]
\]
\[
= \frac{1}{2} \frac{\partial}{\partial t} |u|^2 + \nabla \cdot \left( \left( \frac{1}{2} |u|^2 + p \right) u \right)
\]
in the sense of distributions.

**Remark:** One can show also that with

\[
D_{e} [u] = \frac{1}{4 \pi} \int_{\omega} \left( \delta G \right)_2 \cdot \delta_{\varepsilon} u |\delta_{\varepsilon} u|^2
\]

that
\[
\Pi[u] = \lim_{\varepsilon \to 0} D_{\varepsilon} [u]
\]
in the sense of distributions.
Theorem: (Constantin - E. Titi, 1994) If \( u \in L^3(0, T; L^3) \) and
\[
\int_0^T \| \nabla u \|_{L^3}^2 \, dt \leq C \| u \|_{L^3}^{1+}, \quad \text{"third of a derivative in } L^3,\]
then \( \Pi u \equiv 0. \) No anomalous dissipation.

Proof: \( D_x \Pi u = \frac{1}{4} \int_\Omega (\nabla \cdot u) \cdot (\nabla \cdot \nabla u) \, dx \)

Thus \( \int_0^T \| D_x \Pi u \|_{L^3}^3 \, dt \leq \frac{1}{16} \| u \|_{L^3}^3 \| \nabla u \|_{L^3}^3 \rightarrow 0. \)

Remark: \( u \in B^{s_0}_p \) means \( u \in L^p \) and \( \sup_{t \in [0, T]} \frac{1}{t} \| u \|_{L^p} < \infty. \)
\( u \in L^3(0, T; B^{s_0}_p (\mathbb{T}^d)) \) \( s > \frac{1}{3}. \)

Remark: what is required for the theorem is
\( u \in L^3_{t,x} \) and \( \lim_{t \to 0} \int_0^t \| \nabla u \|_{L^3}^3 \, dt \rightarrow 0. \)

This is the sharpest criterion known for conservation.
\( u \in L^3(0, T; B_3^{1/3, \text{co}(W)} (\mathbb{T}^d)) \) critical Onsager space.
\[ \| u \|_{L^p} \leq \| u \|_{L^p} \leq \| u \|_{L^p} \leq \| u \|_{L^p} \leq \| u \|_{L^p} \]

Landau: 1942

The rate of energy dissipation is intermittent. I.e., it is spatially/temporally inhomogeneous. Thus, \( 3p \) should not be a constant multiple of \( p \), i.e. \( u \notin L^p(0, T; B^{3p/3}) \).

Meneveau & 1991

Sreenivasan

Surrogate: \[ \varepsilon' = \left( \frac{du}{dt} \right)^2 \]

Frisch 1995

\[ \zeta_p \]

\[ \zeta_p \]
THEOREM: (Jaarett, 2018): Let $u$ be a weak solution of Euler of class $u \in L^p(0,T; \mathcal{B}^{1/3})$ for some $p > 3$. i.e., $u$ has a $1/3$ derivative in $L^p$, then the distribution $\Pi[u]$ 

$$-\Pi[u] = \frac{1}{2} \text{div} u u^2 + \Delta \left( \frac{1}{2} |u|^2 + p \right) u$$

is a (signed) measure. If, furthermore $p > 3$, then this measure is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is of class $\Pi[u] \in L^{p/3}$.

Thus, if $u$ has $1/3$ derivative in $L^p$, $p > 3$, then the energy dissipation would need to take place on a full measure set.

This, in accord with Landau’s remark, would contradict experimental evidence of intermittent dissipation. Rigorous version of Landau’s objection!

Best experimental measurements suggest dissipation takes place on a set of Hausdorff dimension $D < 3$.

Meneveau & Sreenivasan 1991

2.8
Remark: Proof does not give absolute continuity in the case of $p=3$. Indeed, inviscid Burgers

\[ u \in L^\infty_x(L^\infty_y \cap BV_x) \]

\[ L^\infty_x \cap BV \subset B^{1/p, \infty}_p \text{, } p > 1. \]

and the dissipation takes place at just the shock locations (not absolutely continuous).

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Proof: Recall that

\[ -\Pi \bar{u} \mathbf{z} = \frac{1}{2} \bar{u} \mathbf{1} \mathbf{1}^2 + \nabla \cdot \left( \left( \frac{1}{2} \bar{u} \mathbf{1} \mathbf{1}^2 + p \right) u \right) \]

\[ = \lim_{\varepsilon \to 0} \nabla \bar{u}_e : \tau_e(u, u) \]

where \( \tau_e(u, u) = \overline{(u \circ \eta)_e - \bar{u}_e \circ \eta_e} \)

and \( \overline{f_e} = G_e \ast \mathbf{f} \quad \quad G_e(y) = \frac{1}{\varepsilon} e^{-G(y/e)} \)
Thus, by Hölder's inequality with $\frac{2}{p} = \frac{1}{p} + \frac{2}{p}$

$$\left| \prod_k l_k \right|_{L^{p/3}}^{1/3} \leq 10 \bar{u}_2 l_p \left( \sum |u|_{L_x L_y}^p \right)^{1/3} \sim \left( \ell^{1/3} \right)^{1/2} l_p B_{y,3}^2 \sim 10 \bar{u}_2 l_p B_{y,3}^2 \sim 10 \bar{u}_2 l_p B_{y,3}^2$$

Consequently, the sequence $\{ \prod_k [u]^3 \}_{k=0}^\infty$ is uniformly bounded in $L^{p/3}$ for independent of $k$. Thus, using $p=3$, the weak limit $\prod_k u_k = \frac{\lim_{l \to \infty} \prod_k u_k}{\lim_{l \to \infty} l_0 u_k}$ is a Radon measure. More explicitly, by (38) and Hölder (with the characteristic function of $K$ as a factor), for any compact $K$ and any test $\phi \in C^\infty_0(K)$:

$$\left< \phi, \prod_k u_k \right>_{D'(\mathbb{R}^3 \times T^d)} \leq C_K \| \phi \|_{C^\infty_0} \| u \|_{L^p B_{y,3}^2}^3$$
Moreover, when \( p > 3 \), the measure \( \Pi \in \mathcal{U} \) is absolutely continuous with density function in \( L^{p/3} \).

This follows from the duality characterization of \( L^p \). Namely, if \( q \in (1, \infty) \) is the dual exponent
\[
\frac{1}{q} + \frac{3}{p} = 1,
\]
we have
\[
| \langle \varphi, \Pi \mathcal{U}\mathcal{T} \rangle_{D'(\mathbb{R}^3 \times T^d)} | \leq C \| \varphi \|_{L^q} \| u \|_{L^p_{t,x}} \|
\]
By density of test functions in \( L^2 \), we have that \( \Pi \mathcal{U}\mathcal{T} \) is in dual of \( L^q \), which is \( L^{p/3} \).

**Question:** Can this argument be quantified?

If the dissipation is supported on a set of Hausdorff dimension \( D_1 \) must satisfy \( u \in L^p_{t,x} \) for

\[
\sigma_p \leq \bar{\sigma}(D) < \frac{1}{3} \quad \text{for} \quad p > 3
\]
In fact, the information on the support of the dissipation can be used to get information on the singular support of $u$.

**THEOREM (Isett, 2018)** Let $u$ be any weak solution of Euler in the class $u \in L^p B_{p, \infty}^{3/2}$ for some $p > 3$ that does not conserve energy. Then $u$ must be singular on a subset of spacetime with strictly positive $(d+1)$-dimensional Lebesgue measure.

This shows the necessary complexity of singularities in the Onsager endpoint class.

**Corollary:** Any solution with lower-dimensional $p > 3$ singularity which does not conserve cannot be $L^p B_{p, \infty}^{3/2}$. Consistent with all available evidence!

**LEMMA:** Let $u$ be a weak solution of Euler in the class $L^3_{4, \alpha}$. Then the distribution

$$-\Pi u u_{3} = \frac{1}{2} \partial_i u_{1} u^2 + \nabla \cdot (\frac{1}{2} u_{1} u^2 + p) u$$

has support contained in the singular support of $u$ relative to the critical space $L^3_{4, \beta_3_1}$. 


DEF: The singular support of \( u \) relative to class \( \text{dist } X \) is the complement of those points \( q \in (t,x) \) for which there exists an open neighborhood \( \mathcal{O}_q \) of \( q \) on which \( u \) is represented by a generalized function of the class \( X \).

Recall:

\[
\| u \|_{B_p^{1/3},\infty} := \| u \|_p + \sup_{|r| < 1} \frac{\| u(x+r) - u(x) \|_p}{|r|^{1/3}}
\]

and \( B_p^{1/3}, c_0(M) \) is the class of all distributions \( s.t.

\[
\lim_{|r| \to 0} \frac{\| u(x+r) - u(x) \|_p}{|r|^{1/3}} = 0.
\]

- The singular support of \( u \) relative to \( L^p B_p^{1/3}, c_0(M) \) is a subset of the usual singular support of \( u \) as a distribution.
Proof of Theorem assuming Lemma:

Let \( u \in L^3_{t,x} \) a weak solution of Euler such that
\[
e(\cdot) = \frac{1}{2} \int (u(x,\cdot))^2 \, dx
\]
is not constant. Then the distribution \( \Pi u \):
\[
- \Pi u = \frac{1}{2} \frac{d}{dt} |u|^2 + \nabla \cdot \left( (\frac{1}{2} |u|^2 + p) u \right)
\]
is well defined, and is nontrivial (not zero dist).

For a solution of class \( u \in B^p_{r,\infty} B^{1/3, \infty}_p \) with \( p > 3 \),
we have that \( \Pi u \) is of class \( L^{p/3}_{t,x} \).
Thus, for \( \Pi u \) to be non-zero, the support of \( \Pi u \) as a distribution must occupy a closed set with positive Lebesgue measure.

Since the nontrivial support of \( \Pi u \) gives a lower bound for singular support of \( u \) as a distribution, this concludes the proof. \( \Box \)
Proof of Lemma:

Let $u \in L^3_{\text{loc}}$ be a weak Euler solution, so that $p \in L^{3/2}_{\text{loc}}$. Let $q$ be a point in the complement of the S.S. relative to $L^3_{\text{loc}}$. That is, $q$ any open neighborhood of $\Theta$ s.t. $u \in L^3_{\text{loc}} (\Theta)$. Let $\phi \in C_0^\infty (\Theta)$.

Then by $-\Pi_\text{int} = \lim_{\epsilon \to 0} \nabla u_\epsilon : T_\epsilon (u, u)$

$$\langle \phi, -\Pi_\text{int} \rangle = \lim_{\epsilon \to 0} \int \phi \nabla u_\epsilon : T_\epsilon (u, u) \, dx$$

where, by assumption, $L^3_{\text{loc}} (\Theta)$. Note:

$$\left| \langle \phi, -\Pi_\text{int} \rangle \right| \leq \limsup_{\epsilon \to 0} \int (1 + | \nabla u_\epsilon |^2)^{1/2} 1_{T_\epsilon (u, u)} \, dx$$

Note, integrand is bounded by $|p|^2_{L^{3/2}_{\text{loc}}}$, which is integrable. Moreover, a.e. $t$ one has that $u(t, x) \in B^{1/3}_{3}$. Combining with $|p|^2_{L^{3/2}_{\text{loc}}}$, we have $\langle \phi \rangle = 0$ by dominated convergence theorem.