Anomalous Dissipation, Spontaneous Stochasticity & Onsager's Conjecture

by

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Abstract

Turbulence displays a number of remarkable features. It is a *super dissipator*, able to efficiently deplete its energy without the direct aid of viscosity. Non-vanishing energy dissipation in the limit of zero viscosity is termed *anomalous dissipation* and it is so fundamental to our modern understanding that it is ofter referred to as the "zeroth law of turbulence". Turbulent fluids are also exceptionally *strong mixers*, capable of very rapidly separating nearby particles within the flow. This is related to the phenomenon of spontaneous stochasticity, or the non-uniqueness of Lagrangian particle trajectories at infinite Reynolds number. Though seemingly distinct, these features are conjectured to be closely connected:

> "*There seems to be a strong relation between the behavior of the Lagrangian trajectories and the basic hydrodynamic properties of developed turbulent flows: we expect the appearance of non-unique trajectories for $Re \to \infty$ to be responsible for the dissipative anomaly, the direct energy cascade, the dissipation of higher conserved quantities and the pertinence of weak solutions of hydrodynamical equations at $Re = \infty$.*" — K. Gawędzki & M. Vergassola (2000)

This dissertation contains detailed and mathematically rigorous investigations of these properties of turbulence for number of hydrodynamic models, with a particular

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focus on establishing the connections conjectured above. Specifically:

(Chapter 2): We prove spontaneous stochasticity of trajectories backward-in-time for the Burgers equation. This is the first such proof for a deterministic PDE problem and the randomness of trajectories accounts for dissipation of all convex “energies”.

(Chapter 3): We prove short-time particle dispersion in coarse-grained fields is related to the turbulent energy cascade. The direction of the cascade (upscale/downscale) determines whether such particles spread faster forward or backward in time.

(Chapter 4) & (Chapter 5): We prove that spontaneous stochasticity is necessary and sufficient for anomalous scalar dissipation with any advecting velocity field whatsoever. Chapter 4 does this for domains without boundaries (e.g. tori, spheres), and chapter 5 extends the framework to wall-bounded flows. The proof exploits a novel Lagrangian fluctuation-dissipation relation for scalars, both passive and active.

(Chapter 6): We prove an Onsager-type singularity theorem which shows that dissipative anomalies can appear for strong limits of compressible Navier-Stokes solutions only if the limiting weak Euler solutions have low regularity of the type observed empirically in compressible turbulence.

Primary Reader: Gregory L. Eyink

Secondary Reader: Avanti Athreya

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Dedication

This thesis is dedicated to my grandparents, John and Despina Frangos and Theodoros and Iphygenia Drivas.

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Chapter 1

Introduction

Everything has been said before, but since nobody was listening, we have to start again.
— André Gide, Le traité du Narcisse (1891)

Turbulent flows are ubiquitous in the world around us, from trailing airplane wakes to swirling cream in our morning coffee. Despite its prevalence, basic questions about this complex non-linear phenomenon persist. Some notable features include:

1. *Enhanced Dissipation*,
2. *Strong Mixing*,
3. *(Un)predictability*.

This thesis contains an account of some detailed investigations of these three seemingly distinct features in a number of hydrodynamic models, with a focus on drawing connections between them.

The motion of a viscous incompressible fluid is governed by the Navier-Stokes equations. This is a system of $d+1$ equations (where d denotes the dimension of space) to determine a velocity field $\boldsymbol{u} := \boldsymbol{u}(\boldsymbol{x}, t)$:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1.2}$$

where the mass-density is constant and set to unity, ν is the kinematic viscosity which measures the 'stickiness' of the fluid and p is the pressure required to retain incompressibility. Equation (1.1) is Newton's law of motion, i.e. $m\boldsymbol{a} = \boldsymbol{f}$, written for infinitesimal parcels of fluid and the Eq. (1.2) is conservation of mass. These equations are simple to write, but hidden within them is a description of complex turbulent motions.

The Navier-Stokes equations (1.1) are non-linear; the influence of the nonlinearity on the fluid motion is determined by a competition between the strength of the inertial term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ relative to the linear friction term $\nu \Delta \boldsymbol{u}$. The dominant behavior depends on the length scale ℓ at which the dynamics is observed and is captured by the dimensionless *scale-based Reynolds number* defined as $Re(\ell) := \frac{\delta u(\ell)\ell}{\nu}$ where $\delta u(\ell)$ is the magnitude of the typical velocity difference across scale ℓ. If ℓ is the *integral scale*, L, representing a measure of the domain occupied by the fluid, then $Re(L) := Re$ is the Reynolds number. The scale η such that $Re(\eta) \approx 1$ is called the viscous (or Kolmogorov) scale and it represents the scale at which the inertial nonlinearity and dissipative linear term are of comparable strengths. The range of scales $\eta \ll \ell \ll L$ is called the *inertial range*. In the inertial range, the direct effects of viscosity and large scale forcing are not 'seen', and the effect of the inertial term on the dynamics is dominant. The relative importance of the various terms in (1.1) can be more clearly seen by scaling our variables using L as a length scale and $\delta u(L)$ as a velocity scale. These imply a scale $\tau_L := L/\delta u(L)$ which represents the time it takes for the largest eddy to substantially deform and thus it is called the "large-eddy turnover time". The non-dimensionalized (1.1) reads:

$$\tilde{\partial}_t \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = -\tilde{\nabla} \tilde{p} + \frac{1}{Re} \tilde{\Delta} \tilde{\boldsymbol{u}}. \tag{1.3}$$

Thus, we see that $1/Re$ controls the effect of the dissipation on the dynamics and therefore varying Re is mathematically equivalent to varying ν. As Re increases (ν decreases) the inertial range of scales increases, the non-linearity becomes ever more dominant and, as a result, turbulence is enhanced. The limit $Re \to \infty$, in the words of Lars Onsager, describes 'ideal' turbulence. Throughout this thesis, we are interested in the properties of ideal turbulent flow.

In §1.1 – 1.3, we discuss the characteristics: (1) enhanced dissipation, (2) strong mixing and (3) unpredictability. In these sections we briefly discuss the novel contributions of this thesis and postpone a more detailed outline to §1.4.

1.1 Energy Dissipation Anomaly & Onsager's Conjecture

Kinetic energy is not conserved for solutions of the Navier-Stokes equations governing viscous fluid dynamics. The energy balance for smooth solutions reads:

$$\partial_t \left(\frac{1}{2}|\boldsymbol{u}|^2\right) + \nabla \cdot \left[\left(\frac{1}{2}|\boldsymbol{u}|^2 + p\right)\boldsymbol{u} - \nu\nabla\left(\frac{1}{2}|\boldsymbol{u}|^2\right)\right] = -\nu|\nabla\boldsymbol{u}|^2. \tag{1.4}$$

Viscosity acts as a frictional force that serves as the only available mechanism to convert mechanical energy into internal energy by heating the fluid.

Remarkably, observations from experiments and simulations of fluid turbulence show that *kinetic energy dissipation is non-vanishing in the limit of zero viscosity*:

$$\langle\varepsilon(t)\rangle := \lim_{\nu\to 0}\int d\boldsymbol{x}\ \nu|\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 > 0. \tag{1.5}$$

There has been a wealth of studies, both numerical and experimental, which confirm this surprising phenomenon (e.g. Dryden [1943], Cadot et al. [1997], Sreenivasan [1984, 1998], Pearson et al. [2002], Kaneda et al. [2003]). Figure 1.1 shows a compilation of some recent numerical evidence.[1]

[1] We must make some qualifying remarks here regarding the state of experimental and numerical evidence for a *finite-time* dissipation anomaly starting with *smooth initial data*. The data presented in Figure 1.1 of obtained by Kaneda et al. [2003] represent a compilation of numerical experiments, all with different initial data and forcing schemes. Focusing on the data from the new simulations in that paper (denoted by symbols ■ and ▲), they represent the dissipation rates measured once the fluids have reached a statistically quasi-stationary state (judged by monitoring one-point statistics).

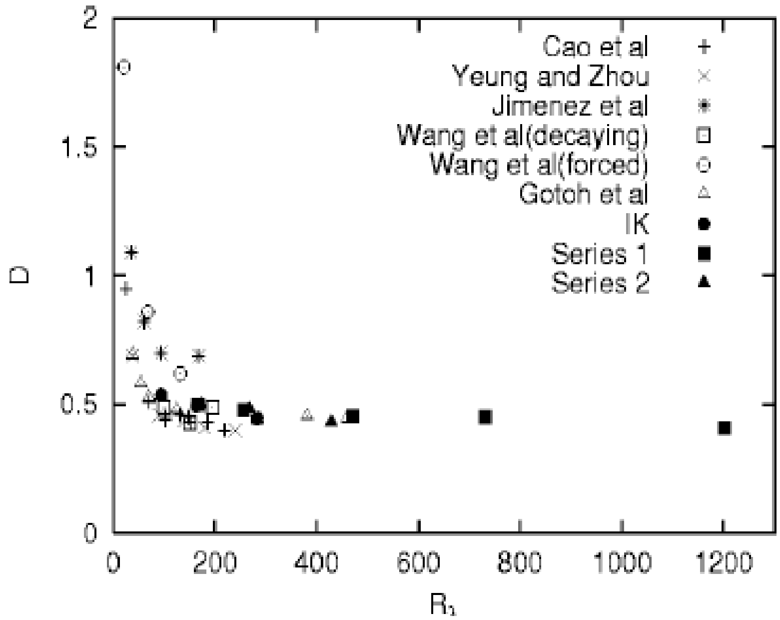

Figure 1.1: Evidence for the "zeroth law": direct numerical simulation results from Kaneda et al. [2003] showing the global dissipation "D" $:= \langle \varepsilon^\nu \rangle$, defined as a non-dimensionalized version of the energy dissipation rate tending to a constant as $R_\lambda \sim 1/\nu \to \infty$.

This phenomenon, dubbed *anomalous dissipation*, is so fundamental to our modern understanding of turbulence that it is often termed the "zeroth law."

Without viscosity, fluid motion is governed by the Euler equations (that is, Eq. (1.1) with $\nu = 0$), strong solutions of which conserve kinetic energy. This fact is seemingly at odds with the zeroth law. As a resolution to this apparent paradox, Lars

Higher Re simulations are initiated with the final slice of the lower Re simulations. Since these slices at the steady state display a Kolmogorov-type inertial range which extends for higher Re, this procedure corresponds to studying the initial value problem (IVP) with 'roughening' initial data. In particular, in the limit of infinite Re the data not smooth but only (presumably) has a Kolmogorov-type inertial range extending down to zero length scales. The time it takes for a given flow to reach such a state is typically on the order of a few large-eddy turnover times which is only weakly dependent on the Reynolds number. Thus, this data represents the limit as $Re \to \infty$ with dissipations measured at times $t(Re) \approx$ (const), initiated with data $\boldsymbol{u}_0$ such that $\|\nabla \boldsymbol{u}_0(Re)\| \to \infty$. In this way, singularities may effectively be 'introduced' in finite time by the initial data! For this reason, caution must be taken when interpreting the data displayed in Fig. 1.1 as direct evidence for a finite-time anomaly starting with smooth initial conditions.

Onsager [1949] conjectured that in 3D turbulent flows, energy dissipation may be non-vanishing in the zero-viscosity limit due to a lack of smoothness of the limiting *Euler* velocity field $\boldsymbol{u}$.[2]

> "...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available." — Onsager [1949]

Onsager realized that in order to be consistent with the observations of a dissipative anomaly, high-Reynolds number turbulent velocity fields must approximate weak Euler solutions $\boldsymbol{u}$ which satisfy a global energy balance of the form:

$$\frac{d}{dt}\int d\boldsymbol{x}\,\left(\frac{1}{2}|\boldsymbol{u}|^2\right) = -\int d\boldsymbol{x}\; D(\boldsymbol{u}). \tag{1.6}$$

In (1.6), the "defect" or "anomaly" $D(\boldsymbol{u})$ is a distribution that depends only on the weak Euler field $\boldsymbol{u}$. As we will see, it is related to Onsager's physical mechanism for the dissipation "without the aid of viscosity" that is observed experimentally. For smooth solutions, $D(\boldsymbol{u}) \equiv \boldsymbol{0}$, but this may fail to be the case for less regular weak solutions. To gain an intuition for this defect, one follows the same steps to derive the global energy balance for smooth solutions - by multiplying the Euler equations by $\boldsymbol{u}$ and integrating over the entire domain. Formally, this results in:

$$\frac{d}{dt}\int d\boldsymbol{x}\,\left(\frac{1}{2}|\boldsymbol{u}|^2\right) = -\int d\boldsymbol{x}\;\boldsymbol{u}\cdot\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}). \tag{1.7}$$

[2] Indeed, from the observation (1.5) we see that the velocity gradient must become singular $|\nabla\boldsymbol{u}| \sim \nu^{-1/2}$ as $\nu \to 0$ and so $\boldsymbol{u}$ cannot be differentiable in this limit. Onsager made a much more refined conjecture about the regularity, as will be discussed presently.

If the solution $\boldsymbol{u}$ is sufficiently smooth, an argument using integration-by-parts shows that the integrand on the right-hand-side of (1.7) is a total-divergence and thus vanishes by Gauss' theorem. Onsager realized that for this to fail and be consistent with anomalous dissipation $D(\boldsymbol{u}) > 0$, these solutions can possess essentially at most "a third of a derivative". Specifically, if there is anomalous dissipation in the zero viscosity limit, then velocity field cannot satisfy any bound of the form:

$$|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})| \leq (\text{const.})|\boldsymbol{r}|^{\alpha} \quad \text{for any} \quad \alpha > 1/3$$

i.e. it cannot be Hölder continuous $\boldsymbol{u} \notin C^{1/3+}$. Indeed, treating the gradient ∇ formally as a multiplicative operator, one observes

$$\int d\boldsymbol{x}\ \boldsymbol{u} \cdot \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \sim \int d\boldsymbol{x}\ (|\nabla|^{1/3}\boldsymbol{u})^3. \tag{1.8}$$

This suggests that if $\boldsymbol{u}$ is Hölder continuous with exponent 1/3, one can make sense of this term. Any better regularity would be sufficient to justify the integration-by-parts necessary to show that the right-hand-side of (1.7) vanishes and thus $D(\boldsymbol{u}) \equiv 0$.

Mathematically, *Onsager's conjecture(s)* can be precisely stated as follows. Let $\boldsymbol{u}$ be a weak (distributional) solution of the incompressible Euler equations. Then:

(weak): Any $\boldsymbol{u} \in C^{\alpha}$ with $\alpha > 1/3$ necessarily conserves energy.

(strong): There exists a $\boldsymbol{u} \in C^{\alpha}$ with $\alpha \leq 1/3$ which fails to conserve energy.

(physical): Such dissipative solutions $\boldsymbol{u}$ can arise from the zero-viscosity limit.

Eyink [1994] published the first rigorous results towards settling the **(weak)** form of Onsager's conjecture. Then Constantin et al. [1994] proved the sharp result that 3D Euler flows with regularity $L^3_t B_3^{1/3+,\infty}$ conserve energy. In later work, Duchon and Robert [2000] proved a beautiful result relating dissipative anomalies (1.5) that arise in vanishing viscosity limit to those (1.6) which may occur in weak Euler solutions:

Theorem 1.1.1 (Duchon and Robert [2000]) *Any weak solution of the Euler equations $\boldsymbol{u} \in L^3((0,T);L^3)$ which is also a strong limit of a sequence of dissipative weak solutions of Navier-Stokes $\boldsymbol{u}^\nu$ as $\nu \to 0$ has the following property:*

$$\lim_{\nu\to 0} \nu |\nabla \boldsymbol{u}^\nu|^2 = \lim_{\ell \to 0} \frac{1}{4\ell} \int \mathrm{d}\boldsymbol{r}\ (\nabla G)_\ell(\boldsymbol{r}) \cdot \delta\boldsymbol{u}(\boldsymbol{r}) |\delta \boldsymbol{u}(\boldsymbol{r})|^2 \tag{1.9}$$

where G is an arbitrary standard mollifier and the equality (1.9) *is interpreted in the sense of distributions.*

This theorem identifies $D(\boldsymbol{u})$ from (1.6) as, in the sense of distributions:

$$D(\boldsymbol{u}) = \lim_{\ell \to 0} \frac{1}{4\ell} \int \mathrm{d}\boldsymbol{r}\ (\nabla G)_\ell(\boldsymbol{r}) \cdot \delta\boldsymbol{u}(\boldsymbol{r}) |\delta \boldsymbol{u}(\boldsymbol{r})|^2. \tag{1.10}$$

This makes precise the heuristic (1.8). This formula actually appears in Onsager's unpublished notes and was derived by him sometime before his letter to C.C. Lin in 1945 (see Eyink and Sreenivasan [2006b]). The equality (1.10) depends only on the weak solution of Euler[3] but for zero-viscosity limits, Theorem 1.1.1 identifies $D(\boldsymbol{u})$ with the anomaly arising from viscous dissipation (plotted, for example, in Figure 1.1 above), which validates[4] Onsager's picture that the Euler equation has an inviscid mechanism that accounts for the observed viscous anomaly. As Duchon and Robert point out, the formula (1.9) is essentially a spatially local version of the celebrated Kolmogorov 4/5th law which is valid without the need to appeal to any notion of ensembles of random velocities or spatial homogeneity and isotropy. For a detailed discussion and precise correspondence, see Eyink [2002].

Work on the **(strong)** form of Onsager's conjecture has a long history beginning with a paper by Scheffer [1993], and later Shnirelman [1997] (although they were not motivated by Onsager's original work). They showed the existence of nontrivial weak Euler solutions with compact support in time. Pioneering work of De Lellis and Székelyhidi Jr [2009], [2010], [2012a,b], [2013] established a program to build Hölder continuous weak Euler solutions based on the Nash-Kuiper Theorem and Gromov's h-principle. These original constructions fell short of obtaining solutions in space $C^{1/3-}$, and there was much further work Buckmaster [2015], Buckmaster et al. [2013,

[3] A-priori $D(\boldsymbol{u})$ can have either sign. In particular, the weak Euler solution could as easily produce energy as dissipation since the Euler equations are time reversible. In fact, the original constructions of Shnirelman [1997] are of compact support in space-time and therefore do produce energy. However, the identification (1.9) shows that for weak solutions $\boldsymbol{u}$ obtained as vanishing viscosity limits, $D(\boldsymbol{u}) \geq 0$.

[4] The connection is not fully rigorous since the strong convergence assumption of the Theorem 1.1.1 is a very strong one and cannot yet be established from first principles. However, P. Isett pointed out in a remark during the IPAM workshop "Turbulent Dissipation, Mixing and Predictability" (January 2017) that such convergence follows from Besov regularity with exponent $0 < s \leq 1$ of Navier-Stokes solutions uniform in viscosity. This too cannot yet be established from first principles, but is entirely consistent with all known experimental evidence.

2015], Isett [2012], Isett and Oh [2016b,c] devoted to closing the gap. Finally, P. Isett [2016] resolved the strong form of Onsager's conjecture, proving that weak solutions of compact support in time exist in the class $C^{1/3-}$ and Buckmaster et al. [2017] showed that for any smooth positive energy profile (in particular, those that strictly decrease) there exists a corresponding weak Euler solution in the Onsager critical class.

It is important to note that none of these constructed solutions are obtained from the zero-viscosity limit. Thus, the **(physical)** form of Onsager's conjecture remains wide open and none of these results thus far are of direct relevance to experimental observations of high Reynolds number turbulence, e.g. to explain the observation of Figure 1.1. There exist, however, some partial results which establish the vanishing viscosity limit (at least along subsequences) if one admits weaker notions of what it means to be an "Euler solution". In particular, DiPerna and Majda [1987] introduced a notion of measure-valued weak solution which replaces $\boldsymbol{u}(\boldsymbol{x}, t)$ by a probability measure $\nu_{\boldsymbol{x},t}(d\boldsymbol{u})$ and Lions [1996] introduced "dissipative Euler solutions" by setting conditions based on a generalized error energy inequality. Crucially, both these notions of solutions satisfy weak-strong equivalence, proved by Brenier et al. [2011] and Lions [1996] respectively.

So far, our discussion has been restricted only to incompressible flow. Compressibility, however, is crucial to many applications in applied physics (internal confinement fusion), engineering (high-temperature reactive flows, internal combustion engines and supersonic aircraft) and astrophysics (interstellar medium, star formation). In Chapter 6 of this thesis, we extend Onsager's picture of ideal turbulence to compressible fluids. In particular, we relate possible inertial range anomalies for energy and entropy with dissipative anomalies arising in the high Re limit (analogous to Theorem 1.1.1) and prove a theorem about the required singularities to sustain dissipation without viscosity.

In the following sections we will discuss other special property of turbulent flows – strong mixing and unpredictability. Although these properties seem quite distinct from Onsager's conjecture and the dissipative anomaly, we will see in the course of this thesis that there are numerous close connections.

1.2 Mixing & Spontaneous Stochasticity

Meteorologist Lewis Fry Richardson [1926] observed that particle pairs advected by a turbulent flow (such as a pair of soot particles in a volcanic plume, Figure 1.2) have mean-square separation increasing with time as the cube power:

$$\langle |x_1(t) - x_2(t)|^2 \rangle \sim \langle \varepsilon \rangle t^3, \tag{1.11}$$

where $\langle \varepsilon \rangle$ is the same dissipation rate as in Eq. (1.5). The relationship (1.11) holds for intermediate times[5] in a turbulent flow and for particles initially separated by scales

[5] At shorter times, particles separate in ballistically – this is the so-called Batchelor regime. At very long times particles are diffusive with Taylor [1921] diffusivity given by the Lagrangian

Figure 1.2: Taken from the article *36 years ago, Mount St. Helens erupted, has helped scientists since.* which appeared in The Oregonian, 05/2016.

in or near the inertial range. One remarkable aspect of this observation is that, as Reynolds number increases (viscosity decreases), the inertial range of scales increases and particles can initially start closer and closer and yet, on average, spread to a distance independent of their initial separation. Extrapolating from this behavior, Bernard, Gawędzki and Kupiainen [1998] in a foundational paper inferred that *a Lagrangian fluid particle in a deterministic Navier-Stokes fluid nevertheless moves randomly at infinite Reynolds number even with infinitely precise initial data.* This is more chaotic than chaos!

This phenomenon, dubbed *spontaneous stochasticity*, is similar to anomalous dissipation in the respect that mathematically it is also a consequence of roughness of the velocity and is responsible for the strong mixing properties of turbulence. One way to

autocorrelation of the velocity field.

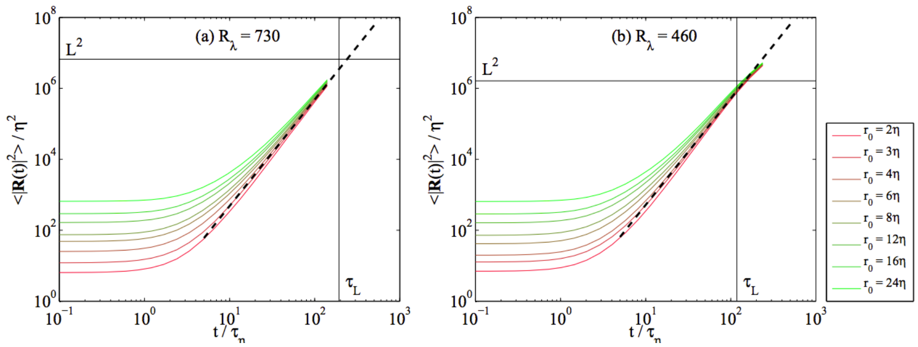

Figure 1.3: Figure taken from the paper Bitane et al. [2013]. The Reynolds number is increases going from right to left and the dashed line represents Richardson behavior of $\sim gt^3$. The fact that the plots look more or less identical while the Kolmogorov scales are correspondingly decreasing, means that particles start off closer together and reach finite distances independent of their separation sooner. Extrapolating, at high enough Reynolds number particles can start of arbitrarily close and reach finite distances arbitrarily soon – spontaneous stochasticity!

understand spontaneous stochasticity is by considering stochastically perturbed fluid trajectories in the limit of vanishing noise, together with the infinite Reynolds number limit. Given a vector field $\boldsymbol{u}^\nu$ which is smooth for all $\nu > 0$ but possibly becomes irregular as $\nu \to 0$, we define trajectories perturbed by an additive Brownian motion with strength $\sqrt{2\kappa}$ by:

$$d\boldsymbol{x}_t(\boldsymbol{a}) = \boldsymbol{u}^\nu(\boldsymbol{x}_t(\boldsymbol{a}), t)dt + \sqrt{2\kappa}\, d\mathbf{W}_t \tag{1.12}$$

$$\boldsymbol{x}_{t_0}(\boldsymbol{a}) = \boldsymbol{a} \tag{1.13}$$

If one takes the limit $\kappa \to 0$ with ν fixed, then standard arguments show that these noisy trajectories converge to the solution of $d\boldsymbol{x}_t(\boldsymbol{a}) = \boldsymbol{u}^\nu(\boldsymbol{x}_t(\boldsymbol{a}), t)dt$, which is necessarily unique because $\boldsymbol{u}^\nu$ is smooth. On the other hand, if $\boldsymbol{u} = \lim_{\nu\to 0} \boldsymbol{u}^\nu$ is sufficiently

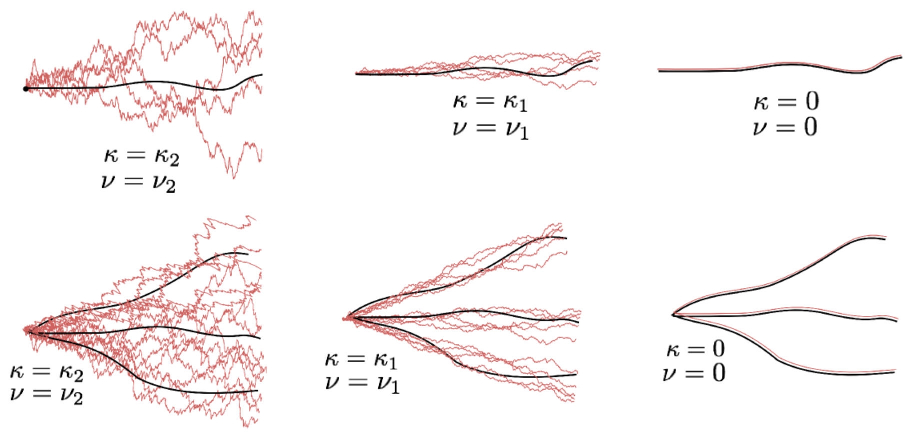

Figure 1.4: *Top*: if the advecting velocity $\boldsymbol{u}^\nu$ remains smooth in the limit as $\nu \to 0$, then there are necessarily unique characteristics of the limiting field and all the stochastic trajectories collapse around this as the noise vanishes i.e. $\kappa \to 0$.
Bottom: If, on the other hand, $\boldsymbol{u}^\nu$ is not Lipshitz in the limit $\nu \to 0$, there are possibly non-unique integral curves and as the noise vanishes, the particle remains 'spread out', concentrated along possibly infinitely many non-unique characteristics.

rough, then there may be non-unique solutions to the equation $d\boldsymbol{x}_t(\boldsymbol{a}) = \boldsymbol{u}(\boldsymbol{x}_t(\boldsymbol{a}), t)dt$ and the noisy trajectories need not collapse onto a unique curve but may remain spread out and stochastic in the joint limit $\nu, \kappa \to 0$. See Figure 1.4 for a cartoon of these possibilities.

Spontaneous stochasticity may be relevant to a range of real-world practical problems such as predicting and modeling the spread of oil in the oceans or pollutants in the atmosphere [Sawford et al., 2005] to explaining high-speed magnetic reconnection, see e.g. Eyink et al. [2013b] and Lalescu et al. [2015].

Although there is only limited empirical evidence of spontaneous stochasticity in Navier-Stokes turbulence (e.g. Figure 1.3), it has been theoretically demonstrated by Bernard et al. [1998] and rigorously proved to occur by Le Jan and Raimond [2002, 2004] in the Kraichnan [1968] model for turbulent advection. This model consists of a passive scalar advected by a realization of a Gaussian white noise velocity field which is only Hölder regular in space and which serves as a synthetic turbulent flow field. In Kraichnan's model there is a dissipative anomaly analogous to (1.5) for passive scalars in the zero-diffusion limit and its origin was shown to be spontaneous stochasticity. This fact demonstrated that, at least for the Kraichnan model, these two seemingly disparate phenomena are actually deeply connected.

In Chapter 2 of this thesis, we demonstrate the connection between anomalous dissipation and spontaneous stochasticity in the Burgers equation. More precisely, we find a relation between the sign of conservation-law anomalies in Burgers and spontaneous stochasticity backward in time. In turbulence language, the direct cascade of energy to small scales in Burgers is due to stochastic particle splitting backward in time. The empirical observations on particle dispersion in Navier-Stokes turbulence lead us to conjecture a deep relation between cascade direction and the time-asymmetry of particle dispersion. Indeed, 3D Navier-Stokes turbulence has a forward cascade of energy, just as Burgers, and likewise a faster particle dispersion backward in time than forward. This conjecture is supported by the numerical observation of a reversed asymmetry for the inverse energy cascade of 2D turbulence, with Richardson particle dispersion in inverse cascade instead faster forward in time than backward Sawford et al. [2005]. In Chapter 3, we explore this issue for short-time dispersion in coarse-grained fields, rigorously relating the cascade direction to enhanced forward/backwards dispersion.

Despite this connection between spontaneous stochasticity and anomalous dissipation observed now in both the Kraichnan model and Burgers, there was some doubt as to whether this mechanism would generalize beyond these models. In particular, there was a question as to whether the connection depends crucially on the white-in-time correlation of the Kraichnan velocity or the compressibility of the Burgers velocity field. In Chapters 4 and 5, we settle this issue by proving that spontaneous stochasticity is necessary and sufficient for anomalous dissipation of transported scalars and, moreover, that this holds, to some extent, for flows in containers with and without walls.

1.3 Predictability and Control of Turbulent Flows

Finally, we discuss the important issue of predictability of turbulent flows, with a focus on the connection to anomalous dissipation and spontaneous stochasticity.

A problem of great interest for both scientists and engineers is that of predicting, controlling and modeling high-Reynolds number turbulent flows. These issues are inseparably related to the uniqueness of the infinite Reynolds number limit. The Kraichnan model described in Section 1.2 is a good toy system to study these issues because there are infinitely many weak solutions of the ideal scalar equations advected by a rough Kraichnan velocity field. However, it was proved that spontaneous stochasticity provides a natural admissibility criterion which selects a unique viscosity solution [Bernard et al. [1998], Le Jan and Raimond [2002, 2004]]. These results led us to conjecture that for the Burgers equation, spontaneous stochasticity can be used to select unique entropy solutions [Eyink and Drivas [2015a]].

Similarly, for real fluids, recent work on the strong form of Onsager's conjecture described in Section 1.1 has shown that there is a huge zoo of non-unique dissipative weak solutions for given initial data. In particular, even energy dissipating solutions (a useful selection criteria for selecting entropy solutions of one-dimensional conservation laws) are not unique. One natural question is, how can we select the physically relevant velocity fields? Motivated by work on the Kraichnan model, Eyink [2006] conjectured a "martingale hypothesis". This hypothesis states that Euler solutions possessing certain statistical conservation laws related to 'enhanced mixing' selects those solutions achieved by the zero-viscosity limit. In more detail, Eyink conjectured that generalized Euler solutions obtained in the limit $\nu \to 0$ are selected by the property that their circulations are backwards martingales of the spontaneously stochastic flow:

$$\oint_C u_t \cdot d\ell = \mathbb{E} \oint_{\xi_t^{-1}(C)} u_0 \cdot d\ell \qquad \text{for all rectifiable loops } C. \tag{1.14}$$

Lending support to this conjecture, Constantin and Iyer [2008] proved that such a "stochastic Kelvin theorem" uniquely characterizes strong solutions of Navier-Stokes and this result was recently generalized by Rezakhanlou [2014] to a certain class of weak solutions.

However, it is also possible that some of this non-uniqueness seen in the weak Euler solutions is "real" and that physically relevant solutions with exactly prescribed initial data are not unique but random in the high Reynolds-number limit. In support of this idea, consider a result of Leith and Kraichnan [1972]. Assuming a closure hypothesis, they demonstrated a Richardson-type spreading of pairs of Navier-Stokes solutions in the L^2-space of velocities for 2D and 3D turbulence. They predicted

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{L^2}^2 \sim \langle\varepsilon\rangle t. \tag{1.15}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions with slightly different initial data and where $\langle\varepsilon\rangle$ is again same dissipation rate as in Eq. (1.5) and Eq. (1.11)! This relation is analogous to Richardson dispersion for Lagrangian particles (1.11). In particular, the separation predicted by (1.15) is also independent of the initial discrepancy of the velocities $\|\boldsymbol{u}_1(0) - \boldsymbol{u}_2(0)\|_{L^2}^2$ and is suggestive of 'spontaneous stochasticity' in the function space of velocities. In particular, adding small random perturbations to the Navier-Stokes initial data or equations (accounting for uncertainty in initial conditions or the effects of molecular noise on the dynamics) should yield Euler solutions that remain non-unique and stochastic, even as those perturbations vanish in the zero-viscosity limit. In fact, Ruelle [1979] showed that molecular noise can manifest in the dissipation range of turbulence and thence, the Leith–Kraichnan mechanism allows an inverse cascade of 'error' up to arbitrarily large scales.[6] The behavior (1.15) is difficult to test experimentally since it is often not possible to achieve detailed control of the initial or inflow conditions, but Boffetta and Musacchio [2001] have verified this prediction for two-dimensional turbulence using direct numerical simulation.

Recently Mailybaev [2015, 2016], proposed a picture of spontaneously stochastic solutions in the Gledzer-Ohkitani-Yamada (GOY) and Sabra shell models of turbulence. Shell models are simplified dynamical systems aimed at describing the behavior of a typical velocity fluctuation u_n at a scale $\ell \sim 2\pi/k_n$ where the wave numbers form 'shells' as a geometric progression $k_n = k_0 2^n$. The ideal (zero viscosity) GOY model form a singularity at a certain finite time, after which the dynamics are not a priori defined. A. Mailybaev showed through theoretical argument and careful numerics that the inviscid limit of viscous Sabra solutions along different subsequences yields different limiting solutions after singularity, thereby demonstrating that the solutions remain stochastic (see Figure 1.5). A. Mailybaev called this phenomenon of non-uniqueness a *stochastic anomaly*.

Stochastic anomalies, if they exist in real hydrodynamic turbulence, would have obvious implications for predictability and control of turbulent flows. In particular, if the above speculations for Navier-Stokes turbulence are correct, the evolution cannot

[6] This observation was made by Gregory Eyink (private communication) at the IPAM workshop on mathematical turbulence in September 2015.

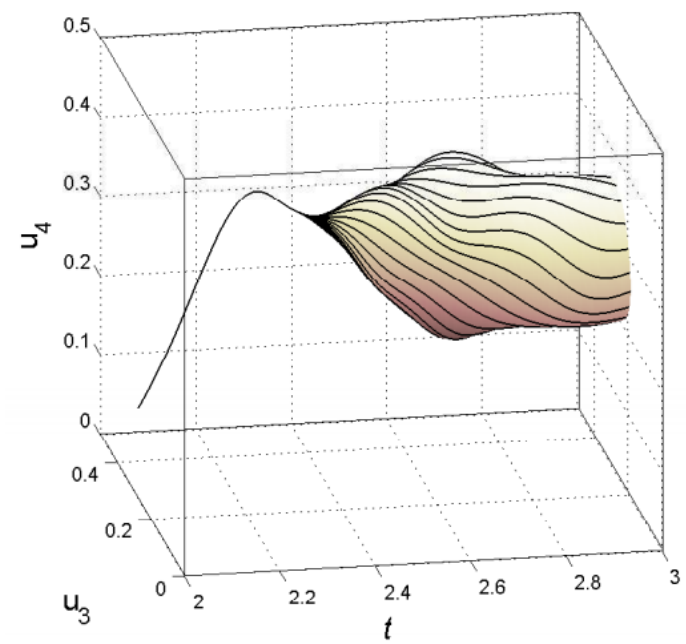

Figure 1.5: Plotted are many "ideal" evolutions of two shells of the GOY model. Different curves represents the evolution for distinct positive (but very small) viscosities. Before the blowup time, these solutions collapse. After blowup, minute changes in the viscosity lead to wildly different solution behavior. This figure is taken from the paper of Mailybaev [2016].

be determined even if the initial data is specified to infinite precision. However, it is possible that "almost every" possible evolution will share certain features and that novel statistical methods of prediction could be developed to exploit this. We do not discuss this difficult issue in any detail in this thesis, but its study will undoubtably be important for deepening our understanding hydrodynamic turbulence.

1.4 Outline

Below we break down the content of each chapter. Much of the content of these chapters is drawn from the associated published journal articles. However, some additional details not appearing in the published versions are included here, often as appendices, and Chapter 3 is based entirely on unpublished material. We now give a short summary of the contents of each chapter:

Chapter 2: Spontaneous Stochasticity and
Anomalous Dissipation in Burgers

Based on: Eyink and Drivas [2015a]

Pgs 28 – 76

We develop a Lagrangian approach to conservation-law anomalies in weak solutions of inviscid Burgers equation, motivated by previous work on the Kraichnan model of turbulent scalar advection. We show that the entropy solutions of Burgers possess Markov stochastic processes of (generalized) Lagrangian trajectories backward in time for which the Burgers velocity is a backward martingale. This property is shown to guarantee dissipativity of conservation-law anomalies for general convex functions of the velocity. The backward stochastic Burgers flows with these properties are not unique, however. We construct infinitely many such stochastic flows, both by a geometric construction (in Eyink and Drivas [2015a]) and by the zero-noise limit of the Constantin-Iyer stochastic representation of viscous Burgers solutions. The latter proof yields the spontaneous stochasticity of Lagrangian trajectories backward in time for Burgers, at unit Prandtl number. It is conjectured that existence of a backward stochastic flow with the velocity as martingale is an admissibility condition which selects the unique entropy solution for Burgers. We discuss the relation of our results for Burgers with incompressible Navier-Stokes turbulence, especially Lagrangian admissibility conditions for Euler solutions and the relation between turbulent cascade directions and time-asymmetry of Lagrangian stochasticity.

Chapter 3: Cascade Direction and Lagrangian
Time-Asymmetry in Turbulence
Pgs 76 – 113

We discuss the relationship between energy cascade direction and time-asymmetry of Lagrangian particles, focusing only on short time behavior. Our study is facilitated by a rigorous version of the Ott-Mann-Gawędzki relation – sometimes described as the "Lagrangian analogue of the 4/5ths law" – which remains valid for arbitrarily large Reynolds numbers. With this relation established, we investigate the short-time dispersion of trajectories in coarse-grained Navier-Stokes fields. Our version of the Ott-Mann-Gawędzki relation allows us to exactly identify the t^3 behavior in a short-time expansion of the particle dispersion with the energy flux-through-scale (cascade) term, thereby providing a rigorous justification for the relations appearing in Bitane et al. [2012], Falkovich and Frishman [2013] and Jucha et al. [2014], if these are interpreted for particles in coarse-grained fields. We then show that this flux-through-scale term matches on to the dissipative anomaly (1.5) for 3D turbulence (known by Onsager sometime before 1945 and first proved in the literature by Duchon and Robert [2000]), whereas for 2D we show that it matches on to a possible anomalous input from a force acting at infinitesimally small scales. Finally, using this result, we follow the argument of Jucha et al. [2014] and show that if the cascade is downscale as in 3D turbulence, particles initial spread faster backward-in-time whereas the reverse is true if the cascade is upscale, as in 2D turbulence. This rigorously establishes a connection between these phenomena which was conjectured by Eyink and Drivas [2015a], although the conjecture there was based on the properties of Richardson dispersion which occurs at longer times and is possibly distinct from the short time mechanism established here.

Chapter 4: A Lagrangian Fluctuation-Dissipation Relation
for Scalar Turbulence: Domains without Boundaries

Based on: Drivas and Eyink [2016]

Pgs 113 – 164

An exact relation is derived between scalar dissipation due to molecular diffusivity and the randomness of stochastic Lagrangian trajectories for flows without bounding walls. This "Lagrangian fluctuation-dissipation relation" equates the scalar dissipation for either passive or active scalars to the variance of scalar inputs associated to initial scalar values and internal scalar sources, as those are sampled backward in time by the stochastic Lagrangian trajectories:

$$\frac{1}{2t}\left\langle \text{Variance}\begin{pmatrix}\text{scalar inputs sampled}\\ \text{by noisy trajectories}\end{pmatrix}\right\rangle_{\text{space ave}} = \Big\langle \text{scalar dissipation} \Big\rangle_{\text{space}-\text{time ave}} \tag{1.16}$$

where the trajectories are perturbed tracer particles defined by (1.12). As an important application, we reconsider the phenomenon of "Lagrangian spontaneous stochasticity" or persistent non-determinism of Lagrangian particle trajectories in the limit of vanishing viscosity and diffusivity. Previous work on the Kraichnan (1968) model of turbulent scalar advection has shown that anomalous scalar dissipation is associated in that model to Lagrangian spontaneous stochasticity. There has been controversy, however, regarding the validity of this mechanism for scalars advected by an actual turbulent flow. We here completely resolve this controversy by exploiting the fluctuation-dissipation relation. For either a passive or active scalar advected by any divergence-free velocity field, including solutions of the incompressible Navier-Stokes equation, and away from walls, we prove that anomalous scalar dissipation requires Lagrangian spontaneous stochasticity. For passive scalars we prove furthermore that spontaneous stochasticity yields anomalous dissipation for suitable initial scalar distributions, so that the two phenomena are there completely equivalent. These points are illustrated by numerical results from a database of homogeneous, isotropic turbulence, which provide both additional support to the results and physical insight into the representation of diffusive effects by stochastic Lagrangian particle trajectories.

Chapter 5: A Lagrangian Fluctuation-Dissipation Relation
for Scalar Turbulence: Wall Bounded Domains

Based on: Drivas and Eyink [2017b]

Pgs 164 – 210

In Chapter 5, we derive Lagrangian fluctuation-dissipation relations for advected scalars in wall-bounded flows. The relations equate the dissipation rate for either passive or active scalars to the variance of scalar inputs from the initial values, boundary values, and internal sources, as those are sampled backward in time by stochastic Lagrangian trajectories. New probabilistic concepts are required to represent scalar boundary conditions at the walls: the boundary local-time density at points on the wall where scalar fluxes are imposed and the boundary first hitting-time at points where scalar values are imposed. These concepts are illustrated both by analytical results for the problem of pure heat conduction and by numerical results from a database of channel-flow flow turbulence, which also demonstrate the scalar mixing properties of near-wall turbulence. As an application of the fluctuation-dissipation relation, we examine for wall-bounded flows the relation between anomalous scalar dissipation and Lagrangian spontaneous stochasticity, i.e. the persistent non-determinism of Lagrangian particle trajectories in the limit of vanishing viscosity and diffusivity. In Chapter 4, we showed that spontaneous stochasticity is the only possible mechanism for anomalous dissipation of passive or active scalars, away from walls. Here it is shown that this remains true when there are no scalar fluxes through walls. Simple examples show, on the other hand, that a distinct mechanism of non-vanishing scalar dissipation can be thin scalar boundary layers near the walls. Nevertheless, we prove for general wall-bounded flows that spontaneous stochasticity is another possible mechanism of anomalous scalar dissipation. The framework we develope also allows us to connect the classical problem of Nusselt-Rayleigh scaling in Rayleigh-Bénard convection to the duration of a boundary mixing time of noisy particles. This elucidates the necessary mixing properties required for any scaling law $Nu \sim Ra^{\beta}$ and openes a new avenue for analysis of this long-standing problem. This work is described in Eyink and Drivas [2017a] but is not discussed in this thesis.

Chapter 6: An Onsager Singularity Theorem for Turbulent Solutions of Compressible Euler Equations

Based on: Drivas and Eyink [2017c]

Pgs 210 – 267

We prove that bounded weak solutions of the compressible Euler equations will conserve thermodynamic entropy, unless the solution fields have sufficiently low space-time Besov regularity. A quantity measuring kinetic energy cascade will also vanish for such Euler solutions, unless the same singularity conditions are satisfied. It is shown furthermore that strong limits of solutions of compressible Navier-Stokes equations that are bounded and exhibit anomalous dissipation are weak Euler solutions. These inviscid limit solutions have non-negative anomalous entropy production and kinetic energy dissipation, with both vanishing when solutions are above the critical degree of Besov regularity. Stationary, planar shocks with an ideal-gas equation of state provide simple examples of compressible Navier-Stokes solutions that satisfy the conditions of our theorems and which demonstrate sharpness of our L^3-based conditions. These conditions involve space-time Besov regularity, but we show that they are satisfied by Euler solutions that possess similar space regularity uniformly in time. Two other papers which extensively discuss the physics of these anomalies and their connection to turbulent cascades are Eyink and Drivas [2017b,c], but these are not discussed in detail in this thesis.

Chapter 2

Spontaneous Stochasticity and Anomalous Dissipation in Burgers

2.1 Introduction

As discussed in the Introduction, energy dissipation in incompressible Navier-Stokes turbulence is, within experimental errors, independent of viscosity at sufficiently high Reynolds numbers. This empirical observation motivated Onsager in 1949 to conjecture that incompressible fluid turbulence is described by singular (distributional) Euler solutions that dissipate energy by a nonlinear cascade mechanism [Onsager, 1949, Eyink and Sreenivasan, 2006a]. Physically, the Lagrangian interpretation of the Eulerian energy cascade is usually in terms of Taylor's vortex-stretching picture [Taylor and Green, 1937, Taylor, 1938]. However, Taylor's ideas depend on the validity of the Kelvin circulation theorem, which is very unlikely to hold in the conventional sense for high-Reynolds-number turbulent fluids [Lüthi et al., 2005, Guala et al., 2005, Eyink, 2006]. Mathematically, Onsager's conjectured Euler solutions have not yet been obtained as zero-viscosity limits of Navier-Stokes solutions. While weak Euler solutions have been constructed which dissipate kinetic energy and have the spatial Hölder regularity of observed turbulent fields (see discussion of the "strong" version of Onsager's conjecture in Introduction), such solutions are wildly non-unique.

These problems have been resolved, on the other hand, in a toy turbulence model, the Kraichnan model of passive scalar advection by a Gaussian random velocity field which is white-noise in time and rough (only Hölder continuous) in space [Kraichnan, 1968, Falkovich et al., 2001]. In this model there is anomalous dissipation of the scalar energy due to a turbulent cascade process. In Lagrangian terms the turbulent dissipation is explained by "spontaneous stochasticity" of the fluid particle trajectories, associated to Richardson explosive dispersion [Bernard et al., 1998]. As quoted in the epigraph of the thesis abstract, Gawędzki and Vergassola [2000] suggested that this non-uniqueness and intrinsic stochasticity of Lagrangian trajectories should also underlie the anomalous dissipation in weak solutions of hydrodynamic equations relevant to actual fluid turbulence. Weak solutions of the passive advection equation in the Kraichnan model have been rigorously constructed and shown to coincide with solutions obtained by smoothing the velocity or adding scalar diffusivity and then removing these regularizations [E and Vanden-Eijnden, 2000, 2001, Jan and Raimond, 2002, 2004]. One can characterize these weak solutions by the property that the scalar values are backward martingales for Markov random processes of Lagrangian trajectories.

This successful theory for the Kraichnan model motivated one of us to conjecture a similar "martingale hypothesis" for fluid circulations in the weak solutions of incompressible Euler equations that are believed to be relevant for turbulence [Eyink, 2006]. For smooth solutions of Euler equations, the backward martingale property reduces to the usual Kelvin Theorem on conservation of circulations. However, for singular solutions it imposes an "arrow of time" which was proposed as an infinite set of admissibility conditions to select "entropy" solutions of the Euler solutions [Eyink, 2007, 2010]. This conjecture assumes spontaneous stochasticity in high-Reynolds Navier-Stokes turbulence, for which numerical evidence has been obtained in studies of 2-particle dispersion [Sawford et al., 2008, Eyink, 2011, Bitane et al., 2013]. Subsequently, in very beautiful work, Constantin and Iyer [2008] established a characterization of the solutions of the incompressible Navier-Stokes solutions as those space-time velocity fields for which the fluid circulations are backward martingales of a stochastic advection-diffusion process (see also [Eyink, 2010]). This "stochastic Kelvin Theorem" is the exact analogue for Navier-Stokes of the property proposed earlier for entropy solutions of Euler equations. Of course, the result for the Navier-Stokes equation does not imply its analogue for the Euler equations and, at this time, the zero-viscosity limit is so poorly understood that a mathematical proof (or counterexample) for Euler does not seem to be forthcoming.

There are simpler PDE problems, however, where the zero-viscosity limit is much better understood. These include scalar conservation laws in one space dimension [Bressan, 2012, Dafermos, 2006], with the Burgers equation [Burgers, 1939, Bec and Khanin, 2007] as a prominent example. The scalar conservation laws possess weak solutions that are uniquely selected by entropy admissibility conditions and which coincide with solutions obtained by the zero-viscosity limit. The Burgers equation, in particular, has long been a testing ground for ideas about Navier-Stokes turbulence[1]. It is therefore a natural question whether the known entropy solutions of inviscid Burgers satisfy a version of the martingale property conjectured for "entropy solutions" of incompressible Euler. Since smooth solutions of inviscid Burgers preserve velocities along straight-line characteristics, the natural conjecture for Burgers is that the Lagrangian velocity is a backward martingale. As a matter of fact, Constantin and Iyer [Constantin and Iyer, 2008] established exactly such a characterization of the solutions of the viscous Burgers equation. In order for such a representation to hold also for the zero-viscosity limit, there must be a form of "spontaneous stochasticity" for Burgers flows. It has been argued that these flows are only coalescing and that stochastic splitting is absent [Bauer and Bernard, 1999]. This is true,however, only

[1]Note, furthermore, that the general scalar conservation law in one-dimension $u_t + (f(u))_x = 0$ with a strictly convex flux function f is equivalent for smooth solutions to Burgers equation for the associated velocity field $v = f'(u)$. This equivalence extends to viscosity-regularized equations in a slightly modified form. A simple calculation shows that $u_t + (f(u))_x = \varepsilon\ u_{xx}$ is equivalent to $v_t + \left[\frac{1}{2}v^2 + \varepsilon g(v)\right]_x = \varepsilon\ v_{xx}$, where $g(v) = 1/\hat{f}''(v)$ and $\hat{f}(v)$ is the Legendre dual of the convex function $f(u)$. Hence, an entropy solution u of $u_t + (f(u))_x = 0$ should give an entropy solution $v = f'(u)$ of Burgers, and inversely.

forward in time. The natural martingale property involves instead flows backward in time and it is plausible that there should exist a suitable stochastic inverse of the forward coalescing flow.

A main result of this chapter is that there are indeed well-defined Markov inverses of the forward coalescing flows for the entropy solutions of inviscid Burgers, such that the Burgers velocity is a backward martingale of these stochastic processes. This result implies a stochastic representation of the standard entropy solutions of inviscid Burgers exactly analogous to the Constantin-Iyer (C-I) representation of viscous Burgers solutions. Interestingly, there is more than one way to construct such a stochastic inverse (in contrast to the Kraichnan model, where the stochastic process of backward Lagrangian trajectories appears to be essentially unique [Jan and Raimond, 2002, 2004]) [2]. We obtain one set of stochastic inverses by a direct geometric construction, closely related to recent work of Moutsinga [Moutsinga, 2012]. We obtain another stochastic inverse by the zero-viscosity limit of the backward diffusion processes in the Constantin-Iyer representation, demonstrating spontaneous stochasticity for Burgers flows backward in time at unit Prandtl number. The stochastic inverse flows we obtain are (backward) Markov jump-drift processes supported on generalized solutions of the Lagrangian particle equations of motion (generalized characteristics

[2] The "essential uniqueness" is that of the stochastic backward process for a given weak solution of the passive-scalar advection equation. The Kraichnan model for an intermediate regime of compressibility has distinct weak solutions in the simultaneous limit $\nu, \kappa \to 0$, obtained by holding fixed different values of the "turbulent Prandtl number" [E and Vanden-Eijnden, 2000, 2001]. The backward stochastic process is uniquely fixed by that limit, however, which fully specifies the boundary conditions at zero-separation. We shall see that the case is otherwise with Burgers, which has infinitely many distinct stochastic inverse flows for the same, unique dissipative weak solution.

in the sense of Dafermos [Dafermos, 2006].) Although not themselves unique, each constructed backward stochastic flow enjoys the properties discussed above and provides a representation of the unique entropy solutions of Burgers. Furthermore, we show that the backward martingale property of the Burgers velocity is exactly what is required to make the solutions dissipate convex entropies. For this purpose, we derive a novel Lagrangian formula for inviscid Burgers dissipation. We conjecture that existence of a stochastic process of generalized characteristics with the backward martingale property for velocities is an admissibility condition for inviscid Burgers which uniquely selects the standard entropy solution.

Finally, we discuss the importance of the time-irreversibility of Burgers equation and the associated differences with the time-reversible Kraichnan model. We suggest that the direction of turbulent cascades is related generally in irreversible fluid models to the time-asymmetry of Lagrangian particle behavior.

2.2 Lagrangian Formulation of Anomalous Dissipation

We derive a Lagrangian expression for dissipative anomalies of inviscid Burgers.

2.2.1 Basic Burgers Facts

Before beginning, we remind the reader of some standard results about Burgers, many quite elementary (see also Evans [1988]). For example, see [Bec and Khanin, 2007]. Let u be a smooth solution of the inviscid Burgers equation for initial data u_0 at time t_0. Using the standard method of characteristics, one can see that

$$x = a + (t - t_0)u_0(a), \quad u(x,t) = u_0(a). \tag{2.1}$$

Note that

$$\xi_{t_0,t}(a) = a + (t - t_0)u_0(a)$$

is the Lagrangian flow map of fluid mechanics, with inverse $\alpha_{t_0,t} = \xi_{t_0,t}^{-1}$ the "back-to-labels"' map so that $u(x,t) = u_0(\alpha_{t_0,t}(x))$. All of the following are simple consequences of (2.1):

$$u'(x,t) = u_0'(\alpha_{t_0,t}(x))\alpha_{t_0,t}'(x)$$

$$\xi_{t_0,t}'(a) = 1 + (t - t_0)u_0'(a)$$

$$\alpha_{t_0,t}'(x) = 1 - (t - t_0)u'(x,t) = [\xi_{t_0,t}'(a)]^{-1}$$

and thus

$$u'(x,t) = \frac{u_0'(a)}{1 + (t - t_0)u_0'(a)}.$$

It follows from the latter formula that, wherever $u'(a) < 0$ at any initial point a, a shock will form in finite time from smooth initial data $u_0(a)$. The singularity will occur (unless the particle is absorbed first by another shock) at time

$$t = t_0 + \frac{1}{\max\{0, -u_0'(a)\}}.$$

The first shock occurs at the minimum of the above quantity, related to the maximum of the negative velocity gradient. At later times, all of the previous results for smooth solutions are valid at points between shocks.

We consider Burgers solutions of bounded variation with countably many shocks located at coordinates $\{x_i^*\}_{i=1}^{\infty}$ at time t. Let u_i^- be the velocity immediately to the left of the ith shock and u_i^+ the velocity immediately to the right. The *Rankine-Hugoniot jump conditions* require that the shock velocity $u_i^* = dx_i^*/dt$ for any weak solution be an average:

$$u_i^* = \frac{u_i^- + u_i^+}{2}. \tag{2.2}$$

Entropy solutions of inviscid Burgers have the property $u_i^- > u_i^+$. As a matter of fact, it is well-known that the energy conservation anomaly at a Burgers shock is $\frac{1}{12}(u_i^+ - u_i^-)^3$, which is negative (dissipative) precisely when $u_i^- > u_i^+$. This is also the Lax admissibility condition for weak solutions [Lax, 1957] in the context of Burgers. Thus, each shock corresponds to a Lagrangian interval $[a_i^-, a_i^+]$ such that $u_i^\pm = u_0(a_i^\pm)$

and

$$x_i^* = a_i^- + tu_i^- = a_i^+ + tu_i^+. \tag{2.3}$$

The union of shock intervals in the Lagrangian space is denoted below as $S = \bigcup_{i=1}^{\infty}[a_i^-, a_i^+]$.

2.2.2 Dissipative Anomalies

Our goal in this section is to derive fundamentally Lagrangian expressions for dissipative anomalies in inviscid Burgers, analogous to those obtained for integral invariants of passive scalars in the Kraichnan model [Bernard et al., 1998, Gawędzki and Vergassola, 2000] . Thus let ψ be a continuous function and Ψ its anti-derivative. Take $t_0 = 0$ for simplicity. Then

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) &= \int_{\mathbb{R}\backslash\{x_i^*\}_{i=1}^{\infty}} \mathrm{d}x\ \psi(u(x,t)) = \int_{\mathbb{R}\backslash\{x_i^*\}_{i=1}^{\infty}} \mathrm{d}x\ \psi(u_0(\alpha_{t_0,t}(x))) \\
&= \int_{\mathbb{R}\backslash S} \mathrm{d}a\ \psi(u_0(a))\ \xi'_{t_0,t}(a) \\
&= \int_{\mathbb{R}} \mathrm{d}a\ \psi(u_0(a))\big(1 + tu_0'(a)\big) - \int_S \mathrm{d}a\ \psi(u_0(a))\big(1 + tu_0'(a)\big) \\
&= \int_{\mathbb{R}} \mathrm{d}a\ \psi(u_0(a)) + t\int_{\mathbb{R}} \mathrm{d}a\ \frac{d}{da}\Psi(u_0(a)) - \int_S \mathrm{d}a\ \psi(u_0(a))\big(1 + tu_0'(a)\big) \\
&= \int_{\mathbb{R}} \mathrm{d}a\ \psi(u_0(a)) - \int_S \mathrm{d}a\ \psi(u_0(a))\big(1 + tu_0'(a)\big)
\end{aligned}$$

We used the assumption $\lim_{a\to\pm\infty} u_0(a) = u_\infty$ to set $\int_{\mathbb{R}} \mathrm{d}a\ \frac{d}{da}\Psi(u_0(a)) = 0$. We see that $\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t))$ is conserved for a smooth Burgers solution, when $S = \emptyset$.

We now consider the case of weak solutions with shocks. We can rewrite the second term:

$$\begin{aligned}\int_S \mathrm{d}a\ \psi(u_0(a))\big(1 + tu_0'(a)\big) &= \int_S \mathrm{d}a\ \psi(u_0(a)) + t\int_S \mathrm{d}a\ \psi(u_0(a))u_0'(a)\\ &= \sum_{i=1}^{\infty}\left[\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a)) + t\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a))u_0'(a)\ \right]\\ &= \sum_{i=1}^{\infty}\left[\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a)) - t\int_{u_i^+}^{u_i^-} \mathrm{d}u\ \psi(u)\right]\end{aligned}$$

Thus,

$$\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) - \int_{\mathbb{R}} \mathrm{d}a\ \psi(u_0(a)) = -\sum_{i=1}^{\infty}\left[\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a)) - t\int_{u_i^+}^{u_i^-} \mathrm{d}u\ \psi(u)\right] \quad (2.4)$$

The right-hand side is a Lagrangian representation of the conservation law anomaly.

A Burgers solution u is a *dissipative* if, for any convex function ψ,

$$\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a)) \geq t\int_{u_i^+}^{u_i^-} \mathrm{d}u\ \psi(u), \quad i = 1, 2, \ldots \quad (2.5)$$

Dividing by $a_i^+ - a_i^-$ and using the relationship (2.3), this is equivalent to

$$\frac{1}{a_i^+ - a_i^-}\int_{a_i^-}^{a_i^+} \mathrm{d}a\ \psi(u_0(a)) \geq \frac{1}{u_i^- - u_i^+}\int_{u_i^+}^{u_i^-} \mathrm{d}u\ \psi(u), \quad i = 1, 2, \ldots \quad (2.6)$$

Since both $\psi(u) = u$ and $\psi(u) = -u$ are convex functions, any dissipative solution must satisfy the relation

$$\frac{1}{a_i^+ - a_i^-} \int_{a_i^-}^{a_i^+} u_0(a)\, da = \frac{1}{2}(u_i^- + u_i^+), \tag{2.7}$$

which will prove fundamental to our later work. Note that (2.7) is equivalent to the standard "Maxwell construction" of the dissipative solution at shocks, in which one chooses the Lagrangian map of the weak solution to satisfy $\xi^*_{t_0,t}(a) = x_i^*(t)$ for $a \in [a_i^-, a_i^+]$, under the constraint

$$\int_{a^-}^{a^+} da\, \left[\xi_{t_0,t}(a) - \xi^*_{t_0,t}(a)\right] = 0, \tag{2.8}$$

with $\xi_{t_0,t}(a) = a + u_0(a)t$ the naive Lagrangian map [Bec and Khanin, 2007]. To see this, substitute the definitions of the maps ξ and integrate to give an equivalent expression of the Maxwell construction as

$$x_i^*(t) = \frac{1}{2}(a_i^- + a_i^+) + \frac{t}{a_i^+ - a_i^-} \int_{a_i^-}^{a_i^+} u_0(a)\ da.$$

On the other hand, the average of the two expressions in (2.3) gives

$$x_i^*(t) = \frac{1}{2}(a_i^- + a_i^+) + \frac{t}{2}(u_i^- + u_i^+), \tag{2.9}$$

from which (2.7) is obviously equivalent to (2.8).

We now show that (2.9) with $u_i^- > u_i^+$ implies (2.5), at any final time t_f. Since the argument applies to every shock, we hereafter drop the i subscript. The argument is best understood graphically, so we refer to the Fig.1 below which plots a typical Burgers shock:

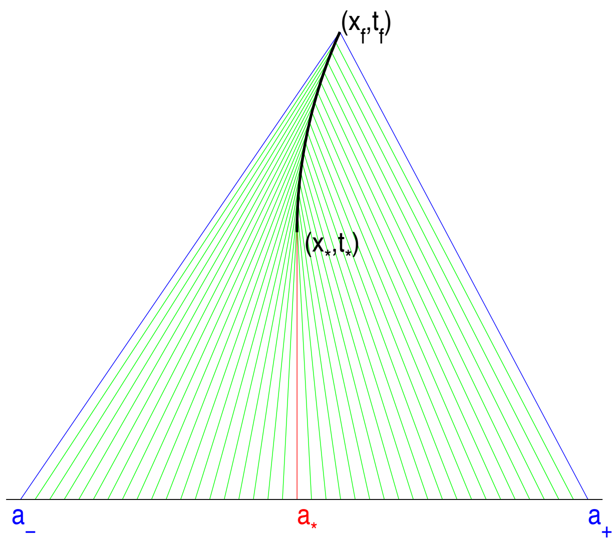

Figure 2.1: *Spacetime Plot of a Burgers Shock.* Shown in green are the straight lines corresponding to the smooth particle motions. These converge onto the shock curve in **black**, which begins at (x_*, t_*) and ends at (x_f, t_f), the final time considered. On the abscissa is the space of Lagrangian positions at time 0, showing the shock interval $[a_-, a_+]$ and, in red, the label a_* and straight-line characteristic where the shock first forms at time t_*.

Note that the straight characteristic passing through the initial point with label a has slope equal to $1/u_0(a)$. Thus, this graph represents the configuration used to obtain the average on the left-hand side of (2.6). On the other hand, the right-hand side of (2.6) is obtained from a uniformized configuration in which the true initial velocity

$u_0(a)$ at each point a is replaced by an "apparent initial velocity" $(x_f^* - a)/t_f$. This configuration is represented in Fig.2 below by the straight line drawn from each point $(a, 0)$ to the final point (x_f^*, t_f). The inequality in (2.6) is the statement that the uniform distribution on the velocity interval $[u_+, u_-]$ is less spread out than the distribution of the true initial velocity, as measured by the convex function ψ. To show this, we can gradually "lift" the characteristic lines along the shock curve $x_*(s)$ from $s = t_*$ to $s = t_f$. We can expect that the integral is successively decreased by this operation. To make this argument analytically, we introduce the function

$$\Delta_\psi(s) = \int_{a_-(s)}^{a_+(s)} \psi\left(\frac{x_*(s) - a}{s}\right)\, \mathrm{d}a + \int_{[a_-, a_-(s)] \cup [a_+(s), a_+]} \psi(u_0(a))\, \mathrm{d}a,$$

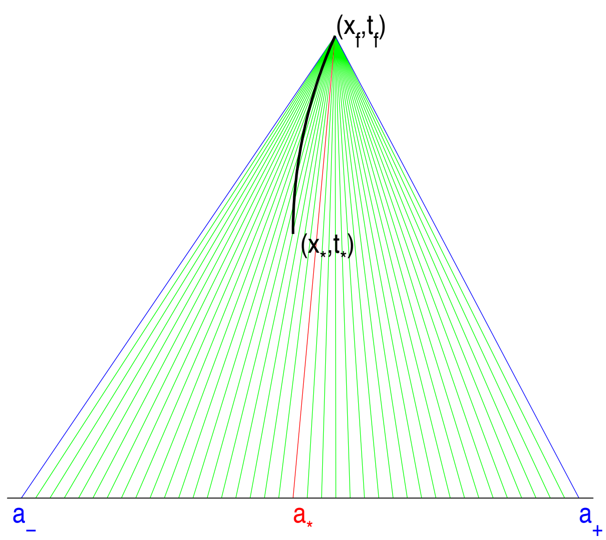

Figure 2.2: *"Uniformized" Burgers Shock.* Compared with the previous Fig.1, all straight-line characteristics have been replaced by straight lines from initial point $(a, 0)$ to the final point (x_f^*, t_f).

for $s \in [0, t]$, where $[a_-(s), a_+(s)]$ is the Lagrangian interval at time 0 for the shock located at $x_*(s)$ at time s. Note that for $s < t_*$, the time of first appearance of the shock,

$$\Delta_\psi(s) = \int_{a_-}^{a^+} \psi(u_0(a))\ \mathrm{d}a,$$

while for $s = t$

$$\Delta_\psi(t) = \int_{a_-}^{a_+} \psi\left(\frac{x_*(t) - a}{t}\right)\ \mathrm{d}a = t \int_{u^+}^{u^-} \psi(u)\ \mathrm{d}u.$$

Thus, the total dissipative anomaly over time interval $[0, t]$ (for a single shock) is the difference $\Delta_\psi(t) - \Delta_\psi(0)$. We shall show that $\Delta_\psi(s)$ is non-increasing in s. Taking the s-derivative and using (2.3) gives

$$\frac{d}{ds}\Delta_\psi(s) = \frac{1}{s}\int_{a_-(s)}^{a_+(s)} \psi'\left(\frac{x_*(s) - a}{s}\right)\left(u_*(s) - \frac{x_*(s) - a}{s}\right)\ \mathrm{d}a$$

Convexity of ψ implies that $\psi\left(\frac{x_*(s)-a}{s}\right) + \psi'\left(\frac{x_*(s)-a}{s}\right)\left(u_*(s) - \frac{x_*(s)-a}{s}\right) \leq \psi(u_*(s))$ and thus

$$\frac{d}{ds}\Delta_\psi(s) \leq \frac{1}{s}\int_{a_-(s)}^{a_+(s)} \left[\psi(u_*(s)) - \psi\left(\frac{x_*(s) - a}{s}\right)\right]\ \mathrm{d}a. \tag{2.10}$$

On the other hand, condition (2.9) for time s can be rewritten as

$$u_*(s) = \frac{1}{a_+(s) - a_-(s)}\int_{a_-(s)}^{a_+(s)} \frac{x_*(s) - a}{s}\ \mathrm{d}a, \tag{2.11}$$

so that convexity of ψ yields also

$$\psi(u_*(s)) \leq \frac{1}{a_+(s)-a_-(s)} \int_{a_-(s)}^{a_+(s)} \psi\left(\frac{x_*(s)-a}{s}\right) \, \mathrm{d}a, \tag{2.12}$$

and hence the non-positivity of the righthand side of (2.10). Thus, Δ_ψ is non-increasing, and $\Delta_\psi(t) \leq \Delta_\psi(0)$, which is equivalent to (2.5).

This proof gives a simple Lagrangian interpretation of the dissipative anomaly for Burgers equation: information about the initial velocity is "erased" as the particles fall into the shock and the initial velocity distribution is replaced by a uniform distribution in the shock interval. This decreases the average value of any convex function of the velocities because, instantaneously, the velocities are mixed (homogenized) by the shock to be closer to its own velocity $u_*(s)$.[3] Note that the above argument yields a new Lagrangian expression for the dissipative anomaly:

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}x \; \psi(u(x,t)) - \int_{\mathbb{R}} \mathrm{d}a \; \psi(u_0(a)) \\
&= \sum_{i=1}^{\infty} \int_0^t \frac{\mathrm{d}s}{s} \int_{a_i^-(s)}^{a_i^+(s)} \psi'\left(\frac{x_i^*(s)-a}{s}\right)\left(u_i^*(s) - \frac{x_i^*(s)-a}{s}\right) \mathrm{d}a \\
&= \sum_{i=1}^{\infty} \int_0^t \frac{\mathrm{d}s}{s} \int_{a_i^-(s)}^{a_i^+(s)} \Big(\psi(u_i^*(s)) - \psi(u_i(s)) - D_\psi^{u_i(s)}(u_i^*(s), u_i(s))\Big) \, \mathrm{d}a.
\end{aligned} \tag{2.13}$$

We have introduced here the notation $u_i(s) = \frac{x_i^*(s)-a}{s}$ and used the definition of

[3] This result is a sort of Burgers-equation version of Landauer's principle in the physical theory of computation, which states that erasure of information implies entropy production [Landauer, 1961]. One may also see some resemblance with the generalized second law of black-hole thermodynamics [Mukohyama, 1997], with the shock being analogous to the event horizon of the black hole.

the *Bregman divergence* between u and u^* with respect to the convex function ψ [Bregman, 1967]:

$$D_\psi^u(u^*, u) = \psi(u^*) - \psi(u) - \psi'(u) \cdot (u^* - u).$$

Instantaneously, one has

$$\frac{d}{dt}\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) = \frac{1}{t}\sum_{i=1}^{\infty}\int_{a_i^-(t)}^{a_i^+(t)} \Big(\psi(u_i^*(t)) - \psi(u_i(t)) - D_\psi^{u_i}(u_i^*(t), u_i(t))\Big)\,\mathrm{d}a. \quad (2.14)$$

Since $D_\psi^u(u^*, u) \geq 0$, we can easily see using inequality (2.12) that the contribution from each shock to the anomaly is non-positive.

We recall the standard Eulerian result for the dissipative anomaly

$$\frac{d}{dt}\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) = \sum_{i=1}^{\infty}\Big(u_i^*(t)(\psi(u_i^-) - \psi(u_i^+)) - (J(u_i^-) - J(u_i^+))\Big).$$

Here (ψ, J) is a Lax entropy pair with entropy flux function defined by

$$J(u) = \int \mathrm{d}u\ u\ \psi'(u).$$

See [Dafermos, 2006, Bressan, 2012]. Our Lagrangian formula is connected to the work of Khanin and Sobolevski [2010] on particle dynamics for Hamilton-Jacobi equations. They defined a "dissipative anomaly" which measured the rate of the difference in the action functional between true action minimizers and trajectories of particles on shocks. For Burgers as a Hamilton-Jacobi equation, the Hamiltonian and Lagrangian coincide with the convex function $\psi(u) = \frac{1}{2}u^2$. With this choice of ψ, the "dissipative anomaly" of [Khanin and Sobolevski, 2010] is the maximum over $\pm$ of the Bregman divergences $D_\psi^{u_i^\pm}(u_i^*, u_i^\pm) = \frac{1}{2}|u_i^\pm - u_i^*|^2$. Further relations with their work will be explored in section 2.5.

In the published paper [Eyink and Drivas, 2015a], we construct, for any entropy solution u of inviscid Burgers, a backward Markov process of generalized solutions of the ODE $dx/dt = u(x,t)$. This process is thus a generalized (stochastic) inverse of the forward coalescing flow for inviscid Burgers, which has been considered by many authors [Bauer and Bernard, 1999, Bogaevsky, 2004]. The essential property of the stochastic inverse considered here is that the velocities of the process are backward martingales, generalizing the result for smooth solutions of inviscid Burgers that velocities are Lagrangian invariants (preserved along characteristics). As we shall see, it is this backward martingale property which implies Lagrangian dissipativity of the entropy solution and it is natural to conjecture that this property uniquely characterizes the entropy solution. On the other hand, the stochastic inverses with the above stated properties are not themselves unique. In fact, in section 3 of the Eyink and Drivas [2015a] (not in this thesis) we construct a set of such inverses by a direct geometric method. In the next section, 2.3, we obtain another such stochastic inverse via the zero-viscosity limit of the backward diffusion process in the

Constantin-Iyer representation of viscous Burgers solutions.

2.3 Zero-Viscosity Limit

We now construct a fundamentally different stochastic inverse by considering the zero-viscosity limit of the stochastic representation of Constantin & Iyer [Constantin and Iyer, 2008] for the viscous Burgers solutions.

2.3.1 The Constantin-Iyer Representation

To make our discussion self-contained, we begin by presenting a new derivation of the Constantin-Iyer representation for viscous Burgers solutions. We then establish the close relation of this stochastic representation to the classical Hopf-Cole representation. These results hold for any space dimension $d \geq 1$, so that we discuss in this section multi-dimensional Burgers.

Consider a solution $\boldsymbol{u}$ to the viscous Burgers equation on the space-time domain $D = \mathbb{R}^d \times [t_0, t_f]$ with initial condition $\boldsymbol{u}_0(\boldsymbol{x})$ and define the *backward stochastic flow* $\tilde{\boldsymbol{\xi}}_{t,s}$ for $t_f \geq t \geq s \geq t_0$ by the solution of the stochastic differential equation

$$\mathrm{d}\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) = \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + \sqrt{2\nu}\,\hat{\mathrm{d}}\tilde{\mathbf{W}}(s) \tag{2.15}$$

with final conditions

$$\tilde{\boldsymbol{\xi}}_{t,t}(\boldsymbol{x}) = \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^d, \ t \in [t_0, t_f]. \tag{2.16}$$

Note that the noise strength is related to the kinematic viscosity of the fluid ν. Here $\tilde{\mathbf{W}}(t)$ denotes an $\mathbb{R}^d$-valued Wiener process and "$\hat{\mathrm{d}}$" in (2.15) implies a backward Ito SDE. These flows enjoy the semigroup property $\tilde{\boldsymbol{\xi}}_{s,r} \circ \tilde{\boldsymbol{\xi}}_{t,s} = \tilde{\boldsymbol{\xi}}_{t,r}$ a.s. for $t \geq s \geq r$. For the basic theory of backward Itō integration and stochastic flows that we use below, see [Friedman, 2006, Kunita, 1997].

The fundamental property of the backward stochastic flows defined above for the viscous Burgers velocity field is given by the following:

Proposition 2.3.1 *The stochastic Lagrangian velocity $\tilde{\boldsymbol{v}}(s|\boldsymbol{x}, t) = \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)$ is a backward martingale of the stochastic flow defined by (2.15), (2.16). That is,*

$$\mathbb{E}\big(\tilde{\boldsymbol{v}}(s|\boldsymbol{x}, t)\big|\mathcal{F}_{t,r}\big) = \tilde{\boldsymbol{v}}(r|\boldsymbol{x}, t), \quad t \geq r \geq s,$$

where $\mathcal{F}_{t,r}$ is the filtration of sigma algebras $\boldsymbol{\sigma}\{\tilde{\mathbf{W}}(u) : t \geq u \geq r\}$.

Proof By the backward Itō formula for flows [Kunita, 1997], we have, for any $\boldsymbol{x} \in \mathbb{R}^d$ and for each t, t' satisfying $t_0 \leq t' \leq t < t_f$ that

$$\mathrm{d}\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}), t') = \boldsymbol{u}_t \mathrm{d}t' + (\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,t'} \cdot \boldsymbol{\nabla}_x)\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}_{x_i x_j} \mathrm{d}\langle \tilde{\xi}^i_{t,t'}, \tilde{\xi}^j_{t,t'} \rangle,$$

where the quadratic variation can be calculated from (2.15) to be $\mathrm{d}\langle \tilde{\xi}^i_{t,t'}, \tilde{\xi}^j_{t,t'} \rangle =$

$2\nu\delta^{ij}\mathrm{d}t'$ where δ^{ij} is the Kronecker delta. Thus,

$$\begin{aligned}\mathrm{d}\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t') &= \Big((\boldsymbol{u}_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nu\triangle\boldsymbol{u})\,\mathrm{d}t' + \sqrt{2\nu}\;\hat{\mathrm{d}}\tilde{\mathbf{W}}(t')\cdot\boldsymbol{\nabla}_x\boldsymbol{u}\Big)\Big|_{(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t')} \\ &= \sqrt{2\nu}\;\hat{\mathrm{d}}\tilde{\mathbf{W}}(t')\cdot\boldsymbol{\nabla}_x\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t')\end{aligned}$$

In passing to the second line, we used the fact that $\boldsymbol{u}$ solves Burgers equation on D. Integrating over $t'\in[s,r]$ gives

$$\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) = \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}),r) + \sqrt{2\nu}\int_s^r \;\hat{\mathrm{d}}\tilde{\mathbf{W}}(t')\cdot\boldsymbol{\nabla}_x\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t')$$

Since $\mathbb{E}\Big(\int_s^r \hat{\mathrm{d}}\tilde{\mathbf{W}}(t')\cdot\boldsymbol{\nabla}_x\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t')\Big|\mathcal{F}_{t,r}\Big) = 0$ for a backward Itō integral, the result follows. $\square$

Note that unconditional expectation gives

$$\mathbb{E}\big(\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\big) = \mathbb{E}\big(\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\big|\mathcal{F}_{t,t}\big) = \boldsymbol{u}(\boldsymbol{x},t). \tag{2.17}$$

This leads to the Constantin-Iyer representation:

Proposition 2.3.2 *A smooth function $\boldsymbol{u}$ on the space-time domain $D=\mathbb{R}^d\times[t_0,t_f]$ is a solution to the viscous Burgers equation with initial data $\boldsymbol{u}(\cdot,t_0)=\boldsymbol{u}_0$ if and only* *if for each* $(\boldsymbol{x}, t) \in D$ *it satisfies*

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathbb{E}\left[\boldsymbol{u}_0(\tilde{\boldsymbol{\xi}}_{t,t_0}(\boldsymbol{x}))\right], \tag{2.18}$$

where the map $\tilde{\boldsymbol{\xi}}_{t,s}$ *is the stochastic flow defined by (2.15), (2.16) for the velocity field* $\boldsymbol{u}$.

Proof First suppose that $\boldsymbol{u}$ solves the viscous Burgers equation with initial condition $\boldsymbol{u}_0$. Using (2.17) with $s = 0$ yields formula (2.18). Now for the other direction, assume that (2.18) holds, together with (2.15),(2.16). Using the semigroup property of the flow maps and the $\mathcal{F}_{t,s}$-measurability of $\tilde{\boldsymbol{\xi}}_{t,s}$, we conclude that for any $t, t' \in [t_0, t_f],\ t \geq t'$

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{u}_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x}))\right] &= \mathbb{E}\left[\boldsymbol{u}_0\left(\tilde{\boldsymbol{\xi}}_{t',0} \circ \tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x})\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{u}_0\left(\tilde{\boldsymbol{\xi}}_{t',0}\left(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x})\right)\right)\Big|\mathcal{F}_{t,t'}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{u}_0\left(\tilde{\boldsymbol{\xi}}_{t',0}(\boldsymbol{y})\right)\right]\Big|_{\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x})=\boldsymbol{y}}\right] \;=\; \mathbb{E}\left[\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}), t')\right].
\end{aligned}$$

We therefore see that the following equivalent representation is implied for $t \geq t'$:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathbb{E}\left[\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}), t')\right].$$

An application of the backward Itō's formula to $\boldsymbol{u} \circ \tilde{\boldsymbol{\xi}}_{t,t'}$ gives:

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t') + \int_{t'}^{t} (\partial_s \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nu\triangle\boldsymbol{u})|_{(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)} \, \mathrm{d}s$$
$$+ \sqrt{2\nu}\int_{t'}^{t} \hat{\mathrm{d}}\tilde{\mathbf{W}}_s \cdot \boldsymbol{\nabla}_x \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)$$

Using the above results for $t > t'$ and computing

$$0 = \frac{\boldsymbol{u}(\boldsymbol{x},t) - \mathbb{E}\left[\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}),t')\right]}{t-t'} = \mathbb{E}\left[\frac{\int_{t'}^{t} (\partial_s \boldsymbol{u}_s + \boldsymbol{u}_s\cdot\boldsymbol{\nabla}\boldsymbol{u}_s - \nu\triangle\boldsymbol{u}_s)|_{(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)} \, \mathrm{d}s}{t-t'}\right]$$

which, at the coincidence limit $t' \nearrow t$, proves that any $\boldsymbol{u}$ satisfying (2.18) must solve the viscous Burgers equation. □

The C-I representation (2.18) for Burgers is exactly analogous to that employed for studies of passive scalar advection the Kraichnan model [Bernard et al., 1998, Gawędzki and Vergassola, 2000] and can be written also as

$$\boldsymbol{u}(\boldsymbol{x},t) = \int \mathrm{d}^d\mathrm{a}\ \boldsymbol{u}(\boldsymbol{a},s)\ p_{\boldsymbol{u}}(\boldsymbol{a},s|\boldsymbol{x},t), \quad s \le t$$

where

$$p_{\boldsymbol{u}}(\boldsymbol{a},s|\boldsymbol{x},t) = \mathbb{E}\left[\delta^d(\boldsymbol{a} - \tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}))\right]$$

is the transition probability for the backward diffusion. Unlike the linear relation for the Kraichnan model, however, the C-I representation is a nonlinear fixed point condition for the Burgers solution, because the drift of the diffusion process is the Burgers velocity itself. There should be close connections with the stochastic control formulation introduced by P.-L. Lions for general Hamilton-Jacobi equations [Lions, 1983, Fleming and Soner, 2006]. Note, however, that the C-I representation requires no assumption that the velocity field is potential. When this latter condition holds it is possible to establish a direct relation with the Hopf-Cole solution [Hopf, 1950, Cole, 1951], by means of the following:

Proposition 2.3.3 *The backward transition probabilities $p_{\boldsymbol{u}}(\boldsymbol{a}, s|\boldsymbol{x}, t)$ of the stochastic Lagrangian trajectories in the C-I representation of viscous Burgers equation have the form*

$$p_{\boldsymbol{u}}(\boldsymbol{a}, s|\boldsymbol{x}, t) = \frac{\exp\left(-\frac{1}{2\nu}S(\boldsymbol{a}, s|\boldsymbol{x}, t)\right)}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2\nu}S(\boldsymbol{a}', s|\boldsymbol{x}, t)\right) \mathrm{d}^d\mathrm{a}'} \tag{2.19}$$

with

$$S(\boldsymbol{a}, s|\boldsymbol{x}, t) = \frac{|\boldsymbol{x} - \boldsymbol{a}|^2}{2(t-s)} + \phi(\boldsymbol{a}, s) - \phi(\boldsymbol{x}, t), \tag{2.20}$$

and ϕ is any solution of the KPZ/Hamilton-Jacobi equations

$$\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 = \nu\triangle\phi + \gamma(t), \tag{2.21}$$
$$\phi(\boldsymbol{x}, t_0) = \phi_0(\boldsymbol{x}) + c_0,$$

where $\boldsymbol{u}_0 = \boldsymbol{\nabla}\phi_0$ but the function $\gamma(t)$ and constant c_0 may be freely chosen.

Proof Calculate the transition probability by the Girsanov transformation

$$
\begin{aligned}
p_{\boldsymbol{u}}(\boldsymbol{a}, s|\boldsymbol{x}, t) &= \mathbb{E}^W \left[\delta^d(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) - \boldsymbol{a})\right] \\
&= \mathbb{E}^{\xi,\nu}_{\boldsymbol{x}} \left[\delta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) - \boldsymbol{a}) \left(\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}^{\xi,\nu}_{\boldsymbol{x}}}\right)\right],
\end{aligned}
$$

where the first expectation $\mathbb{E}^W$ is over the Wiener measure $\mathcal{P}^W$ associated to the Brownian motion $\tilde{\mathbf{W}}$, the second expectation is over the (scaled) Wiener measure $\mathcal{P}^{\xi,\nu}_{\boldsymbol{x}}$ associated to the Brownian motion $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) \sim \boldsymbol{x} + \sqrt{2\nu}\tilde{\mathbf{W}}(s)$, and the Radon-Nikodym derivative (change of measure) is given by the backward Girsanov formula [Kunita, 1997]:

$$
\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}^{\xi,\nu}_{\boldsymbol{x}}} = \exp\left[\frac{1}{2\nu}\int_s^t \left(\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau) \cdot \hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_\tau - \frac{|\boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau)|^2}{2}\mathrm{d}\tau\right)\right]
$$

Now, suppose we are considering potential flow so that $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\nabla}\phi(\boldsymbol{x}, t)$. Demanding that $\boldsymbol{u}$ satisfy the viscous Burgers equation, ϕ must satisfy (2.21). By the backward Itō formula we have that

$$
\begin{aligned}
\mathrm{d}\phi(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau) &= (\partial_\tau\phi - \nu\triangle\phi)\big|_{(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}),\tau)}\mathrm{d}\tau + \hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,\tau} \cdot \boldsymbol{\nabla}\phi(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau) \\
&= \gamma(\tau)\mathrm{d}\tau - \frac{1}{2}|\boldsymbol{\nabla}\phi(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau)|^2\mathrm{d}\tau + \hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,\tau} \cdot \boldsymbol{\nabla}\phi(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}), \tau)
\end{aligned}
$$

The Girsanov formula thus becomes:

$$\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}_{\boldsymbol{x}}^{\xi,\nu}} = \frac{1}{\mathcal{N}} \exp\left(\frac{1}{2\nu}\left(\phi(\boldsymbol{x},t) - \phi(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\right)\right) \tag{2.22}$$

where $\mathcal{N}$ must be chosen to satisfy the normalization condition $\mathbb{E}_{\boldsymbol{x}}^{\xi,\nu}\left[\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}_{\boldsymbol{x}}^{\xi,\nu}}\right] = 1$. Using the equality in distribution $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) \sim \boldsymbol{x} + \sqrt{2\nu}\tilde{\mathbf{W}}(s)$, one obtains

$$\begin{aligned}
p_{\boldsymbol{u}}(\boldsymbol{a},s|\boldsymbol{x},t) &= \frac{1}{\mathcal{N}}\mathbb{E}_{\boldsymbol{x}}^{\xi,\nu}\left[\delta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) - \boldsymbol{a})\exp\left(\frac{1}{2\nu}\left(\phi(\boldsymbol{x},t) - \phi(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\right)\right)\right] \\
&= \frac{1}{\mathcal{N}}\mathbb{E}_{\boldsymbol{x}}^{\xi,\nu}\left[\delta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) - \boldsymbol{a})\right]\exp\left(\frac{1}{2\nu}\left(\phi(\boldsymbol{x},t) - \phi(\boldsymbol{a},s)\right)\right) \\
&= \frac{1}{\mathcal{N}}\frac{1}{(4\pi\nu t)^{d/2}}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu(t-s)} + \frac{1}{2\nu}\left(\phi(\boldsymbol{x},t) - \phi(\boldsymbol{a},s)\right)\right)
\end{aligned}$$

with

$$\mathcal{N} = \frac{1}{(4\pi\nu t)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu(t-s)} + \frac{1}{2\nu}\left(\phi(\boldsymbol{x},t) - \phi(\boldsymbol{a},s)\right)\right)\mathrm{d}^d\mathrm{a}.$$

$\square$

Remark # 1: For related calculations using a forward Girsanov transformation, see [Garbaczewski et al., 1997]. The backward Girsanov formula is equivalent to the Lagrangian path-integral

$$p_{\boldsymbol{u}}(\boldsymbol{a},s|\boldsymbol{x},t) = \int_{\boldsymbol{x}(t)=\boldsymbol{x}}\mathcal{D}\boldsymbol{x}\ \delta^d(\boldsymbol{x}(s)-\boldsymbol{a})\exp\left(-\frac{1}{4\nu}\int_s^t|\dot{\boldsymbol{x}}(\tau) - \boldsymbol{u}(\boldsymbol{x}(\tau),\tau)|^2\,\mathrm{d}\tau\right),$$

which appears in the physics literature [Shraiman and Siggia, 1994, Falkovich et al., 2001]. For a careful discussion of this equivalence, see the Appendix of [Eyink, 2011].

Remark # 2: It is now straightforward to show that the C-I representation is equivalent to the Hopf-Cole formula [Hopf, 1950, Cole, 1951]:

$$\boldsymbol{u}(\boldsymbol{x},t) = -2\nu\boldsymbol{\nabla}_x \ln\left[\frac{1}{(4\pi\nu t)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu t}-\frac{\phi_0(\boldsymbol{a})}{2\nu}\right)\mathrm{d}^d\mathrm{a}\right].$$

Using the chain rule and moving the gradient inside the integral gives

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{x},t) &= \frac{\frac{1}{(4\pi\nu t)^{d/2}}\int_{\mathbb{R}^d}2\nu\boldsymbol{\nabla}_a\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu t}\right)\cdot\exp\left(-\frac{\phi_0(\boldsymbol{a})}{2\nu}\right)\mathrm{d}^d\mathrm{a}}{\frac{1}{(4\pi\nu t)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu t}-\frac{\phi_0(\boldsymbol{a})}{2\nu}\right)\mathrm{d}^d\mathrm{a}} \\
&= \frac{\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu t}-\frac{\phi_0(\boldsymbol{a})}{2\nu}\right)\boldsymbol{\nabla}_a\phi_0(\boldsymbol{a})\mathrm{d}^d\mathrm{a}}{\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu t}-\frac{\phi_0(\boldsymbol{a})}{2\nu}\right)\mathrm{d}^d\mathrm{a}} \\
&= \int_{\mathbb{R}^d}\boldsymbol{u}_0(\boldsymbol{a})\ p_{\boldsymbol{u}}(\boldsymbol{a},0|\boldsymbol{x},t)\mathrm{d}^d\mathrm{a} \qquad (2.23)
\end{aligned}$$

using integration by parts, $\boldsymbol{u}_0(\boldsymbol{a}) = \boldsymbol{\nabla}\phi_0(\boldsymbol{a})$, and the expression:

$$p_{\boldsymbol{u}}(\boldsymbol{a},s|\boldsymbol{x},t) = \frac{\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{4\nu(t-s)}-\frac{1}{2\nu}\phi(\boldsymbol{a},s)\right)}{\int_{\mathbb{R}^d}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{a}'|^2}{4\nu(t-s)}-\frac{1}{2\nu}\phi(\boldsymbol{a}',s)\right)\mathrm{d}^d\mathrm{a}'} \qquad (2.24)$$

with some common factors canceled.

Remark # 3: In Appendix 2.A, we completely characterize when the Girsanov formula becomes path or history independent, i.e. that it can be constructed from knowledge of the process *only* at the endpoints t_0 and t_f as in the formula (2.22). In particular, we allow for general invertible diffusion matrices for the noise strength and prove that the Girsanov formula is history independent if and only if the drift is constructed from the noise-strength tensor and a potential which solves a generalized KPZ/Hamilton-Jacobi equation (2.37). That the potential ϕ of $\boldsymbol{u} = \nabla\phi$ solves (2.21) in the case of Burgers is a special case of this equation when the noise strength tensor is a constant multiple of the identity.

2.3.2 Spontaneous Stochasticity

We now employ the results of the previous section to show that the backward diffusion process associated to the C-I representation remains random (non-deterministic) as $\nu \to 0$, which is exactly the property of spontaneous stochasticty (backward in time). We here explicitly denote the viscosity dependence of the Burgers solution by subscript, as $\boldsymbol{u}_\nu = \boldsymbol{\nabla}\phi_\nu$, and the zero-viscosity limit is denoted by $\boldsymbol{u}_* = \boldsymbol{\nabla}\phi_*$. We also define the measure on $\mathbb{R}^d$

$$P_\nu^{s;\boldsymbol{x},t}(\mathrm{d}\boldsymbol{a}) = \mathrm{d}^d\mathrm{a}\; p_{\boldsymbol{u}_\nu}(\boldsymbol{a}, s|\boldsymbol{x}, t)$$

associated to the backward diffusion with densities (2.19). Our limit result is then stated as:

Proposition 2.3.4 *For any sequence $\boldsymbol{x}_\nu = \boldsymbol{x} + O(\nu)$, the probability measures $P_\nu^{s;\boldsymbol{x}_\nu,t}$*

on $\mathbb{R}^d$ *for each* $\nu > 0$ *converge weakly along subsequences in the limit* $\nu \to 0$ *to probability measures* $P_*^{s;\boldsymbol{x},t}$*, which may depend on the subsequence but which are always supported on atoms in the finite set*

$$\mathcal{A}_{s;\boldsymbol{x},t} = \mathrm{argmin}_{\boldsymbol{a}} \left[\frac{|\boldsymbol{x}-\boldsymbol{a}|^2}{2(t-s)} + \phi_*(\boldsymbol{a},s) \right].$$

If $(\boldsymbol{x},t)$ *is a regular point of the limiting inviscid Burgers solution* $\boldsymbol{u}_*$*, then* $P_\nu^{s;\boldsymbol{x}_\nu,t} \xrightarrow{w} P_*^{s;\boldsymbol{x},t} = \delta_{\boldsymbol{a}(s;\boldsymbol{x},t)}$*, where* $\boldsymbol{a}(s;\boldsymbol{x},t) = \boldsymbol{x} - \boldsymbol{u}_*(\boldsymbol{x},t)(t-s)$ *is the inverse Lagrangian image at time* $s < t$ *of* $\boldsymbol{x}$ *at time* t*. Suppose instead that* $(\boldsymbol{x},t)$ *is a generic point on the shock set of* $\boldsymbol{u}_*$ *and the sequence* $\boldsymbol{x}_\nu$ *satisfies*

$$\lim_{\nu\to 0} \boldsymbol{u}_\nu(\boldsymbol{x}_\nu,t) = p\ \boldsymbol{u}_*(\boldsymbol{x}^-,t) + (1-p)\boldsymbol{u}_*(\boldsymbol{x}^+,t), \quad p \in [0,1] \tag{2.25}$$

where the velocities $\boldsymbol{u}(\boldsymbol{x}^\pm,t)$ *are the limits from the two sides of the shock. Then*

$$P_\nu^{s;\boldsymbol{x}_\nu,t} \xrightarrow{w} P_*^{s;\boldsymbol{x},t} = p\ \delta_{\boldsymbol{a}_+(s;\boldsymbol{x},t)} + (1-p)\delta_{\boldsymbol{a}_-(s;\boldsymbol{x},t)} \tag{2.26}$$

where $\boldsymbol{a}_\pm(s;\boldsymbol{x},t) = \boldsymbol{x} - \boldsymbol{u}_*(\boldsymbol{x}^\pm,t)(t-s)$ *are the two inverse Lagrangian images at time* $s < t$ *of* $\boldsymbol{x}$ *at time* t*, so that* $\mathcal{A}_{s;\boldsymbol{x},t} = \{\boldsymbol{a}_-(s;\boldsymbol{x},t), \boldsymbol{a}_+(s;\boldsymbol{x},t)\}$*. In particular, if* $\boldsymbol{x}_\nu$ *satisfies* $\boldsymbol{u}_\nu(\boldsymbol{x}_\nu,t) = \bar{\boldsymbol{u}}_*(\boldsymbol{x},t)$*, then* $p = 1/2$ *and*

$$P_\nu^{s;\boldsymbol{x}_\nu,t} \xrightarrow{w} P_*^{s;\boldsymbol{x},t} = \frac{1}{2}\ \delta_{\boldsymbol{a}_+(s;\boldsymbol{x},t)} + \frac{1}{2}\delta_{\boldsymbol{a}_-(s;\boldsymbol{x},t)}. \tag{2.27}$$

Proof The solutions ϕ_ν have limits ϕ_* as $\nu \to 0$ given by the Lax-Oleinik formula for the zero-viscosity Burgers solution [Bec and Khanin, 2007]:

$$\phi_*(\boldsymbol{x}, t) = \inf_{\boldsymbol{a}} \left[\frac{|\boldsymbol{x} - \boldsymbol{a}|^2}{2(t-s)} + \phi_*(\boldsymbol{a}, s) \right].$$

This implies existence of the continuous limiting function

$$S_*(\boldsymbol{a}, s|\boldsymbol{x}, t) = \lim_{\nu \to 0} S_\nu(\boldsymbol{a}, s|\boldsymbol{x}, t) = \frac{|\boldsymbol{x} - \boldsymbol{a}|^2}{2(t-s)} + \phi_*(\boldsymbol{a}, s) - \phi_*(\boldsymbol{x}, t) \tag{2.28}$$

with the properties $S_*(\boldsymbol{a}, s|\boldsymbol{x}, t) \geq 0$ and $= 0$ only for the finite set $\mathcal{A}_{s;\boldsymbol{x},t} \subset \mathbb{R}^d$ of $\boldsymbol{a}$-values at which the infinimum in the Lax-Oleinik formula is achieved. Because velocities $\boldsymbol{u}_\nu = \boldsymbol{\nabla}\phi_\nu$ are bounded in the limit as $\nu \to 0$, we observe that $S_\nu(\boldsymbol{a}, s|\boldsymbol{x}_\nu, t) = S_\nu(\boldsymbol{a}, s|\boldsymbol{x}, t) + O(\nu)$ if $\boldsymbol{x}_\nu = \boldsymbol{x} + O(\nu)$, so that

$$\frac{1}{\nu} S_\nu(\boldsymbol{a}, s|\boldsymbol{x}_\nu, t) = \frac{1}{\nu} S_\nu(\boldsymbol{a}, s|\boldsymbol{x}, t) + O(1).$$

It follows that outside the finite set $\mathcal{A}_{s;\boldsymbol{x},t}$, probabilities for $P_\nu^{s;\boldsymbol{x}_\nu,t}$ decay exponentially as $\nu \to 0$. These measures are thus exponentially tight and have weak limits along subsequences which are supported on atoms in the set $\mathcal{A}_{s;\boldsymbol{x},t}$.

In the case where $(\boldsymbol{x}, t)$ is a regular point of $\boldsymbol{u}_*$, the set $\mathcal{A}_{s;\boldsymbol{x},t} = \{\boldsymbol{a}(s;\boldsymbol{x},t)\}$, a singleton, and every weak subsequential limit is the delta measure $\delta_{\boldsymbol{a}(s;\boldsymbol{x},t)}$.

In the case where $(\boldsymbol{x}, t)$ is a generic point in the shock set of $\boldsymbol{u}_*$, the set $\mathcal{A}_{s;\boldsymbol{x},t} =$

$\{\boldsymbol{a}_+(s;\boldsymbol{x},t),\ \boldsymbol{a}_-(s;\boldsymbol{x},t)\}$. Hence, every weak subsequential limit is of the form

$$p_*\ \delta_{\boldsymbol{a}_+(s;\boldsymbol{x},t)} + (1-p_*)\delta_{\boldsymbol{a}_-(s;\boldsymbol{x},t)}$$

for some $p_* \in [0,1]$. However, taking the limit as $\nu \to 0$ of the C-I representation $\boldsymbol{u}_\nu(\boldsymbol{x}_\nu,t) = \int \boldsymbol{u}_\nu(\boldsymbol{a},s)P_\nu^{s;\boldsymbol{x}_\nu,t}(d\boldsymbol{a})$ gives

$$p\,\boldsymbol{u}_*(\boldsymbol{x}^+,t)+(1-p)\boldsymbol{u}_*(\boldsymbol{x}^-,t) = \int \boldsymbol{u}_*(\boldsymbol{a},s)P_*^{s;\boldsymbol{x}_\nu,t}(d\boldsymbol{a}) = p_*\,\boldsymbol{u}_*(\boldsymbol{x}^+,t)+(1-p_*)\boldsymbol{u}_*(\boldsymbol{x}^-,t),$$

or $(p-p_*)(\boldsymbol{u}_*(\boldsymbol{x}^+,t) - \boldsymbol{u}_*(\boldsymbol{x}^-,t)) = \boldsymbol{0}$. Since $\boldsymbol{u}_*(\boldsymbol{x}^+,t) \neq \boldsymbol{u}_*(\boldsymbol{x}^-,t)$, $p_* = p$ for every subsequence $\nu_k \to 0$ and thus (2.26) holds. □

Remark # 1: Following the approach of [Laforgue and R. E. O' Malley, Jr., 1995], we may define the *shock surface* for $\nu > 0$ as

$$\mathcal{S}_\nu(t) = \left\{\boldsymbol{x}:\ |\boldsymbol{x}-\boldsymbol{x}_*| = O\left(\frac{\nu}{|\Delta \boldsymbol{u}|}\right) \text{ for some } \boldsymbol{x}_* \in \mathcal{S}_*(t) \text{ and } \boldsymbol{u}_\nu(\boldsymbol{x},t) = \frac{\boldsymbol{u}_*(\boldsymbol{x}_*^-,t)+\boldsymbol{u}_*(\boldsymbol{x}_*^+,t)}{2}\right\}$$

where $\mathcal{S}_*(t)$ is the shock surface of the inviscid limit $\boldsymbol{u}_*$ and where $\Delta\boldsymbol{u} = \boldsymbol{u}_- - \boldsymbol{u}_+$. The previous proposition thus implies that (2.27) holds for a sequence $\boldsymbol{x}_\nu \in \mathcal{S}_\nu(t)$ such that $\boldsymbol{x}_\nu \to \boldsymbol{x} \in \mathcal{S}_*(t)$. This means that stochastic particles which are "exactly on the shock" at time t for $\nu > 0$ must jump off the shock backward in time as $\nu \to 0$, with equal probability to the left or to the right.

2.3.3 Limiting Backward Process

Of the limiting probabilities obtained in Proposition 2.3.4, there is a distinguished case in which the particle starts "exactly on the shock" for $\nu > 0$ and then jumps off to the right or the left with equal probabilities as $\nu \to 0$. It is only for this case that the particle drift velocity at the shock equals the limiting shock velocity. This case corresponds to a random process $\tilde{x}_*(t)$ which enjoys the same properties as the processes obtained by the geometric construction in section 3 of the published paper. It is clearly Markovian backward in time in an extended state space $X(t) \subset \mathbb{R} \times \{-1, 1\}$ with label $\alpha = +1$ indicating to the right of the shock and $\alpha = -1$ to the left. The Markov generator is

$$L(t)f(x, \pm) = -u(x^{\pm}, t)f'(x, \pm)$$

and initial conditions for (x, t) a shock point of u_* assign $\alpha = +1$ or -1 with probability 1/2. The realizations of this random process move always along straight-line characteristics. The Lagrangian velocity $\bar{u}_*(\tilde{x}_*(t), t)$ is also a martingale backward in time. For (x, t) a regular point of u_*, this is the usual conservation of velocity along straight-line characteristics, while, for (x, t) a shock point of u_*, the martingale property depends also on the definition $\bar{u}(x, t) = \frac{1}{2}\left[u(x^+, t) + u(x^-, t)\right]$. Note that, as for the geometric construction, the martingale property implies in 1D the positivity of dissipation. Indeed, the shock velocity can be represented using the martingale property as

$$\begin{aligned} u_*(t) &= \frac{1}{2}[u_+(t) + u_-(t)] \\ &= \frac{1}{2}\left[\left(\frac{x_*(t) - a_+}{t}\right) + \left(\frac{x_*(t) - a_-}{t}\right)\right] \\ &= \frac{1}{a_+ - a_-}\int_{a_-}^{a_+}\left(\frac{x_*(t) - a}{t}\right)\, \mathrm{d}a, \end{aligned} \tag{2.29}$$

which is exactly the condition (2.11) needed to show positivity. The Lax entropy condition $u^- > u^+$ is implicit in this formulation, since it guarantees that $a^+ > a^-$.

2.4 Non-uniqueness, Dissipation and a Conjecture

The results of the previous two sections can be restated as follows: the entropy (viscosity, dissipative) solution u of inviscid Burgers equation in one space dimension with smooth initial data u_0 satisfies the identity

$$\bar{u}(x,t) = \mathbb{E}[u_0(\tilde{x}(0))|\tilde{x}(t) = x] = \int \mathrm{d}a\ u_0(a) p_u(a, 0|x, t) \tag{2.30}$$

where $\mathbb{E}$ is expectation with respect to any of the random processes $\tilde{x}(\tau)$ backward in time constructed in the previous section or in section 3 of [Eyink and Drivas, 2015a] and $p_u(a, s|x, t)$ is the transition probability for this process. The random process

$\tilde{x}(\tau)$ has the following properties:

(i) The realizations of the process projected to coordinate space are generalized solutions of the ODE $d\tilde{x}/d\tau = \bar{u}(\tilde{x}, \tau)$.

(ii) The process is Markov backward in time.

(iii) The velocity process $D_\tau^+ \tilde{x}(\tau) = \bar{u}(\tilde{x}(\tau), \tau)$ is a backward martingale.

The formula (2.30) is the analogue of the representation of the weak solutions for passive scalars in the Kraichnan model [E and Vanden-Eijnden, 2000, Jan and Raimond, 2002] and is an inviscid analogue of the Constantin-Iyer representation of viscous Burgers solutions. Note that it is a consequence of (i) that, at points of smoothness of $u(x,t)$, the process $\tilde{x}(\tau)$ is deterministic and consists of the single characteristic curve which arrives to (x,t). On the other hand, when (x,t) is located on a shock, the previous constructions contain an interesting element of non-uniqueness. The two approaches, the geometric one of section 3 of the published paper and the zero-viscosity limit of section 2.3 (which applies also for dimensions $d \geq 1$), lead to quite different stochastic processes. Even within the geometric approach there is an important element of non-uniqueness, because the time t_0 before formation of the first shock —when the positions are chosen to be uniformly distributed on the Lagrangian interval $[b_-^f, b_+^f]$ — is completely arbitrary. As can be seen from (21),(24) of the published paper, assuming a uniform distribution on particle positions $[b_-^f, b_+^f]$ at time $t_0 < t_*$ does not lead to uniform distributions at other times $t < t_*$. Hence there are uncountably many distinct definitions of random processes for which (2.30) is valid and all of the properties (i),(ii),(iii) hold.

To the non-uniqueness of the stochastic process of backward particle motions there corresponds a similar non-uniqueness in the Lagrangian expression (2.14) of anomalous dissipation. In that expression, one may likewise chose any initial time $t_0 < t_*$ and represent the dissipation by integrals over the Lagrangian intervals $[b_i^-(t), b_i^+(t)]$ of particle positions at time t_0 that will have fallen into shocks at time t:

$$\frac{d}{dt}\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) = \frac{1}{t-t_0}\sum_{i=1}^{\infty}\int_{b_i^-(t)}^{b_i^+(t)} \Big(\psi(u_i^*(t)) - \psi(u_i(t)) - D_\psi^{u_i}(u_i(t), u_i^*(t))\Big)\ \mathrm{d}b,$$
$$u_i(t) = \frac{x_i^*(t) - b}{t-t_0}. \tag{2.31}$$

Note that this anomalous dissipation may be directly represented in terms of the corresponding random process discussed in the published paper (with uniform distribution on positions in shock intervals at time t_0) as

$$\frac{d}{dt}\int_{\mathbb{R}} \mathrm{d}x\ \psi(u(x,t)) = \sum_{i=1}^{\infty}\Delta u_i$$
$$\mathbb{E}\left[\psi(u_i^*(t)) - \psi\left(\frac{\tilde{x}(t)-\tilde{x}(t_0)}{t-t_0}\right) - D_\psi^{u_i}\left(\frac{\tilde{x}(t)-\tilde{x}(t_0)}{t-t_0},\ u_i^*(t)\right)\Big|\dot{\tilde{x}}(t) = u_i^*(t)\right] \tag{2.32}$$

where $\Delta u_i = u_i^- - u_i^+ > 0$. The negative sign of the dissipation is then seen to be directly due to the backward martingale property (iii) of the random process. Note indeed that it is a consequence of the Bauer-Bernard [Bauer and Bernard, 1999] definition that solutions of $\dot{x} = \bar{u}(x,t)$ in their sense satisfy

$$\tilde{x}(t') - \tilde{x}(t'') = \int_{t''}^{t'} \mathrm{d}\tau \ \bar{u}(\tilde{x}(\tau),\tau).$$

Therefore, integrating with respect to time t in Proposition 3.2 of the published paper gives as a direct corollary

$$\mathbb{E}\Big(\frac{\tilde{x}(t') - \tilde{x}(t'')}{t'-t''}\Big|\dot{\tilde{x}}(t) = u\Big) = u \ \text{ for all } t'' < t' \leq t, \quad t'', t', t \in [0, t_f].$$

This property for $t' = t$, $t'' = t_0$ and, in particular, the consequence that for any convex function ψ

$$\mathbb{E}\Big[\psi\Big(\frac{\tilde{x}(t) - \tilde{x}(t_0)}{t-t_0}\Big)\Big|\dot{x}(t) = u\Big] \geq \psi(u) \ \text{ for all } s \in [0, t_f].$$

is thus the basic property required to show dissipativity of the Burgers solution.

The above fundamental connection between the backward martingale property and dissipation motivates the following:

Conjecture: The only space-time velocity field u on $\mathbb{R} \times [0, t_f]$ which satisfies the identity (2.30) for a stochastic process $\tilde{x}(\tau)$ with the properties (i),(ii),(iii) is the unique viscosity (entropy, dissipative) solution of inviscid Burgers with initial condition u_0 on $\mathbb{R}$. [4]

[4] The results of section 2.3 suggest that the conjecture should also hold for space dimensions $d \geq 1$, although the entropy conditions are no longer valid.

In particular, it should follow directly from the dissipation implied by the backward martingale property that the field defined by the stochastic representation satisfies the conditions to be an "admissible weak solution" (e.g. see [Dafermos, 2006], Def. 6.2.1). The condition imposed is highly implicit, since the velocity u which appears as the result of the average in (2.30) is the same as the velocity u which appears in the ODE in (i) governing particle motion. The conjecture as stated above is not explicit enough to be subject to proof or disproof or even to be entirely well-formulated, without additional conditions. A natural requirement on spatial regularity is that the velocity field be of bounded-variation at each fixed time t, $u(\cdot,t) \in BV(\mathbb{R})$ for all $t \in [0,t_f]$. In that case $u(\cdot,t)$ is continuous except at a countable set of points where right- and left-hand limits exist, so that the field $\bar{u}(x,t) = \frac{1}{2}(u(x-,t) + u(x+,t)$ is well-defined. Some temporal regularity must also certainly be assumed, such as $u \in C([0,t_f]; L^1(\mathbb{R}))$.

It is important to emphasize that the uniqueness claim in the conjecture above is for the weak solution u only and not for the random process $\tilde{x}$. As we have already seen by explicit construction, there is more than one such random process $\tilde{x}$ for the same entropy solution u. There is thus an arbitrariness in how the Burgers velocities can be regarded to be transported by their own flow. This is similar to the arbitrariness that exists even for some smooth problems, e.g. the Lie-transport of a magnetic field

$\mathbf{B}$ (closed 2-form) by a smooth velocity field $\boldsymbol{u}$, governed by the induction equation

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \mathbf{B}).$$

It has long been known (e.g. [Newcomb, 1958]) that there is more than one "motion" which can be consistently ascribed to the magnetic field-lines governed by the above equation. This arbitrariness holds in that case even for the linear problem of passive transport of the magnetic field by a smooth velocity.

2.5 Time-Asymmetry of Particle Stochasticity

We have shown in the published paper that there is spontaneous stochasticity in the zero-noise limit at finite-Pr for Lagrangian particles in a Burgers flow moving *backward in time*. On the contrary, the zero-noise limit forward in time at any Pr should lead to a natural coalescing flow for Burgers [Bauer and Bernard, 1999], as has been proved rigorously at $Pr = 0$ [Khanin and Sobolevski, 2010, 2012]. The Burgers system is thus quite different from the time-reversible Kraichnan model, where strongly compressible flows lead to coalescence both forward and backward in time [Gawędzki and Vergassola, 2000, E and Vanden-Eijnden, 2001]. Based on studies in the Kraichnan model, the difference between stochastic splitting or sticking of particles has been viewed as a consequence of the degree of compressibility of the velocity field, with weakly compressible/near-solenoidal velocities leading to splitting and strongly compressible/near-potential velocities leading to sticking [Gawędzki and Vergassola, 2000, E and Vanden-Eijnden, 2000, 2001, Jan and Raimond, 2002, 2004]. However, the Burgers equation with a velocity that is pure potential can produce both sticking and splitting, in different directions of time.

The Burgers system appears in fact to have a remarkable similarity in particle behaviors to incompressible Navier-Stokes turbulence, even though the Burgers velocity is pure potential and the Navier-Stokes velocity is pure solenoidal. Because the Navier-Stokes equation just as viscous Burgers is not time-reversible, it can exhibit distinct particle behaviors forward and backward in time. Navier-Stokes turbulence appears to lead to Richardson 2-particle dispersion and, consequently, stochastic particle splitting, both forward and backward in time. Remarkably, however, the rate of dispersion is found in empirical studies of three-dimensional Navier-Stokes turbulence to be greater backward in time than forward [Sawford et al., 2005, Berg et al., 2006, Eyink, 2011]. This is the same tendency seen in a very extreme form in Burgers, where there is particle splitting backward in time but only coalescence forward in time.

We have also shown in this work, at least for Burgers, that there is a direct connection between spontaneous stochasticity and anomalous dissipation for hydrodynamic equations, as had been suggested earlier in [Gawędzki and Vergassola, 2000]. More precisely, the relation we have found is between the sign of conservation-law anomalies in Burgers and spontaneous stochasticity backward in time. In turbulence language, the *direct cascade* of energy to small scales in Burgers is due to stochastic particle splitting backward in time. The empirical observations on particle dispersion in Navier-Stokes turbulence cited above lead us to suggest more generally *a deep relation between cascade direction and the time-asymmetry of particle dispersion.* Indeed, three-dimensional Navier-Stokes turbulence has a forward cascade of energy, just as Burgers, and likewise a faster particle dispersion backward in time than forward. This conjecture is strengthened by the numerical observation of a reversed asymmetry for the inverse energy cascade of two-dimensional turbulence, with Richardson particle dispersion in 2D inverse cascade instead faster forward in time than backward [Faber and Vassilicos, 2009].

There is a well-known connection in statistical physics between dissipation/entropy production and the asymmetry between forward and backward processes, embodied in so-called *fluctuation theorems*. For a recent review of this theory, see [Gawędzki, 2013]. Since it is natural to suspect a relation with our conjectures above, we briefly recall here that the fluctuation theorems state that

$$\mathbb{E}(e^{W[\tilde{\boldsymbol{x}}]}) = 1 \tag{2.33}$$

where $e^W = d\mathcal{P}'/d\mathcal{P}$ is a Radon-Nikodým derivative of the path measure for a time-reversed process with respect to the path measure for the direct process. Physically, $-k_B W$ has often the meaning of "entropy production" and the consequence of Jensen's inequality,

$$\mathbb{E}(W[\tilde{\boldsymbol{x}}]) \leq 0,$$

implies the sign of energy dissipation or entropy production in the 2nd law of thermodynamics. However, the fluctuation theorems are a considerable refinement of the 2nd law, since they state not only the existence of entropy production on average but also provide information about the likelihood of 2nd-law violations.

Fluctuation theorems are straightforward to derive for stochastic particle motion in Burgers governed by the SDE

$$d\tilde{\boldsymbol{x}} = \boldsymbol{u}(\tilde{\boldsymbol{x}}, t)\mathrm{d}t + \sqrt{2\nu}\, \mathrm{d}\tilde{\mathbf{W}}(t), \quad t \in [t_0, t_f],$$

especially when the velocity is potential with $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\nabla}\phi(\boldsymbol{x}, t)$. In this case, the time-reverse process is the same as the direct process with merely the time-change $t' = t_0 + t_f - t$ [Kolmogorov, 1935]. Also, for any gradient dynamics with additive noise, the accompanying measures (or instantaneously stationary measures) at time t are

$$n_t(\mathrm{d}\boldsymbol{x}) = \frac{1}{Z_t} \exp\left(\frac{\phi(\boldsymbol{x}, t)}{\nu}\right).$$

The standard recipes [Gawędzki, 2013] then give (2.33) with

$$W[\tilde{\boldsymbol{x}}] = \frac{1}{\nu}\phi(\tilde{\boldsymbol{x}}(t_f), t_f) - \ln \rho_f(\tilde{\boldsymbol{x}}(t_f))$$
$$+\frac{1}{\nu}\int_{t_0}^{t_f} \partial_t\phi(\tilde{\boldsymbol{x}}(t), t)\mathrm{d}t - \frac{1}{\nu}\phi(\tilde{\boldsymbol{x}}(t_0), t_0) + \ln \rho_0(\tilde{\boldsymbol{x}}(t_0))$$

where $\rho_0(\boldsymbol{x})$ and $\rho_f(\boldsymbol{x})$ are starting probability densities for the forward and backward processes which may be freely chosen. The trajectories $\tilde{\boldsymbol{x}}(t)$ in the expectation $\mathbb{E}$ of (2.33) are sampled from solutions of the forward SDE with initial data chosen from ρ_0.

An intriguing question is whether such fluctuation theorems for stochastic particle motion in Burgers have any relation with anomalous dissipation in the limit $\nu \to 0$. For Burgers the potential satisfies the KPZ/Hamilton-Jacobi equation

$$\partial_t\phi_\nu(\tilde{\boldsymbol{x}}(t), t) = -\frac{1}{2}|\boldsymbol{\nabla}\phi_\nu(\tilde{\boldsymbol{x}}(t), t)|^2 + \nu\triangle\phi_\nu(\tilde{\boldsymbol{x}}(t), t).$$

Also, the forward stochastic flow $\tilde{\boldsymbol{x}}_\nu(t)$ converges to the coalescing flow $\boldsymbol{x}_*(t)$ for Burgers as $\nu \to 0$. Note that the laplacian term has been shown [Khanin and Sobolevski, 2010, 2012] to have the limit along the trajectories of the forward coalescing flow given by

$$\lim_{\tau\downarrow 0}\lim_{\nu\downarrow 0} \nu\triangle\phi_\nu(\boldsymbol{x}_*(t+\tau), t+\tau) = -\min_{\pm} D_L^{\boldsymbol{u}_\pm(t)}\left(\boldsymbol{u}_*(t), \boldsymbol{u}_\pm(t)\right),$$

the Bregman divergence for the free-particle Lagrangian $L(t, \boldsymbol{x}, \boldsymbol{v}) = \frac{1}{2}|\boldsymbol{v}|^2$, or just the kinetic energy. (Note the sign error in [Khanin and Sobolevski, 2010], p.1591) The quantity $\partial_t \phi_\nu(\tilde{\boldsymbol{x}}(t), t)$ then has an enticing similarity to our expression (2.13) for the dissipative anomaly, when $\psi = L$. Unfortunately, we are skeptical that any general connection exists. A counterexample[5] is the stationary shock solution of viscous Burgers considered in our published paper, which has a kinetic energy anomaly $-\frac{2}{3}u_0^3\delta(x)$ in the limit $\nu \to 0$, but for which $\partial_t \phi(x) = 0$! It remains to be seen whether any ideas related to the fluctuation theorems can be at all connected with dissipative anomalies in Burgers or elsewhere.

2.6 Final Discussion

Our work has verified that many of the relations suggested by the Kraichnan model [Bernard et al., 1998, Gawędzki and Vergassola, 2000], between Lagrangian particle stochasticity, anomalous dissipation, and turbulent weak solutions, remain valid for the inviscid Burgers equation. Our results for Burgers give, as far as we are aware, the first proof of spontaneous stochasticity for a deterministic PDE problem. There is some similarity with the results of Brenier [Brenier, 1989] on global-in-time existence of action minimizers for incompressible Euler fluids via “generalized flows”. However, unlike Brenier’s work which dealt with a two-time boundary-value problem,

[5] There is also a physical puzzle what quantity would constitute the “temperature” to relate the “entropy production” $-k_B W[\tilde{\boldsymbol{x}}]$ to energy dissipation.

our stochastic representation (2.30) is valid for solutions of the Cauchy problem, like the similar representations for weak solutions in the Kraichnan model. As in Brenier's work, however, and unlike in the Kraichnan model, we find that the stochastic Lagrangian flows for inviscid Burgers are generally non-unique (even for entropy solutions). An important question left open by our work is whether existence of suitable stochastic processes of Lagrangian trajectories, which are backward Markov and for which the velocity is a backward martingale, uniquely characterize the entropy solution of Burgers.

The most important outstanding scientific issue is certainly the validity of similar results for more physically realistic hydrodynamic equations, such as the incompressible Navier-Stokes equation. It is an entirely open mathematical question whether standard weak solutions of incompressible Euler can be obtained by the zero-viscosity limit of incompressible Navier-Stokes solutions and whether these Euler solutions are characterized by a backward martingale property for the fluid circulations, as earlier conjectured by us [Eyink, 2006, 2007, 2010]. It is not even known whether the "arrow of time" specified by the martingale property is the same as the arrow specified by dissipation of energy. That is to say, it is unknown whether weak Euler solutions (if any) satisfying the backward martingale property for circulations must have kinetic energies always decreasing in time. There is not even a formal physicists' argument that this is so, let alone a rigorous proof.

The existence of non-vanishing energy dissipation in the limit of zero viscosity has been termed the "zeroth law of turbulence" [Frisch, 1995]. Explaining such anomalous dissipation is indeed the zeroth-order problem for any theory of turbulence. While much is known about turbulent energy cascade in Eulerian representation from a synthesis of experiment, simulations and theory, the Lagrangian aspects remain rather mysterious. G. I. Taylor's vortex-stretching picture [Taylor and Green, 1937, Taylor, 1938] is still the most common and popularly taught Lagrangian view of turbulent dissipation (e.g. see Feynman's undergraduate lectures [Feynman et al., 1964], volume II, section 41-5). Taylor's line-stretching mechanism is exemplified by the Kazantsev-Kraichnan model of kinematic magnetic dynamo in its "free decay regime", but this example also shows that Taylor's mechanism becomes much more subtle in the presence of spontaneous stochasticity [Eyink and Neto, 2010]. We believe that the possibility exists for fundamentally new Lagrangian perspectives on turbulent energy dissipation for Navier-Stokes and related equations. We hope that the current work may provide some useful hints in that direction.

Appendix

2.A History-Independence of Girsanov Formula

In this Appendix, we completely characterize we prove necessary and sufficient conditions for the Girsanov change-of-measure formula to be independent of the history of the stochastic trajectories.

Consider the following general backwards stochastic differential equation:

$$\begin{aligned} \mathrm{d}\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) &= \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + \boldsymbol{\sigma}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s, \\ \tilde{\boldsymbol{\xi}}_{t,t}(\boldsymbol{x}) &= \boldsymbol{x}. \end{aligned} \tag{2.34}$$

Here $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$, $\tilde{\boldsymbol{\xi}} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ and $\boldsymbol{\sigma} : \mathbb{R}^n \times \mathbb{R}^+ \to M_n(\mathbb{R})$ where $M_n(X)$ is the space of $n \times n$ matrices with entries taking values in X. We assume that $\boldsymbol{\sigma}$ is invertible and all the functions are smooth enough to take the required derivatives. Consider also purely diffusive trajectories:

$$
\begin{aligned}
\mathrm{d}\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}) &= \boldsymbol{\sigma}(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}), s) \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s, \\
\tilde{\boldsymbol{\zeta}}_{t,t}(\boldsymbol{x}) &= \boldsymbol{x}.
\end{aligned}
\tag{2.35}
$$

We wish to change measure between the Wiener measure $\mathcal{P}^W$ associated to the Brownian motion $\tilde{\mathbf{W}}$ and the (scaled) Weiner measure $\mathcal{P}^{\zeta}_{\boldsymbol{x}}$ associated to the Brownian motion $\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x})$. The Radon-Nikodym derivative (change of measure) is given by the backward Girsanov formula:

$$
\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}^{\zeta}_{\boldsymbol{x}}} = \exp\left[\int_s^t \left(\boldsymbol{\sigma}(\tilde{\boldsymbol{\zeta}}_{t,\tau}(\boldsymbol{x}), \tau)^{-1}\boldsymbol{u}(\tilde{\boldsymbol{\zeta}}_{t,\tau}(\boldsymbol{x}), \tau) \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_\tau - \frac{|\boldsymbol{\sigma}(\tilde{\boldsymbol{\zeta}}_{t,\tau}(\boldsymbol{x}), \tau)^{-1}\boldsymbol{u}(\tilde{\boldsymbol{\zeta}}_{t,\tau}(\boldsymbol{x}), \tau)|^2}{2}\mathrm{d}\tau\right)\right].
$$

Theorem 2.A.1 *The backward Girsanov formula is history independent*

$$
\frac{\mathrm{d}\mathcal{P}^W}{\mathrm{d}\mathcal{P}^{\zeta}_{\boldsymbol{x}}} = \exp\left(\psi(\boldsymbol{x}, t) - \psi(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}), s) - \int_s^t c(\tau)\mathrm{d}\tau\right) \tag{2.36}
$$

for some $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ if and only if the drift $\boldsymbol{u}$ is specified as $\boldsymbol{u} = \mathbf{D}\nabla\psi$ with $\mathbf{D} = \boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}$ and scalar function ψ satisfies the following partial differential equation:

$$
\partial_t\psi + \frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2 = \frac{1}{2}\mathbf{D} : \nabla \otimes \nabla\psi + c(t). \tag{2.37}
$$

Proof First assume that (2.36) holds and prove that $\boldsymbol{u} = \mathbf{D}\nabla\psi$ and ψ solves equation

(2.37). From the backward Girsanov formula, it follows that

$$\mathrm{d}\psi(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s) = \left(-\left(\frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} + c(s)\right)\mathrm{d}s + \boldsymbol{\sigma}^{-1}\boldsymbol{u}\big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s.$$

However, by (backwards) Ito's formula, we know

$$\begin{aligned}\mathrm{d}\psi(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s) &= \left(\partial_s\psi - \frac{1}{2}\mathbf{D}:\nabla\otimes\nabla\psi\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)}\mathrm{d}s + \nabla\psi(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)\cdot\boldsymbol{\sigma}(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)\hat{\mathrm{d}}\tilde{\mathbf{W}}_s \\ &= \left(\partial_s\psi - \frac{1}{2}\mathbf{D}:\nabla\otimes\nabla\psi\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)}\mathrm{d}s + \boldsymbol{\sigma}^{\mathrm{T}}\nabla\psi\big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)}\cdot\hat{\mathrm{d}}\tilde{\mathbf{W}}_s.\end{aligned}$$

Taking the difference of the above two formulas, we find

$$\left(\partial_s\psi + \frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2 - \frac{1}{2}\mathbf{D}:\nabla\otimes\nabla\psi - c(s)\right)\Big|_{(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)}\mathrm{d}s = \left(\boldsymbol{\sigma}^{-1}\boldsymbol{u} - \boldsymbol{\sigma}^{\mathrm{T}}\nabla\psi\right)\Big|_{(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)}\cdot\hat{\mathrm{d}}\tilde{\mathbf{W}}_s$$

This implies that for all times t and $s < t$ the following two conditions must be met:

$$\boldsymbol{u} = \mathbf{D}\nabla\psi, \tag{2.38}$$

$$\partial_s\psi + \frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2 = \frac{1}{2}\mathbf{D}:\nabla\otimes\nabla\psi + c(s). \tag{2.39}$$

Note that substituting in (2.38) into (2.39) yields proves that ψ satisfies Eq. (2.37).

Now suppose that the drift $\boldsymbol{u} = \mathbf{D}\nabla\psi$ and scalar function ψ satisfies equation (2.37).

Again we have by the (backwards) Ito's formula

$$\begin{aligned}
\mathrm{d}\psi(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}), s) &= \left(\partial_s \psi - \frac{1}{2}\mathbf{D} : \nabla \otimes \nabla \psi\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \mathrm{d}s + \boldsymbol{\sigma}^{\mathrm{T}} \nabla \psi\big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s \\
&= -\left(\frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2 + c(s)\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \mathrm{d}s + \boldsymbol{\sigma}^{\mathrm{T}} \nabla \psi\big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s \\
&= -\left(\frac{1}{2}|\boldsymbol{\sigma}^{-1}\boldsymbol{u}|^2 + c(s)\right)\Big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \mathrm{d}s + \boldsymbol{\sigma}^{-1} \boldsymbol{u}\big|_{(\tilde{\boldsymbol{\zeta}}_{t,s}(\boldsymbol{x}),s)} \cdot \hat{\mathrm{d}}\tilde{\mathbf{W}}_s.
\end{aligned}$$

The result follows directly upon integration from s to t. □

Note that the KPZ/Hamilton-Jacobi equation formulation for the burgers potential ϕ, (2.21), is one such equation in the class (2.37), with $\phi = 2\kappa\psi$ and $\boldsymbol{\sigma} = \sqrt{2\kappa}\mathbf{I}$. Our analysis of limiting spontaneously stochastic trajectories could likely be carried over for a wider class of models (velocities which have the special form $\boldsymbol{u} = \mathbf{D}\nabla\psi$ with function ψ satisfying (2.37)).

Chapter 3

Turbulent Cascade Direction and Lagrangian Time-Asymmetry

Perhaps the most notable difference between $2d$ and $3d$ turbulence is the direction of the energy cascade. In $3d$, energy is transferred from large to small scales where it is dissipated by molecular viscosity whereas in $2d$, energy cascades from small to large scales where it builds up until it is depleted, for example, by a linear damping or (very ineffectually) by viscosity. Said succinctly, the energy cascade is down-scale or *direct* in $3d$ and upscale or *inverse* in $2d$. Somewhat mysteriously, Richardson dispersion is observed to be faster backward-in-time for $3d$ turbulence and forward-in-time in $2d$ [Faber and Vassilicos, 2009, Sawford et al., 2005]. This observation, as well as insight from toy models, led to the conjecture that the cascade direction and time-asymmetry exhibited by Richardson dispersion are intimately related [Eyink and Drivas, 2015a].

For a detailed discussion of this conjecture and existing evidence for it, see §8 of Eyink and Drivas [2015a] or §2.5 of Chapter 2 of this thesis.

Recent work on mean-squared particle dispersion by Bitane et al. [2012], Falkovich and Frishman [2013], Jucha et al. [2014] has shed new light on Lagrangian manifestations of time asymmetry and its connection to the turbulent cascade. See the review Xu et al. [2016]. These studies employ the so-called *Ott-Mann-Gawędzki* relation [Ott and Mann, 2000, Falkovich et al., 2001], sometimes described as the "Lagrangian analog of the 4/5th law",[1] to obtain an explicit short-time expansion for the two-particle dispersion in terms of purely Eulerian quantities. For separations in the inertial range $\ell_\nu \ll |\boldsymbol{r}| \ll L$ (where ℓ_ν is the viscous scale and L is the integral scale), the Ott-Mann-Gawędzki relation states:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle|\delta\boldsymbol{v}(\boldsymbol{r};\boldsymbol{x},t)|^2\Big\rangle\Big|_{t=t_0} \simeq -2\langle\varepsilon\rangle \tag{3.1}$$

where $\delta\boldsymbol{v}(\boldsymbol{r};\boldsymbol{x},t) := \boldsymbol{u}(\boldsymbol{X}_{t_0,t}(\boldsymbol{x}+\boldsymbol{r}),t) - \boldsymbol{u}(\boldsymbol{X}_{t_0,t}(\boldsymbol{x}),t)$, and $\langle\cdot\rangle$ is some averaging procedure, space, time or ensemble and where $\langle\varepsilon\rangle$ is the energy dissipation rate. Standard derivations of the relationship assume spatial isotropy and the average must be either interpreted as over the spatial domain, or as a time/ensemble average provided the fields are homogenous.

With the Ott-Mann-Gawędzki relation in hand, the mean–squared dispersion of

[1] A better term might be pseudo-Lagrangian as it involves only Eulerian quantities, although resulting from an instantaneous Lagrangian time-derivative.

ideal Lagrangian tracers for short-times can be calculated using only local (in time) Eulerian quantities in closed form [Bitane et al., 2012]:

$$\left\langle |\delta \boldsymbol{X}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle \approx S_2^{\boldsymbol{u}}(\boldsymbol{r},t_0)\ \tau^2 - 2\langle \varepsilon(\boldsymbol{x},t_0)\rangle \tau^3 + \mathcal{O}(\tau^4). \tag{3.2}$$

where $\tau := t - t_0$, $\delta \boldsymbol{X}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) := \boldsymbol{X}_{t_0,t}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{X}_{t_0,t}(\boldsymbol{x})$ is the Lagrangian deviation vector and $S_2^{\boldsymbol{u}}(\boldsymbol{r},t) := \langle |\delta \boldsymbol{u}(\boldsymbol{r},t)|^2 \rangle$ is the second-order structure function. In simulation of three-dimensional turbulence, Bitane et al. [2012] verified the leading order quadratic and cubic behavior for time differences of order the local eddy turnover time at scale $|\boldsymbol{r}|$.

Recently Jucha et al. [2014] discovered that for $3d$ turbulent flows, Eq. (3.2) predicts that pairs of Lagrangian particles initially spread faster backward-in-time than forward-in-time. This is deduced by inspecting the transformation of (3.2) under time reversal $\tau \to -\tau$ and noting that the $\mathcal{O}(\tau^2)$ term is invariant whereas the $\mathcal{O}(\tau^3)$ changes sign. Since $\langle \varepsilon \rangle > 0$ for high-Reynolds number $3d$ turbulence in accordance with the 'zeroth law', this term enhances the dispersion backwards-in-time and whereas forwards-in-time it depletes it. This observation is important because it introduces a Lagrangian "arrow of time" – given a time-history of a fluid velocity field, one could in principle deduce which direction time is running by comparing the magnitudes of short-time mean-squared dispersion.

Unfortunately, for high-Reynolds numbers, the realm of validity of the expansion

(3.2) becomes vanishing small and the argument of Jucha et al. [2014] cannot be naïvely applied. In particular, the Taylor series expansion of the particle trajectories used to derive (3.2) is only guaranteed to converge in a neighborhood of times on the order of the Kolmogorov time-scale $\tau = \mathcal{O}(Re^{-1})$. Thus, it is desirable to have an alternative Lagrangian measure of time–asymmetry that remains valid for arbitrarily large Reynolds numbers. As we will show, there is such a measure involving dispersion of tracer particle in *spatially coarse-grained fields* instead of their fine-grained counterparts.

Furthermore, the arguments of Jucha et al. [2014] do not apply to $2d$ turbulence as stated. It is well known that for smooth initial data in $2d$, solutions stay smooth globally for both the Navier-Stokes and Euler equations (see Proposition 3.1.5 of this chapter). It follows that, in this setting, the short time expansion (3.2) does remain valid for finite times. However, under very mild assumptions on the initial data, the viscous dissipation vanishes in the high-Reynolds number limit and therefore $\mathcal{O}(\tau^3)$ term in the expansion disappears Taylor [1915, 1917b]. However, it has been recognized that what plays the role of $\langle \varepsilon \rangle$ is the "energy flux through scales" and is non-vanishing for both $2d$ and $3d$ flows, if one considers the appropriate setup [Falkovich et al., 2001, Falkovich and Frishman, 2013, Xu et al., 2016]. For $3d$ turbulence, forced or unforced, the flux-through-scale term matches directly onto the anomalous dissipation arising from viscosity (see discussion around Eq. (3.14) and [Duchon and Robert, 2000]). For $2d$ turbulence, this flux matches the input from a small scale forcing, necessary to drive an inverse energy cascade, and not to the viscous dissipation (see discussion around Eq. (3.20)). Here, following their general approach, we give a rigorous argument for the $2d$ inverse cascade under analogous convergence assumptions in the limit of increasingly small scale forcing. Since the contribution of this forcing is typically to input rather than deplete energy, the $2d$ energy flux term has the opposite sign as it does in $3d$. This indeed suggests that for short times dispersion is faster backwards in $3d$ and forwards in $2d$, following the same trend as in Richardson dispersion at later times [Sawford et al., 2005].

We will provide rigorous justification of this observation here, using a coarse-graining approach. In particular, we derive an exact Ott-Mann-Gawędzki relation for ideal tracer particles moving in fields coarse-grained at a scale ℓ. We find that (3.1) holds for separations $|\boldsymbol{r}|$ satisfying $\ell \ll |\boldsymbol{r}| \ll L$, with precisely the flux-through-scale term appearing on right-hand-side. For large Reynolds number (as well as small scale forcing in 2d) we show that this flux matches on to the fine-grained quantities (energy dissipation for $3d$ and forcing input for $2d$ turbulence). We then derive a short–time expansion for such tracers which holds for times of order the local eddy turnover time set by the filter scale and is independent of the details of viscosity and forcing. Our description is local in both space $\boldsymbol{x}$ and as an average over initial separations $\boldsymbol{r}$. Moreover, the generalized Ott-Mann-Gawędzki relation that we prove, and consequently the short-time relative forwards/backwards dispersion, is shown to be universal in that it is completely independent of the details of the filtering operation employed. As a result, we provide rigorous support for the conclusions of Jucha et al. [2014], Falkovich and Frishman [2013], Xu et al. [2016], if these are appropriately interpreted.

3.1 Dissipation anomalies in $2d$ and $3d$ turbulence

Consider smooth solutions $\boldsymbol{u}$ of the incompressible Navier-Stokes equation on $\mathbb{T}^d$:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \tag{3.3}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3.4}$$

In any dimension d, for such solutions the local kinetic energy evolves according to

$$\partial_t \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) + \nabla \cdot \left(\left(\frac{1}{2} |\boldsymbol{u}|^2 + p \right) \boldsymbol{u} - \nu \nabla \frac{1}{2} |\boldsymbol{u}|^2 \right) = -\nu |\nabla \boldsymbol{u}|^2 + \boldsymbol{u} \cdot \boldsymbol{f}. \tag{3.5}$$

The above balance equation makes it apparent that only two features can change the global energy; the molecular viscosity through the viscous heating term $-\nu|\nabla \boldsymbol{u}|^2$ and the forcing through the input term $\boldsymbol{u} \cdot \boldsymbol{f}$. We now discuss some rather well known but surprising properties of the energy balance for high Reynolds-number turbulence.

Freely-decaying and externally-forced incompressible turbulence appear substantially similar for $d > 2$; there is a direct (or forward) cascade of energy from large to small scales. Moreover for $d = 3$, as discussed in the introduction, it is a well known experimental observation that at large Reynolds numbers (small viscosities) the dissipation rate becomes independent of ν:

$$\lim_{\nu \to 0} \nu |\nabla \boldsymbol{u}|^2 := \varepsilon > 0. \tag{3.6}$$

The physical picture is that energy is input, possibly by an external force, at large scales and then it cascades down until it is dissipated by the action of viscosity. Anomalous dissipation requires, as Re increases, that the fluid efficiently transfer energy through the inertial range down to the ever decreasing scales at which viscosity is relevant. As discussed in the introduction, Onsager realized that, in order to sustain an energy cascade to arbitrarily small scales at large Reynolds number, the fluid velocity must develop Hölder-type singularities as $Re \to \infty$. In §3.1.1, we review known results about Onsager's ideas about the dissipative anomaly in $d > 2$ for the direct energy cascade by studying the limit $\nu \to 0$ for a fixed forcing.

On the other hand, in incompressible $2d$ turbulence with smooth data, the viscous energy dissipation always tends to zero [Taylor, 1915, 1917b]. It has long been recognized that the source of major differences between $d = 2$ and $d > 2$ is the presence of an additional invariant – the enstrophy $\Omega(t) = (1/2)\langle|\omega(\boldsymbol{x}, t)|^2\rangle_x$, see e.g. Lee [1951], Fjørtoft [1953]. It has been proposed by Kraichnan [1967], Leith [1968] and

Batchelor [1969] that this extra constraint in $2d$ results in two simultaneous inertial ranges in the flow, a inverse energy cascade range and a forward enstrophy cascade. In this picture, the energy accumulates in large-scales by quite different mechanisms in freely-decaying and externally-forced turbulence: "vortex merger" [Onsager, 1949, Mcwilliams, 1984] for decaying turbulence and "inverse energy cascade" of [Kraichnan, 1967, Leith, 1968, Batchelor, 1969] for forced turbulence.

In §3.1.1, we investigate the inverse energy cascade in $d = 2$ of forced turbulence. In order to study an extended inverse cascade range originating at arbitrarily small scales, one considers the limit where the force acts at increasingly high wavenumber. This can be accomplished, for example, by using a force with compact spectral support and taking the "typical" forcing wavenumber k_f off to infinity (or equivalently the typical forcing length scale $\ell_f = 2\pi/k_f$ is taken to zero). Such a force will have "infinite frequency" and be zero from the distributional point of view, as we see in Proposition 3.1.1 below. One expects that there will be anomalous input of energy from the forcing:

$$\lim_{\ell_f,\nu\to 0} \boldsymbol{u}^{\nu,\ell_f} \cdot \boldsymbol{f}^{\ell_f} := I. \tag{3.7}$$

We call I the "anomalous input" which is typically positive[2] and is fed into the flow

[2] Although this will depends, of course, on the choice of forcing scheme. For example, the forcing could be chosen to be solution–dependent to insure that energy is injected, e.g. Lundgren-type forcing $\boldsymbol{f} = \alpha \boldsymbol{u}$. Another attractive choice of force is to take $\boldsymbol{f}$ to be a homogenous Gaussian random field which is white-noise correlated in time, i.e. $\langle f_i(\boldsymbol{x},t) f_j(\boldsymbol{x}',t')\rangle_f = 2F_{ij}(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$. This has the theoretical advantage that, after averaging over the forcing statistics, the mean injection rate of energy is *solution independent*, i.e. after averaging the balance (3.10), the injection term is $\langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle_f = F_{ii}(0) > 0$, thereby insuring input of energy on average.

at infinitely small scales, where irregular turbulent motion facilitate energy transfer up through the inertial range and into the largest scales of the flow. Many results of this Chapter will require only some uniform integrability of the forcing scheme used, and that $\boldsymbol{f}^{\ell_f} \to 0$ distributionally as $\ell_f \to 0$. Such distributional convergence is ensured, for example, if the forcing acts on a band of increasingly small scales:

Proposition 3.1.1 *Let $\boldsymbol{f}^{\ell_f}$ have compact spectral support inside a band $[k_f/2, 2k_f]$ around some 'typical' wavenumber $k_f = 2\pi/\ell_f$. Further, assume that $\boldsymbol{f}^{\ell_f} \in L^2((0,T); L^2)$ for all $\ell_f > 0$ and $\boldsymbol{f}^{\ell_f} \in L^1((0,T); L^1)$ uniformly in ℓ_f. Then $\boldsymbol{f}^{\ell_f} \to 0$ distributionally as $\ell_f \to 0$.*

Proof For $k_f > 0$, the force is constructed to satisfy $\text{supp}(\widehat{\boldsymbol{f}^{\ell_f}}(\boldsymbol{k})) \subseteq S(k_f)$ where $S(k_f) = \{\boldsymbol{k} \mid k_f/2 \leq |\boldsymbol{k}| \leq 2k_f\}$ is a shell in wavenumber space. Note that we do not explicitly specify how the amplitudes of the forcing depend on k_f; we need only that the family of forces are uniformly L^1 in space-time. Since $\boldsymbol{f}^{\ell_f} \in L^2((0,T); L^2)$ we have,

$$\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \ \varphi(\boldsymbol{x},t)\boldsymbol{f}^{\ell_f}(\boldsymbol{x},t) = \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \ \mathbf{P}_{\ell_f}[\varphi(\boldsymbol{x},t)]\boldsymbol{f}^{\ell_f}(\boldsymbol{x},t), \tag{3.8}$$

where $\mathbf{P}_{\ell_f}$ is the projection onto the shell $S(k_f)$ of wavenumbers in the force $\boldsymbol{f}^{\ell_f}$. Since the Fourier transform of the C^∞ function decays faster than any polynomial, i.e. $|\widehat{\varphi}(\boldsymbol{k},t)| = O(|\boldsymbol{k}|^{-n})$ as $|\boldsymbol{k}| \to \infty$ for any $n \in \mathbb{N}$ and $t \in (0,T)$ and $\boldsymbol{f}^{\ell_f} \in L^1((0,T); L^1)$

uniformly in ℓ_f,

$$\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\ \mathbf{P}_{\ell_f}[\varphi(\boldsymbol{x},t)]\boldsymbol{f}^{\ell_f}(\boldsymbol{x},t) = \|\mathbf{P}_{\ell_f}\varphi\|_\infty \|\boldsymbol{f}^{\ell_f}\|_1 \leq \|\boldsymbol{f}^{\ell_f}\|_1 \sum_{\boldsymbol{k}\in S(k_f)} |\widehat{\varphi}(\boldsymbol{k})| \xrightarrow{\ell_f\to 0} 0.$$

By the equality (3.8) that $\boldsymbol{f}^{\ell_f} \to 0$ distributionally as $\ell_f \to 0$.

Remark In Proposition 3.1.1, we do not assume that $\boldsymbol{f}^{\ell_f} \in L^2$ uniformly in ℓ_f. In fact, if this were the case, the forcing would be unable to sustain an inverse cascade asymptotically! To see, we additionally assume that $\boldsymbol{u}^{\ell_f} \to \boldsymbol{u}$ strongly in L^2 and $\boldsymbol{f}^{\ell_f} \in L^2$ uniformly. Then:

$$\begin{aligned}\int \mathrm{d}\boldsymbol{x}\ \varphi(\boldsymbol{x})\boldsymbol{u}^{\ell_f}(\boldsymbol{x})\boldsymbol{f}^{\ell_f}(\boldsymbol{x}) &= \int \mathrm{d}\boldsymbol{x}\ \varphi(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})\boldsymbol{f}^{\ell_f}(\boldsymbol{x}) + \int \mathrm{d}\boldsymbol{x}\ \varphi(\boldsymbol{x})\left(\boldsymbol{u}^{\ell_f}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x})\right)\boldsymbol{f}^{\ell_f}(\boldsymbol{x}) \\ &= \int \mathrm{d}\boldsymbol{x}\ \mathbf{P}_{\ell_f}\left[\varphi(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})\right]\boldsymbol{f}^{\ell_f}(\boldsymbol{x}) + \int \mathrm{d}\boldsymbol{x}\ \varphi(\boldsymbol{x})\left(\boldsymbol{u}^{\ell_f}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x})\right)\boldsymbol{f}^{\ell_f}(\boldsymbol{x}).\end{aligned}$$

The second integral vanishes due to the strong convergence $\boldsymbol{u}^{\ell_f}$ as $\ell_t \to 0$:

$$\left|\int \mathrm{d}\boldsymbol{x}\ \varphi(\boldsymbol{x})\left(\boldsymbol{u}^{\ell_f}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x})\right)\boldsymbol{f}^{\ell_f}(\boldsymbol{x})\right| \leq \|\varphi\|_\infty \|\boldsymbol{u}^{\ell_f} - \boldsymbol{u}\|_2 \|\boldsymbol{f}^{\ell_f}\|_2 \xrightarrow{\ell_f\to 0} 0.$$

On the other hand, the first term also vanishes because:

$$\left|\int \mathrm{d}\boldsymbol{x}\ \mathbf{P}_{\ell_f}\left[\varphi(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})\right]\boldsymbol{f}^{\ell_f}(\boldsymbol{x})\right| \leq \|\mathbf{P}_{\ell_f}\left[\varphi\boldsymbol{u}\right]\|_2 \|\boldsymbol{f}^{\ell_f}\|_2$$

A simple application of Hölder's inequality shows $\varphi\boldsymbol{u} \in L^2$, and thus the Fourier-transform decays at least as $\widehat{\varphi \boldsymbol{u}} = O(|\boldsymbol{k}|^{-1})$ as $|\boldsymbol{k}| \to \infty$. Thus, $\|\mathbf{P}_{\ell_f}[\varphi \boldsymbol{u}]\|_2 \xrightarrow{\ell_f \to 0} 0$ and the power input from the force will vanish distributionally if $\|\boldsymbol{f}^{\ell_f}\|_2$ is bounded uniformly in ℓ_f. [3]

3.1.1 Inertial dissipation and Kolmogorov 4/5ths-type laws

We now investigate conditions for such anomalies to exist as distributions and derive 'inertial range' expressions for them. These expressions are essentially Kolmogorov 4/5th's laws in that they identify the anomaly terms (3.6) and (3.7) arising from the direct effects of viscosity or forcing with the anomalous dissipation/input facilitated by turbulent cascade. This section is primarily an exposition of the work of Duchon and Robert [2000]. Many of the stated propositions are quoted directly from their paper with at most some simple extensions necessary for the present study. We also use the methods employed by Duchon and Robert [2000] to study $2d$ inverse cascade, which they never considered, and prove a theorem analogous to Proposition 4 of their paper.

The first result we quote asserts that weak Euler solutions with some space-time integrability satisfy a distributional energy balance with a possible 'turbulent cascade defect'.

[3] We thank Gregory Eyink for pointing this out.

Proposition 3.1.2 (Proposition 2 of Duchon and Robert [2000]) *Let $\boldsymbol{u} \in L^3((0,T);L^3)$ be a weak solution to the Euler equation with forcing $\boldsymbol{f} \in L^2((0,T);L^2)$ on $\mathbb{T}^d$. Let G_ℓ be a standard mollifier and set*

$$\Pi_\ell(\boldsymbol{u}) := \frac{1}{4\ell}\int_{\mathbb{T}^d} d\boldsymbol{r} (\nabla G)_\ell(\boldsymbol{r}) \cdot \delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})|\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})|^2. \tag{3.9}$$

Then as $\ell \to 0$, the functions $\Pi_\ell(\boldsymbol{u}) \in L^1((0,T)\times\mathbb{T}^d)$ converge in the sense of distributions on $(0,T)\times\mathbb{T}^d$ to a distribution $\Pi(\boldsymbol{u})$ independent of the mollifying sequence and the following local kinetic energy balance holds:

$$\partial_t\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + \nabla\cdot\left(\left(\frac{1}{2}|\boldsymbol{u}|^2 + p\right)\boldsymbol{u}\right) = -\Pi(\boldsymbol{u}) + \boldsymbol{u}\cdot\boldsymbol{f}. \tag{3.10}$$

Remark As discussed in the introduction of this thesis, Constantin et al. [1994] proved an 'Onsager singularity theorem' for incompressible Euler, i.e. in order to sustain a turbulent cascade to arbitrarily small scales, a weak Euler solution cannot be in the class $L^3_t B_3^{1/3+,\infty}$. Duchon and Robert [2000] used the formula (3.9) for Π_ℓ to derive a slightly sharper version of the result of Constantin et al. [1994].

Having established a distributional energy balance (3.10) for general weak Euler solutions, we now do the same for Leray solutions of Navier-Stokes, which is also directly quoted:

Proposition 3.1.3 (Proposition 1 of Duchon and Robert [2000]) *Let $\boldsymbol{u}^\nu \in L^2((0,T);H^1)\cap L^\infty((0,T);L^2)$ be a weak solution to the Navier-Stokes equation on $\mathbb{T}^d$ with forcing $\boldsymbol{f} \in L^\infty((0,T);L^2)$. Then as $\ell \to 0$, the functions $D_\ell(\boldsymbol{u}^\nu)$ are defined by* (3.9) *(but renamed) and converge in the sense of distributions on $(0,T)\times\mathbb{T}^d$ to a distribution $D(\boldsymbol{u}^\nu)$ independent of the mollifying sequence and the following local kinetic energy balance holds in the sense of distributions:*

$$\partial_t\left(\frac{1}{2}|\boldsymbol{u}^\nu|^2\right)+\nabla\cdot\left(\left(\frac{1}{2}|\boldsymbol{u}^\nu|^2+p^\nu\right)\boldsymbol{u}^\nu-\nu\nabla\frac{1}{2}|\boldsymbol{u}^\nu|^2\right)=-\nu|\nabla\boldsymbol{u}^\nu|^2-D(\boldsymbol{u}^\nu)+\boldsymbol{u}^\nu\cdot\boldsymbol{f}. \tag{3.11}$$

Remark The distribution $D(\boldsymbol{u}^\nu)$ represents anomalous dissipation due to possible Leray singularities. In particular, if global existence and uniqueness of Navier-Stokes were established, then $D(\boldsymbol{u}^\nu)=0$. Indeed, for $d=2$ with smooth data (e.g. $\omega_0\in C^{k,\alpha}$ for $k\geq 1$ and $\alpha\in(0,1)$) this is established *a-priori*, see Proposition 3.1.5 below.

Using Proposition 3.1.3, we may now state a result valid for Euler solutions obtained by strong space-time L^3 limits of Navier-Stokes solutions as $\nu\to 0$. This will allows us to identify the inertial cascade $\Pi(\boldsymbol{u})$ of Proposition 3.1.2 with a possible dissipation anomaly appearing in the limit of infinite Reynolds number (zero viscosity).

Proposition 3.1.4 (Proposition 4 of Duchon and Robert [2000]) *Let $\boldsymbol{u} \in L^3((0,T); L^3)$ be a weak solution to the Euler equation with forcing $\boldsymbol{f} \in L^\infty((0,T); L^2)$ on $\mathbb{T}^d$ which is obtained as a strong limit in L^3 of (forced) Navier-Stokes solutions as $\nu \to 0$. Then the following local kinetic energy balance holds in the sense of distributions:*

$$\partial_t \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) + \nabla \cdot \left(\left(\frac{1}{2} |\boldsymbol{u}|^2 + p \right) \boldsymbol{u} \right) = -\varepsilon + \boldsymbol{u} \cdot \boldsymbol{f} \tag{3.12}$$

where the following necessarily exists:

$$\varepsilon := \mathcal{D}\text{-}\lim_{\nu \to 0} \left(\nu |\nabla \boldsymbol{u}^\nu|^2 + D(\boldsymbol{u}^\nu) \right). \tag{3.13}$$

Remark Combining Propositions 3.1.2 and 3.1.4, we find that strong L^3 limits of Navier-Stokes solutions have the property that, as distribution:

$$\Pi(\boldsymbol{u}) = \varepsilon \geq 0. \tag{3.14}$$

See e.g. Duchon and Robert [2000], Eyink [2002, 2015c]. Together with Remark 3.1.1, (3.14) shows that anomalous dissipation requires inviscid limit Euler solutions (if they exist) to have L^3 Besov exponent $s \leq 1/3$.

All results thus far hold in arbitrary spatial dimension d. We now restrict our considerations to two spatial dimensions. In $2d$, much more is known *a-priori* about solutions. For example, weak Navier-Stokes solutions in the class $\boldsymbol{u}^\nu \in L^2([0,\infty); H^1) \cap$

$L^\infty([0,\infty);L^2)$ are unique and satisfy equation (3.11) with $D(\boldsymbol{u}^\nu)=0$. In fact, for suitably smooth initial data and forcing, one has global existence for Navier-Stokes and Euler both, as well as guaranteed strong convergence. A standard result is (see e.g. Majda and Bertozzi [2002], Lions and Moulden [1997], Kato [1984], Constantin [1986].):

Proposition 3.1.5 *Let $\boldsymbol{f}\in C([0,T],C^k)$ and $\boldsymbol{u}_0\in C^{1+k,\alpha}$ be divergence free with $k\geq 1$ and $\alpha\in(0,1)$. Then there exists a unique strong solution $\boldsymbol{u}\in C^1([0,T],C^{1+k,\alpha})$ to the forced 2d Euler equations (correspondingly, 2d Navier-Stokes) on any time interval $[0,T]$. Furthermore, $\boldsymbol{u}^\nu\to\boldsymbol{u}$ in $L^\infty([0,T];C^{k,\alpha})$.*

For less regular data of the type $\boldsymbol{u}_0\in L^2$ with $\omega_0\in L^r$ for $r\in(1,\infty)$, it is known at least that there exists a weak Euler solution (possibly non-unique) in the class $\boldsymbol{u}\in C([0,\infty);W^{1,r})$. Provided that $r>6/5$, by Sobolev embedding $W^{1,r}\subset L^3$, such a weak solution is subject to the previous considerations in this section. Moreover, provided that $r>3/2$, it follows that $\Pi=0$ with Π defined in Proposition 3.1.2. These facts appear in Duchon and Robert [2000].

Finally, we consider the case of high-Reynolds number, two-dimensional fluids with increasingly small scale forcing. As discussed previously, this scenario is of the type relevant to the inverse energy cascade in $2d$ turbulence. More precisely, we consider the order of limits $\nu\to 0$ first and subsequently the limit where typical forcing scale $\ell_f\to 0$ so that $\boldsymbol{f}^{\ell_f}\to 0$ distributionally (details of the forcing discussed in the beginning of §3.1). [4] We take the forcing and initial data to be as in Proposition 3.1.5 so that global solutions exist for both Euler and Navier-Stokes and strong convergence as $\nu \to 0$ is guaranteed. For such Euler solutions obtained in the high-Re limit, we then show that as the forcing scale vanishes (see Proposition 3.1.1), if L^3 strong limits exist, they converge to weak solutions of unforced Euler and they must satisfy an kinetic energy balance with possible anomalous input from the forcing acting at infinitesimal scales.

Proposition 3.1.6 *Let $\boldsymbol{f}^{\ell_f}$ and $\boldsymbol{u}_0$ be as in Proposition 3.1.5 with the additional property that $\boldsymbol{f}^{\ell_f} \to 0$ distributionally as $\ell_f \to 0$. Suppose that strong L^3 limits of forced Euler solutions $\boldsymbol{u}^{\ell_f} \in C^1([0,T], C^{1+k,\alpha})$ exist as $\ell_f \to 0$. Then $\boldsymbol{u} \in L^3((0,T); L^3)$ obtained in this limit is a weak solution to the unforced Euler equation on $(0,T)\times\mathbb{T}^2$ and satisfies the local kinetic energy balance in the sense of distributions:*

$$\partial_t \left(\frac{1}{2}|\boldsymbol{u}|^2\right) + \nabla\cdot\left(\left(\frac{1}{2}|\boldsymbol{u}|^2 + p\right)\boldsymbol{u}\right) = I \tag{3.15}$$

[4] It is also natural to study the joint limit $\ell_f, \nu \to 0$ together in some way. To obtain the results of Proposition 3.1.6, it is necessary to control the energy dissipation rate in this joint limit. This will undoubtably depend on the details of how the limit is taken. For example, if ℓ_f, ν together but with $\nu \to 0$ at an asymptotically faster rate then $\ell_f \to 0$, it is reasonable to expect there will be no anomalous dissipation from viscosity and that the statement of Proposition 3.1.6 will remain true. However, for steady turbulence, if $\ell_f \propto \nu^{1/2}$, a finite fraction of the energy will always cascade to high wavenumber [Eyink, 1996]. This is expected to be true because $\ell_d := \nu^{1/2}/\eta^{1/6}$ is the Kraichnan-Batchelor dissipation length in steady-state 2D turbulence (where η is the enstrophy cascade rate). If so, the balance in Proposition 3.1.6 would need to be modified to include the contribution from anomalous dissipation arising from viscosity. We do not address these issues here, however the joint limit is arguably very relevant for comparison with experimental/numerical observations and tests.

where the 'anomalous input' necessarily exists as a distribution:

$$I := \mathcal{D}\text{-}\lim_{\ell_f \to 0} \boldsymbol{u}^{\ell_f} \cdot \boldsymbol{f}^{\ell_f}. \tag{3.16}$$

Proof We first show that any strong space-time L^3 limits of forced Euler solutions $\boldsymbol{u}^{\ell_f}$ as $\ell_f \to 0$ are weak solutions $\boldsymbol{u}$ of unforced Euler. We start from for the forced Euler equation:

$$\partial_t \boldsymbol{u}^{\ell_f} + \nabla \cdot (\boldsymbol{u}^{\ell_f} \otimes \boldsymbol{u}^{\ell_f}) - \nabla p^{\ell_f} + \boldsymbol{f}^{\ell_f} \tag{3.17}$$

interpreted in the sense of distributions. Since $\boldsymbol{f}^{\ell_f} \to 0$ distributionally by assumption, we must only check convergence of the remaining terms. It is easy to see that if $\boldsymbol{u}^{\ell_f} \to \boldsymbol{u}$ strongly in L^2, then as distributions

$$\partial_t \boldsymbol{u}^{\ell_f} + \nabla \cdot (\boldsymbol{u}^{\ell_f} \otimes \boldsymbol{u}^{\ell_f}) \ \xrightarrow{\mathcal{D}}\ \partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \qquad \text{as} \qquad \ell_f \to 0. \tag{3.18}$$

The pressure is fixed by the incompressibility constraint. Note that $p^{\ell_f} \to p$ strongly in $L^{3/2}$ since the difference satisfies the following Poisson equation:

$$-\Delta(p^{\ell_f} - p) = (\nabla \otimes \nabla) : \left[(\boldsymbol{u}^{\ell_f} \otimes \boldsymbol{u}^{\ell_f}) - (\boldsymbol{u} \otimes \boldsymbol{u}) \right].$$

By boundedness of Calderon-Zygmund operators on L^p, we have the followings estimate which follows from an application of Cauchy-Schwartz inequality, using the fact that $\boldsymbol{u}^{\ell_f} \in L^3$ uniformly in ℓ_f:

$$\|p^{\ell_f} - p\|_{3/2} \leq C\|\boldsymbol{u}^{\ell_f} - \boldsymbol{u}\|_3.$$

Whence, we obtain the desired convergence of the pressure to some $p \in L^{3/2}((0,T); L^{3/2})$. Thus, we have that verified that strong limits $\boldsymbol{u}$ satisfy

$$\partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p$$

in the sense of distributions, as desired.

We now turn to the energy conservation. For any given $\ell_f > 0$, the considerations of Proposition 3.1.5 apply and the Euler solution satisfies the balance equation

$$\partial_t \left(\frac{1}{2}|\boldsymbol{u}^{\ell_f}|^2\right) + \nabla \cdot \left(\left(\frac{1}{2}|\boldsymbol{u}^{\ell_f}|^2 + p^{\ell_f}\right) \boldsymbol{u}^{\ell_f}\right) = \boldsymbol{u}^{\ell_f} \cdot \boldsymbol{f}. \tag{3.19}$$

Since $\boldsymbol{u}^{\ell_f} \to \boldsymbol{u}$ in $L^3((0,T); L^3)$ and $p^{\ell_f} \to p$ in $L^{3/2}((0,T); L^{3/2})$, it is straightforward to verify by the same arguments as in Duchon and Robert [2000] convergence of all the 'fluxes' in L^1 and therefore:

$$\begin{aligned} &\partial_t \left(\frac{1}{2}|\boldsymbol{u}^{\ell_f}|^2\right) + \nabla \cdot \left(\left(\frac{1}{2}|\boldsymbol{u}^{\ell_f}|^2 + p^{\ell_f}\right) \boldsymbol{u}^{\ell_f}\right) \xrightarrow{\mathcal{D}} \\ &\qquad \partial_t \left(\frac{1}{2}|\boldsymbol{u}|^2\right) + \nabla \cdot \left(\left(\frac{1}{2}|\boldsymbol{u}|^2 + p\right) \boldsymbol{u}\right) \quad \text{as} \quad \ell_f \to 0. \end{aligned}$$

From the local energy balance equation (3.19) and the above convergence result, it follows that I defined by (3.16) exists as a distribution.

Remark Since $\boldsymbol{u} \in L^3((0,T);L^3)$ of Proposition 3.1.6 is a weak Euler solution with $\boldsymbol{f} = 0$ in the sense of distributions, it follows by comparison of Eq. (3.10) of Proposition 3.1.2 and Eq. (3.15) of Proposition 3.1.6 that

$$\Pi(\boldsymbol{u}) = -I. \tag{3.20}$$

From the formula (3.20), we can derive an 'Onsager singularity theorem' for the inverse-cascade: in order sustain transport of energy from infinitely small scales (i.e. non-zero energy-flux Π), according to Remark 3.1.1, the limiting weak Euler solution cannot be in the space $L_t^3 B_3^{1/3+,\infty}$. Note that Kraichnan's theory for energy spectrum in the inverse-cascade range predicts a scaling $E(k) \sim k^{-5/3}$, which is equivalent to a second-order structure function scaling $S_2(\ell) \sim \ell^{1/3}$. This is consistent with Onsager's original prediction (and the improvements due to Constantin et al. [1994]) of the required regularity to sustain the turbulent energy cascade.

Remark In the next section, we introduce a different scale-flux term $Q_\ell(\boldsymbol{u})$ (Eq. (3.37)) which is distinct from $\Pi_\ell(\boldsymbol{u})$ (Eq. (3.9)) at finite $\ell > 0$. However, $Q_\ell(\boldsymbol{u})$ appears as the energy scale-transfer term in the balance equation for *resolved* kinetic energy:

$$\partial_t\left(\frac{1}{2}|\overline{\boldsymbol{u}}_\ell|^2\right)+\nabla\cdot\left(\left(\frac{1}{2}|\overline{\boldsymbol{u}}_\ell|^2+\bar{p}_\ell\right)\cdot\overline{\boldsymbol{u}}_\ell+\tau_\ell(\boldsymbol{u},\boldsymbol{u})\cdot\overline{\boldsymbol{u}}_\ell\right)=-Q_\ell(\boldsymbol{u})+\overline{\boldsymbol{u}}_\ell\cdot\overline{\boldsymbol{f}}_\ell. \quad (3.21)$$

Hence, by the assumption that $\boldsymbol{u}\in L^3((0,T);L^3)$ and the same types of arguments employed above, one can show that $Q_\ell(\boldsymbol{u})$ has a distributional limit as $\ell\to 0$ and moreover that

$$\Pi(\boldsymbol{u})=\mathcal{D}\text{-}\lim_{\ell\to 0}Q_\ell(\boldsymbol{u}) \quad (3.22)$$

where Π defined by (3.9) in Proposition 3.1.2.

Combining Eqs. (3.20), (3.14), and (3.22) from Remarks 3.1.1 3.1.1 and 3.1.1, we identify the turbulent energy cascade terms with the dissipation/input anomalies:

$$\Pi=-I, \qquad d=2 \quad (3.23)$$

$$\Pi=\quad \varepsilon\le 0, \qquad d>2. \quad (3.24)$$

To summarize, the identification (3.23) holds in the limit where viscosity goes to zero first for smooth initial data (putting us in the realm of strong solutions), and subsequently the limit as the forcing scale is taken to zero, assuming strong L^3 limits exist as $\ell_f\to 0$. On the other hand, the identification (3.24) is valid for any finite energy forcing in the limit of zero viscosity, again assuming that strong L^3 limits exist as $\nu\to 0$. Note that (3.24) also holds for $2d$ turbulence but is uninteresting there since

$\varepsilon = 0$ (as, for regular enough data, there can be no dissipative anomaly according to Proposition 3.1.5). It has been pointed out by Duchon and Robert [2000], Eyink [2002] that the formulae (3.23) and (3.24) are, in a sense, *non-statistical*, spatially local, versions of the Kolmogorov 4/5ths law Kolmogorov [1941]. In particular, they relate the anomaly arising from the inertial range cascade (3.9) to the viscous/forcing defects. Thus, there can be anomalous energy dissipation $\varepsilon > 0$ or input $I \neq 0$ only if the turbulent cascade persists to arbitrarily small scale. This, because of the results of Constantin et al. [1994], yields testable predictions on the scaling exponents for structure functions consistent with an energy defect.

With these identifications in hand, we now investigate time asymmetry of dispersion of ideal tracers moving in filtered fields.

3.2 A Lagrangian manifestation of the dissipative anomaly

3.2.1 Short-time dispersion in coarse-grained fields

Let $\boldsymbol{u}$ solve the Navier-Stokes equation on $\mathbb{T}^d$ with $d = 2, 3$:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f} + \Lambda \boldsymbol{u} \tag{3.25}$$

where, for $q \geq 0$, $\Lambda = \alpha_q(-\Delta)^{-q}$ for $0 \leq q < d$ is a smoothing operator which acts as a large-scale damping.[5] Here we study properties of particles moving in a filtered Navier-Stokes field. Application of spatial filtering (or coarse-graining) in turbulence is common in the turbulence modelling method of large-eddy simulation and has been discussed systematically in Germano [1992]. A detailed modern discussion of the *filtering approach* can be found e.g. in [Eyink, 2005, 2015a,c]. The basic idea is to define a "large-scale" velocity field $\overline{\boldsymbol{u}}_\ell$ by coarse-graining with a smooth (dilated) filter kernel $G_\ell(\boldsymbol{r}) := \ell^{-d}G(\boldsymbol{r}/\ell)$ for $G \in C_0^\infty$ as follows:

$$\overline{\boldsymbol{u}}_\ell(\boldsymbol{x}, t) = \int_\Omega d\boldsymbol{r}\ G_\ell(\boldsymbol{r})\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}, t). \tag{3.26}$$

The artificial length-scale ℓ of the kernel sets the scale over which the fine-grained fields are smoothed. The operation of filtering can be intuitively thought of as the effect of "taking off one's glasses"; it is a purely passive operation which fuzzes out small scale structures. From this point of view, the velocity $\boldsymbol{u}$ is called the "fine-grained" or " bare" field. The residual small-scale motion or "fluctuation" field is then defined simply by $\boldsymbol{u}'_\ell := \boldsymbol{u} - \overline{\boldsymbol{u}}_\ell$. An evolution equation for the large-scale field

[5] Without large-scale damping (such as Ekman friction or more general inverse-Laplacian dissipation), the steady-state of forced 2d turbulence has energy diverging to infinity as $\nu \to 0$. Therefore, in (3.27), we have included the possibility of a damping term in the Navier-Stokes equation of the form $\alpha_q(-\Delta)^{-q}\boldsymbol{u}$ with $q \geq 0$ and constant α_p, so that our analysis applies to $2d$ steady states. For example, with $q = 0$, this term becomes $\alpha\boldsymbol{u}$ which is a linear Ekman friction term used often in modelling geophysical phenomena. In $3d$ flows, energy generally cascades downscale, not upscale, where viscosity removes it, and so we need not consider additional large scale damping terms ($\alpha_p = 0$ for $3d$).

$\overline{\boldsymbol{u}}_\ell$ follows from filtering Eq. (3.25):

$$\partial_t \overline{\boldsymbol{u}}_\ell + \overline{\boldsymbol{u}}_\ell \cdot \nabla \overline{\boldsymbol{u}}_\ell = -\nabla \bar{p}_\ell - \nabla \cdot \tau_\ell(\boldsymbol{u}, \boldsymbol{u}) + \overline{\boldsymbol{f}}_\ell + \Lambda \overline{\boldsymbol{u}}_\ell + \nu \Delta \overline{\boldsymbol{u}}_\ell \tag{3.27}$$

where coarse-graining cumulant or "turbulent stress term" $\tau_\ell(\boldsymbol{u}, \boldsymbol{u}) = \overline{(\boldsymbol{u} \otimes \boldsymbol{u})}_\ell - \overline{\boldsymbol{u}}_\ell \otimes \overline{\boldsymbol{u}}_\ell$ in (3.27) represents the stress of the filtered out scales $< \ell$ on the retained scales $\geq \ell$ and arises because the filtering does not commute with the nonlinearity. Lagrangian tracers in the coarse-grained field satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{X}^\ell_{t_0,t}(\boldsymbol{x}) = \overline{\boldsymbol{u}}_\ell(\boldsymbol{X}^\ell_{t_0,t}(\boldsymbol{x}), t), \qquad \boldsymbol{X}^\ell_{t_0,t_0}(\boldsymbol{x}) = \boldsymbol{x}. \tag{3.28}$$

These trajectories represent ideal particle paths from the point of view of someone looking at the flow "without glasses" (note, these are *not* particle trajectories of the fine-grained fields which have subsequently been smoothed). The mean-squared dispersion of these trajectories is computed from the formula:

$$\left\langle |\delta \boldsymbol{X}^\ell_{t_0,t}(\boldsymbol{r}; \boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_\varphi = \int_{t_0}^t \int_{t_0}^t \mathrm{d}s \mathrm{d}s' \left\langle \delta \boldsymbol{v}^\ell(\boldsymbol{r}; \boldsymbol{x}, s) \cdot \delta \boldsymbol{v}^\ell(\boldsymbol{r}; \boldsymbol{x}, s') \right\rangle_\varphi$$

with the *Lagrangian* filtered velocity defined as $\boldsymbol{v}^\ell(\boldsymbol{x}, t) \equiv \overline{\boldsymbol{u}}_\ell(\boldsymbol{X}^\ell_{t_0,t}(\boldsymbol{x}), t)$ and where $\langle \cdot \rangle_\varphi$ is a smearing operation in base-points $\boldsymbol{x}$ and separations $\boldsymbol{r}$:

$$\langle f(\boldsymbol{r}; \boldsymbol{x}) \rangle_\varphi := \int_{\mathbb{T}^d} \mathrm{d}\boldsymbol{x} \int_{\mathbb{T}^d} \mathrm{d}\boldsymbol{r} \ \varphi(\boldsymbol{x}, \boldsymbol{r}) f(\boldsymbol{r}; \boldsymbol{x})$$

for $\varphi \in C_0^\infty$. We consider here test functions φ which may be decomposed as $\varphi(\boldsymbol{x},\boldsymbol{r}) = \phi(\boldsymbol{x})\psi_R(\boldsymbol{r})$ with $\phi \in C_0^\infty$ arbitrary and ψ_R designed to be a local average over separations $|\boldsymbol{r}| < R$, i.e. having the properties that $\psi_R \geq 0$, $\|\psi_R\|_1 = 1$ and $\mathrm{supp}(\psi_R) \subset B_R(0)$. We will use the notations $\langle\cdot\rangle_\phi$ and $\langle\cdot\rangle_\psi$ to indicate that the smearing taken only in $\boldsymbol{x}$ with ϕ or only in $\boldsymbol{r}$ with ψ.

Taylor expanding the above integrand for short times increments $\tau = t - t_0$:

$$\left\langle|\delta \boldsymbol{X}^\ell_{t_0,t}(\boldsymbol{r};\boldsymbol{x})-\boldsymbol{r}|^2\right\rangle_\varphi \approx \langle S_2^{\bar{\boldsymbol{u}}_\ell,\varphi}(\boldsymbol{r},t_0)\rangle_\psi\ \tau^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle|\delta \boldsymbol{v}^\ell(\boldsymbol{r};\boldsymbol{x},t)|^2\right\rangle_\varphi\Big|_{t=t_0}\tau^3 + \mathcal{O}(\tau^4). \tag{3.29}$$

This expansion should hold up to times of order $\tau_\ell := \ell/\delta u(\ell)$ (the local eddy turnover time at scale ℓ), although, for the validity of the expansion (3.29) and the proof of Proposition 3.2.2, we will need only that trajectories in filtered fields are C^4 in time. In fact, for strong solutions of some fluid models, even analyticity of trajectories in the fine-grained field is known [Frisch and Villone, 2014, Frisch and Zheligovsky, 2014, Zheligovsky and Frisch, 2014, Constantin et al., 2015], but this is a more refined issue which we do not discuss further here.

3.2.2 Generalized Ott-Mann-Gawędzki relation

We now come to our generalization of the Ott-Mann-Gawędzki in the coarse-grained setting.

Proposition 3.2.1 *Let $\boldsymbol{f}^{\ell_f}$ and $\boldsymbol{u}_0$ be as in Proposition 3.1.5 with the additional property that $\boldsymbol{f}^{\ell_f} \to 0$ distributionally as $\ell_f \to 0$ and $\boldsymbol{f}^{\ell_f} \in L^1$ space-time uniformly in ℓ_f. Let $\boldsymbol{u}^\nu$ be a solution of the incompressible Navier-Stokes equation with forcing $\boldsymbol{f}^{\ell_f}$ on $[0,T]\times\mathbb{T}^d$. Suppose L^3-strong limits exist as $\nu \to 0$ for $d > 2$ and as $\ell_f, \nu \to 0$ for 2d. Then for any $t_0 \in [0,T]$,*

$$\lim_{R\to 0}\lim_{\ell\to 0}\lim_{\ell_f,\nu\to 0}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\delta\boldsymbol{v}_\ell(\boldsymbol{r};\boldsymbol{x},t)|^2\rangle_\varphi\Big|_{t=t_0} = 2\langle I(\boldsymbol{x},t_0)\rangle_\phi \qquad d=2, \tag{3.30}$$

$$\lim_{R\to 0}\lim_{\ell\to 0}\lim_{\nu\to 0}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\delta\boldsymbol{v}_\ell(\boldsymbol{r};\boldsymbol{x},t)|^2\rangle_\varphi\Big|_{t=t_0} = -2\langle \varepsilon(\boldsymbol{x},t_0)\rangle_\phi \qquad d>2. \tag{3.31}$$

Remark The order of limits that we consider for $d > 2$ is as follows

$$R \to 0, \quad \ell \to 0, \quad \nu \to 0.$$

Recall that for $d > 2$, we take the forcing as fixed and assume that $\boldsymbol{u}^\nu \to \boldsymbol{u}$ strongly in L^3. By Proposition 3.1.4, $\boldsymbol{u} \in L^3((0,T);L^3)$ is a weak Euler solution satisfying a kinetic energy balance equation with a possible anomaly term. The $\nu \to 0$ limit is necessarily performed before $\ell \to 0$, since, for irregular high Re turbulent solutions, the direct effects of viscosity can be neglected only on a coarse-grained level.

For $d = 2$, we consider the limits:

$$R \to 0, \quad \ell \to 0, \quad \ell_f \to 0, \quad \nu \to 0.$$

We assume that the initial data and forcing of the type assumed in Proposition 3.1.5. Therefore, taking the inviscid limit first at fixed ℓ_f, for smooth enough initial data, Proposition 3.1.5 guarantees strong convergence in L^3 to a smooth Euler solution. Next we take $\ell_f \to 0$, assuming that L^3 limits exist. According to Proposition 3.1.6, the velocity field obtained in the limit will be an unforced weak Euler solution $\boldsymbol{u} \in L^3((0,T);L^3)$. Again, the direct effects of forcing which acts at infinitesimal scales can be neglected for turbulent solutions only after the filtering.

For both $d = 2$ and $d > 2$, the smoothing scale ℓ is taken to zero, with separation R fixed. Note that if the reverse limit were considered, the the right-hand-sides of (3.30) and (3.31) would necessarily vanish since particle trajectories in smooth fields are locally unique and therefore the Lagrangian velocities must coincide as $t \to t_0$. In general, non-triviality of this limit requires that the fine-grained velocity fields be suitably rough, as discussed in the Introduction and is discussed in Remarks 3.1.1 and 3.1.1.

Finally, at finite R, the right-hand-side of (3.30) and (3.31) measures a relative Lagrangian quantity for particles populating a finite region of space. The coincidence $R \to 0$ limit is taken to remove the effects of local spatial variability in the environment.

Proof Using that $X^{\ell}_{t_0,t_0}(\boldsymbol{x}) = \boldsymbol{x}$, as well as the trajectory equation (3.28), one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\delta\boldsymbol{v}_\ell(\boldsymbol{r};\boldsymbol{x},t)|^2\rangle_\varphi\Big|_{t=t_0} = \langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x},t_0)\cdot\delta\boldsymbol{a}_\ell(\boldsymbol{r};\boldsymbol{x},t_0)\rangle_\varphi. \tag{3.32}$$

The Eulerian acceleration increment from the filtered Navier-Stokes equation is:

$$\delta\boldsymbol{a}_\ell(\boldsymbol{r};\boldsymbol{x}) \equiv -\nabla_x\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x}) + \nu\Delta_x\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) + \delta\overline{(\boldsymbol{f}^{\ell_f})}_\ell(\boldsymbol{r};\boldsymbol{x}) + \Lambda\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) - \nabla_x\cdot\delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})$$

where $\delta\tau_\ell(\boldsymbol{r};\boldsymbol{x}) = \tau_\ell(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{x}+\boldsymbol{r}) - \tau_\ell(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{x})$. Thus we have from (3.32):

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\delta\boldsymbol{v}_\ell(\boldsymbol{r};\boldsymbol{x})|^2\rangle_\varphi\Big|_{t=t_0} =& -\langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\nabla_x\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi + \nu\langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\Delta_x\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi \\ &+ \langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\delta\overline{(\boldsymbol{f}^{\ell_f})}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi + \langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\Lambda\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi \\ &- \langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\nabla_x\cdot\delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi.\end{aligned}$$

We estimate each of these contributions separately. First, the viscous contribution, using Young's inequality for convolutions, is bounded by:

$$|\nu\langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\Delta_x\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi| \leq \frac{4\nu}{\ell^2}\|G_\ell\|_1\|(\Delta G)_\ell\|_1\|\phi\|_\infty\|\boldsymbol{u}\|_2^2 \xrightarrow{\nu\to 0} 0$$

with $\ell_f > 0$ and $\ell > 0$ fixed, since $\boldsymbol{u}$ is uniformly bounded L^2. Next we treat the pressure-work term. By incompressibility:

$$\begin{aligned}\langle \delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\nabla_x\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi =& \langle\nabla_x\cdot[\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})]\rangle_\varphi \\ =& -\langle\nabla_x\varphi(\boldsymbol{x},\boldsymbol{r})\cdot\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_{x,r}\end{aligned}$$

By Hölder's inequality, we have:

$$\begin{aligned}|\langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\nabla_x\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi| \leq\ & C\|\nabla_x\phi\|_\infty \sup_{|\boldsymbol{r}|<R}\|\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\|_3 \sup_{|\boldsymbol{r}|<R}\|\delta\bar{p}_\ell(\boldsymbol{r};\boldsymbol{x})\|_{3/2} \\ \xrightarrow{\ell,\ell_f,\nu\to 0}\ & C\|\nabla_x\phi\|_\infty \sup_{|\boldsymbol{r}|\leq R}\|\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})\|_3 \sup_{|\boldsymbol{r}|\leq R}\|\delta p(\boldsymbol{r};\boldsymbol{x})\|_{3/2} \xrightarrow{R\to 0} 0.\end{aligned}$$

Note that we take the limit $\ell_f\to 0$ only for $d=2$. Here we used the fact that $\boldsymbol{u}\in L^3$ uniformly in ν,ℓ_f by assumption and, as a result, that $p\in L^{3/2}$ uniformly (see proof of Proposition 3.1.6). The upper bound vanished as $R\to 0$ by strong continuity in L^p spaces of the shift operator. The contribution from large-scale friction is estimated in a similar fashion:

$$\begin{aligned}|\langle\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\cdot\Lambda\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi| &\leq C\|\phi\|_\infty \sup_{|\boldsymbol{r}|<R}\|\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\|_2 \sup_{|\boldsymbol{r}|<R}\|\Lambda\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\|_2 \\ &\xrightarrow{\ell,\ell_f,\nu\to 0} C_\Lambda\|\phi\|_\infty \sup_{|\boldsymbol{r}|<R}\|\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})\|_2^2 \xrightarrow{R\to 0} 0.\end{aligned}$$

where we have used boundedness of the smoothing operator Λ on L_2, that $\overline{\boldsymbol{u}}_\ell\to\boldsymbol{u}$ strongly in L^2, as well as strong continuity of shifts in L^2.

Special care must be taken with the forcing term. In the case of $d > 2$, we consider forcing with fixed ℓ_f. Thus, similar arguments show that as $R \to 0$, one has from strong continuity of shifts in L^2 that

$$|\langle \delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \delta\overline{(\boldsymbol{f}^{\ell_f})}_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi| \xrightarrow{\ell,\nu\to 0} C\|\phi\|_\infty \sup_{|\boldsymbol{r}|\le R} \|\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})\|_2 \sup_{|\boldsymbol{r}|\le R} \|(\delta\boldsymbol{f}^{\ell_f})(\boldsymbol{r};\boldsymbol{x})\|_2$$
$$\xrightarrow{R\to 0} 0.$$

This argument fails for $2d$ in the limit of small-scale forcing. In particular, $\boldsymbol{f}^{\ell_f}$ cannot be uniformly L^2 if it is to sustain an inverse cascade to large scales (see Remark 3.1). In this case, we give an alternative argument which shows that this contribution vanishes. In particular, we prove uniform convergence of $\overline{(\boldsymbol{f}^{\ell_f})}_\ell \xrightarrow{\ell_f\to 0} 0$ for all $\boldsymbol{x} \in \mathbb{T}^d$ under the assumption that $\boldsymbol{f}^{\ell_f} \in L^1$ uniformly in ℓ_f. Note that if the force were taken to have compact spectral support, as in Proposition 3.1.1, then:

$$\overline{(\boldsymbol{f}^{\ell_f})}_\ell = \int \mathrm{d}\boldsymbol{r}\; G_\ell(\boldsymbol{r})\boldsymbol{f}^{\ell_f}(\boldsymbol{x}+\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}\; (\mathbf{P}_{\ell_f}G_\ell)(\boldsymbol{r})\boldsymbol{f}^{\ell_f}(\boldsymbol{x}+\boldsymbol{r})$$

where $\mathbf{P}_{\ell_f}$ is the projection onto the spectral support of $\boldsymbol{f}^{\ell_f}$. Thus,

$$|\overline{(\boldsymbol{f}^{\ell_f})}_\ell| \le \|\mathbf{P}_{\ell_f}G_\ell\|_\infty \|\boldsymbol{f}^{\ell_f}\|_1$$

and obviously $\|\mathbf{P}_{\ell_f}G_\ell\|_\infty \le \sum_{\boldsymbol{k}\in S(k_f)} |\widehat{G}_\ell(\boldsymbol{k})| \xrightarrow{\ell_f\to 0} 0$. Thus we obtain convergence of the mollified force to zero, uniform in $\boldsymbol{x}$. If the force does not have compact spectral support, but has Fourier-transform 'concentrated' about k_f, the same argument can be modified and applied so long as there is sufficiently rapid decay whenever $||\boldsymbol{k}| - k_f| \gg 1$. Finally, if one knows only that $\boldsymbol{f}^{\ell_f} \in L^1$ and vanishes distributionally, the Arzela-Ascoli theorem guarantees at least that there exists a subsequence of $\ell_f \to 0$ such that $\overline{(\boldsymbol{f}^{\ell_f})}_\ell \to 0$ that converges uniformly for $\boldsymbol{x} \in \mathbb{T}^d$.

Finally, we estimate the contribution of the turbulent flux:

$$\langle \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \nabla_x \cdot \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi = -\langle \nabla_x \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) : \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi - \langle \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \nabla_x\varphi\rangle_{x,r}$$

The second term is easily seen to vanish as $\ell \to 0$ since

$$\begin{aligned} |\langle \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x}) \cdot \nabla_x\varphi\rangle_x| &\le C\|\nabla_x\phi\|_\infty \sup_{|\boldsymbol{r}|<R} \|\delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\|_3 \sup_{|\boldsymbol{r}|<R} \|\delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\|_{3/2} \\ &\le C\|\nabla_x\phi\|_\infty \sup_{|\boldsymbol{r}|<R} \|\delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x})\|_3 \sup_{|\boldsymbol{r}|<\ell} \|\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})\|_3^2 \xrightarrow{\ell,\ell_f,\nu\to 0} 0, \end{aligned}$$

where the constant C changes line by line and we have used the L^p bound on the cumulant

$$\|\tau_\ell(\boldsymbol{u},\boldsymbol{u})\|_p \le C \sup_{|\boldsymbol{r}|<\ell} \|\delta\boldsymbol{u}(\boldsymbol{r};\boldsymbol{x})\|_{2p}^2. \tag{3.33}$$

See e.g. Eyink [2015c,a] or Proposition 3 of Drivas and Eyink [2017c]. After some minor manipulation, we have:

$$\begin{aligned}
&\langle \nabla_x \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) : \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi \\
&\quad = \langle \nabla_x \overline{\boldsymbol{u}}_\ell(\boldsymbol{x}+\boldsymbol{r}) : \tau_\ell(\boldsymbol{x}+\boldsymbol{r})\rangle_\varphi + \langle \nabla_x \overline{\boldsymbol{u}}_\ell(\boldsymbol{x}) : \tau_\ell(\boldsymbol{x})\rangle_\varphi \\
&\qquad - \langle \nabla_x \overline{\boldsymbol{u}}_\ell(\boldsymbol{x}+\boldsymbol{r}) : \tau_\ell(\boldsymbol{x})\rangle_\varphi - \langle \nabla_x \overline{\boldsymbol{u}}_\ell(\boldsymbol{x}) : \tau_\ell(\boldsymbol{x}+\boldsymbol{r})\rangle_\varphi \\
&\quad = \langle Q_\ell(\boldsymbol{u})(\boldsymbol{x}+\boldsymbol{r})\rangle_\varphi + \langle Q_\ell(\boldsymbol{u})(\boldsymbol{x})\rangle_\varphi \qquad (3.34)\\
&\qquad + \langle \delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r},\boldsymbol{x})\cdot\tau_\ell(\boldsymbol{x})\cdot\nabla_r\varphi(\boldsymbol{x},\boldsymbol{r})\rangle_{x,r} - \langle \delta\overline{\boldsymbol{u}}_\ell(-\boldsymbol{r},\boldsymbol{x})\cdot\tau_\ell(\boldsymbol{x})\cdot\nabla_r\varphi(\boldsymbol{x}-\boldsymbol{r},\boldsymbol{r})\rangle_{x,r} \qquad (3.35)\\
&\qquad - \langle \nabla_r\cdot[\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r},\boldsymbol{x})\cdot\tau_\ell(\boldsymbol{x})\varphi(\boldsymbol{x},\boldsymbol{r})]\rangle_{x,r} + \langle \nabla_r\cdot[\delta\overline{\boldsymbol{u}}_\ell(-\boldsymbol{r},\boldsymbol{x})\cdot\tau_\ell(\boldsymbol{x})\varphi(\boldsymbol{x}-\boldsymbol{r},\boldsymbol{r})]\rangle_{x,r}. \qquad (3.36)
\end{aligned}$$

where, in (3.35), we have introduce the resolved energy flux term (discussed in Remark 3.1.1 and appearing in (3.21)):

$$Q_\ell(\boldsymbol{u})(\boldsymbol{x},t) = -\nabla\bar{\boldsymbol{u}}_\ell(\boldsymbol{x},t) : \tau_\ell(\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{u}(\boldsymbol{x},t)). \qquad (3.37)$$

The two terms in (3.36) vanish by the divergence theorem since the test function ψ_R has compact support. The terms in (3.35) easily are seen to vanish as $\ell \to 0$ for any

$R > 0$ since, using the estimate (3.33) for the coarse-graining cumulant τ_ℓ, we have:

$$|\langle \delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r},\boldsymbol{x}) \cdot \tau_\ell(\boldsymbol{x}) \cdot \nabla_r\varphi(\boldsymbol{x},\boldsymbol{r})\rangle_{x,r}| \le C\|\nabla_r\varphi\|_\infty \sup_{|\boldsymbol{r}|\le R}\|\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r})\|_3 \sup_{|\boldsymbol{r}|\le \ell}\|\delta\boldsymbol{u}(\boldsymbol{r})\|_3^2,$$

$$|\langle \delta\overline{\boldsymbol{u}}_\ell(-\boldsymbol{r},\boldsymbol{x}) \cdot \tau_\ell(\boldsymbol{x}) \cdot \nabla_r\varphi(\boldsymbol{x}-\boldsymbol{r},\boldsymbol{r})\rangle_{x,r}| \le C\max\{\|\nabla_x\varphi\|_\infty, \|\nabla_r\varphi\|_\infty\}\times$$

$$\sup_{|\boldsymbol{r}|\le R}\|\delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r})\|_3 \sup_{|\boldsymbol{r}|\le \ell}\|\delta\boldsymbol{u}(\boldsymbol{r})\|_3^2,$$

which vanish again as $\ell, \ell_f, \nu \to 0$ due to the strong continuity of shifts in L^3. By our strong convergence assumptions, Propositions 3.1.2, 3.1.3, 3.1.6 and Remarks 3.1.1 3.1.1 and 3.1.1, the functions Q_ℓ converge distributionally to Π, where $\Pi = -I$ or ε depending on whether we consider the case for $d = 2$ or $d > 2$. Thus:

$$\langle \nabla_x \delta\overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) : \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi \xrightarrow{\ell,\ell_f,\nu\to 0} \langle \Pi(\boldsymbol{u})(\boldsymbol{x}+\boldsymbol{r})\rangle_\varphi + \langle \Pi(\boldsymbol{u})(\boldsymbol{x})\rangle_\phi.$$

Finally we must analyze the first term above in the limit of $R \to 0$. We consider ψ_R which approximates the identity as $R \to 0$ and note:

$$\langle \Pi(\boldsymbol{u})(\boldsymbol{x}+\boldsymbol{r})\rangle_\varphi = \langle \Pi(\boldsymbol{u})(\boldsymbol{x}), \phi(\boldsymbol{x}-\boldsymbol{r})\psi_R(\boldsymbol{r})\rangle_{x,r} = \langle \Pi(\boldsymbol{u})(\boldsymbol{x}), \psi_R * \phi(\boldsymbol{x})\rangle_x$$

Since $\psi_R, \phi \in D(\mathbb{T}^d) = C_0^\infty(\mathbb{T}^d)$, then in the limit of $R \to 0$, we have that $\psi_R * \phi \to \phi$ in the standard Fréchet topology on test functions. Since $\Pi \in D'(\mathbb{T}^d)$ is, by definition, a continuous linear functional on $D(\mathbb{T}^d)$, we have:

$$\langle \nabla_x \delta \overline{\boldsymbol{u}}_\ell(\boldsymbol{r};\boldsymbol{x}) : \delta\tau_\ell(\boldsymbol{r};\boldsymbol{x})\rangle_\varphi \xrightarrow{R,\ell,\ell_f,\nu\to 0} 2\langle \Pi(\boldsymbol{u})(\boldsymbol{x})\rangle_\phi$$

as claimed.

Proposition 3.2.1 constitutes our rigorous generalization of the Ott-Mann-Gawędzki relation. Of course, physically we are never going to the limit of zero viscosity or smoothing scale. Instead, for any $t \geq 0$ our result holds approximately for a range of scales $\ell_\nu \ll \ell \ll R \ll L$ for $d > 2$ and $\ell_\nu \ll \ell_f \ll \ell \ll R \ll L$ for $d = 2$.

3.2.3 Short-time asymmetry of particle dispersion in turbulence

Now we discuss some interesting properties of short-time forward/backward two-particle dispersion. For any $t > t_0$, a particle initially at $\boldsymbol{x}$ at t_0 traveling forward in time by $\tau \equiv t - t_0$ is denoted by $\boldsymbol{X}_{t_0,t}(\boldsymbol{x})$. Likewise, a particle labeled at t_0 traveling backward in time by $-\tau$ is denoted $\boldsymbol{X}_{t_0,t}(\boldsymbol{x})$. Then, as a corollary to the Ott-Mann-Gawędzki relation, the short time-expansion (3.29), and the identification of the anomalies (3.23),(3.24), we have the following (spatially local) Lagrangian representation of the cascade rate:

Proposition 3.2.2 *Under the same assumptions as Proposition 3.2.1, for any* $t > 0$

we have

$$\lim_{R\to 0}\lim_{\ell\to 0}\lim_{\tau\to 0}\lim_{\ell_f,\nu\to 0}\left[\frac{\left\langle|\delta\boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x})-\boldsymbol{r}|^2\right\rangle_{\varphi}-\left\langle|\delta\boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x})-\boldsymbol{r}|^2\right\rangle_{\varphi}}{4\tau^3}\right]=\quad\langle I\rangle_{\phi}\qquad d=2,$$

$$\lim_{R\to 0}\lim_{\ell\to 0}\lim_{\tau\to 0}\lim_{\nu\to 0}\left[\frac{\left\langle|\delta\boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x})-\boldsymbol{r}|^2\right\rangle_{\varphi}-\left\langle|\delta\boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x})-\boldsymbol{r}|^2\right\rangle_{\varphi}}{4\tau^3}\right]=-\langle\varepsilon\rangle_{\phi}\qquad d>2.$$

Proof First we consider the limit of $\nu\to 0$ for $d>2$ and $\ell_f,\nu\to 0$ for $d=2$ obtaining weak solutions of the Euler equations $\boldsymbol{u}\in L^3((0,T);L^3)$ by Propositions 3.1.4 and 3.1.6. Now note that since $\boldsymbol{u}\in L^1$, the filtered field and all its derivatives are uniformly bounded in $\boldsymbol{x}$ for a fixed $\ell>0$:

$$\|\nabla^{(n)}\bar{\boldsymbol{u}}_\ell\|_\infty\leq\|\nabla^{(n)}G\|_\infty\|\boldsymbol{u}\|_1/\ell^n.$$

Thus, it follows from the equation $\boldsymbol{X}^{\ell}_{t_0,t}=\bar{\boldsymbol{u}}_\ell(\boldsymbol{X}^{\ell}_{t_0,t},t)$ that $\boldsymbol{X}^{\ell}_{t_0,t}\in C^\infty_t$ uniformly in $\boldsymbol{x}$ for fixed $\ell>0$. The rest of the argument proceeds nearly identically to the one given by Jucha et al. [2014] and sketched in the introduction of this Chapter. Since $\boldsymbol{X}^{\ell}_{t_0,t}\in C^4_t$, by Taylor's remainder theorem, the short-time expansion for trajectories both forwards and backwards is given by

$$\left\langle |\delta \boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} = \langle S_2^{\bar{\boldsymbol{u}}_\ell,\varphi}(\boldsymbol{r},t_0)\rangle_{\psi}\, \tau^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |\delta \boldsymbol{v}^{\ell}(\boldsymbol{r};\boldsymbol{x},t)|^2 \right\rangle_{\varphi}\Big|_{t=t_0} \tau^3 + \langle X^{(4)}_{t_0,t^*_F(\boldsymbol{x})}(\boldsymbol{x})\rangle_{\varphi}\tau^4 \tag{3.38}$$

$$\left\langle |\delta \boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} = \langle S_2^{\bar{\boldsymbol{u}}_\ell,\varphi}(\boldsymbol{r},t_0)\rangle_{\psi}\, \tau^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |\delta \boldsymbol{v}^{\ell}(\boldsymbol{r};\boldsymbol{x},t)|^2 \right\rangle_{\varphi}\Big|_{t=t_0} \tau^3 + \langle X^{(4)}_{t^*_B(\boldsymbol{x}),t_0}(\boldsymbol{x})\rangle_{\varphi}\tau^4 \tag{3.39}$$

for some $t^*_F(\boldsymbol{x}) \in [t_0, t]$ and $t^*_B(\boldsymbol{x}) \in [-t, t_0]$ for any $\boldsymbol{x} \in \mathbb{T}^d$. Thus, subtracting the forward dispersion from the backward, dividing by τ^3 and taking the limit $\tau \to 0$, we have:

$$\frac{\left\langle |\delta \boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} - \left\langle |\delta \boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi}}{2\tau^3} \xrightarrow{\tau\to 0} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |\delta \boldsymbol{v}^{\ell}(\boldsymbol{r};\boldsymbol{x},t)|^2 \right\rangle_{\varphi}\Big|_{t=t_0}.$$

Finally, taking $\ell \to 0$ and applying Proposition 3.2.1, we obtain the final formulae in the statement of the proposition.

Theorem 3.2.2 makes rigorous the observation of Jucha et al. [2014]. This result is local in the sense that it holds for arbitrarily small "observing regions" selected by particular test functions φ. As before, physically we are never really "going to the limit". Assuming that there is a bound to the correction at finite ℓ of the form

$$\left\langle |\delta \boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} - \left\langle |\delta \boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} = \langle \Pi_\ell\rangle_{\varphi}\tau^3 \left[1 + O\left(\frac{\tau}{\tau_\ell}\right)\right],$$

where $\tau_\ell = O(\ell/\delta u(\ell))$ is the local eddy turnover time at scale ℓ and $\delta u(\ell)$ is some measure of the seperation at that scale, then for $d = 2$ our result holds in the range of scales:

$$\left\langle |\delta \boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} - \left\langle |\delta \boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} \approxeq -4\langle I\rangle_\phi \tau^3 \qquad \text{for} \quad \ell_\nu \ll \ell_f \ll \ell \ll |\boldsymbol{r}| \ll L \quad \text{and} \quad \tau \ll \tau_\ell,$$

and for $d > 2$ our result is:

$$\left\langle |\delta \boldsymbol{X}^{\ell}_{t,t_0}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} - \left\langle |\delta \boldsymbol{X}^{\ell}_{t_0,t}(\boldsymbol{r};\boldsymbol{x}) - \boldsymbol{r}|^2 \right\rangle_{\varphi} \approxeq 4\langle \varepsilon\rangle_\phi \tau^3 \qquad \text{for} \quad \ell_\nu \ll \ell \ll |\boldsymbol{r}| \ll L \quad \text{and} \quad \tau \ll \tau_\ell.$$

Since the energy dissipation rate is signed, $\langle \varepsilon\rangle_\phi \geq 0$, the above result shows that particles initially spread faster backward-in-time than forward-in-time in $3d$ turbulence. Note that this result also holds for $2d$ turbulence but $\langle \varepsilon\rangle_\phi = 0$ (Proposition 3.1.5). Thus, short-time dispersion is symmetric forwards and backwards for high Re turbulence driven by a fixed body force. However, if the forcing length-scale ℓ_f is taken to vanish, there may be asymmetric in the particle dispersion in $2d$ and it would be reversed compared to the $3d$ case since the forcing "typically" injects energy into the flow ($\langle I\rangle_\phi \geq 0$) rather than depletes it. If there are no anomalies due either to the dissipation or to the forcing ($\varepsilon = 0$ or $I = 0$), Proposition 3.2.2 shows that forwards/backwards dispersion are indistinguishable at short times. In this way, dissipation anomalies are intimately connected to time-asymmetry of Lagrangian particle dispersion.

Thus a spatially local version of the observation of Jucha et al. [2014] is rigorously valid, if one consider trajectories in filtered fields. Moreover, Proposition 3.2.2 offers further support to the conjecture of Eyink and Drivas [2015a] regarding the connection of time–asymmetry in Richardson dispersion and the dissipation anomaly.

Chapter 4

A Lagrangian Fluctuation-Dissipation Relation for Scalar Turbulence: Domains without Boundaries

4.1 Introduction

A fundamental feature of turbulent flows is the enhanced dissipation of kinetic energy. It was suggested by G. I. Taylor [1917a] that kinetic energy can "be dissipated in fluid of infinitesimal viscosity". As discussed in the Introduction §1.2, it is empirically observe that turbulent dissipation becomes independent of molecular viscosity at sufficiently high Reynolds numbers, also known as “dissipative anomaly” or sometimes the “zeroth law of turbulence”. Similar phenomena are expected for other turbulent systems, in particular for scalar fields advected by a turbulent fluid, such as concentrations of dyes or aerosols, temperature fluctuations, etc. It was suggested by Taylor [1922] that diffusion by turbulence should depend “little on the molecular conductivity and viscosity of the fluid” and the asymptotic independence of the dissipation rate of scalar fluctuations from the molecular transport coefficients was a fundamental assumption in the Kolmogorov-style theories of scalar turbulence developed by Obukhov [1949] and Corrsin [1951]. A very comprehensive review of the empirical evidence for this hypothesis on scalar dissipation is contained in the paper of Donzis et al. [2005], whose compilation of data is again consistent with scalar dissipation in the bulk of turbulent flows being insensibly dependent on molecular transport coefficients at sufficiently high Reynolds and Péclet numbers. This phenomenon still requires a complete theoretical explanation.

As discussed in the introduction §1.3, Bernard et al. [1998] discovered the remarkable phenomenon called *spontaneous stochasticity* which shed new light on anomalous dissipation. They carried out a study in the Kraichnan [1968] model of synthetic turbulence. In this model the advecting velocity is a Gaussian random field that has Kolmogorov-type scaling of increments in space but a white-noise correlation in time. It was shown in the Kraichnan model that the dissipative anomaly for a decaying passive scalar is due to spontaneous stochasticity [Chaves et al., 2003]. Simply stated, Bernard et al. [1998] showed that Lagrangian particle trajectories become non-unique and stochastic in the infinite Reynolds-number limit. More precisely, they showed that at very large Reynolds and Péclet numbers, when the velocity field is smooth but approximates a "rough" field over a long range of scales, small stochastic perturbations on Lagrangian trajectories due to molecular diffusivity lead to persistent randomness over any finite times even as the perturbations vanish. This effect is due to the explosive (super-ballistic) dispersion of particle pairs in a turbulent flow predicted by Richardson [1926], which leads to loss of memory of initial particle separations or of amplitudes of stochastic perturbations. For excellent reviews of this and related studies on the Kraichnan model, see Falkovich et al. [2001], Kupiainen [2003], Gawędzki [2008].

Since this pioneering work, however, there have been recurrent doubts expressed concerning the validity of these results for real hydrodynamic turbulence. For example, Tsinober [2009] (section 5.4.5) has argued that in real fluids "the flow field is smooth. In such flows 'phenomena' like 'spontaneous stochasticity' and 'breakdown of Lagrangian flow' do not arise and one has to look at different more realistic possibilities." This is a simple misunderstanding, because spontaneous stochasticity is a phenomenon that appears for smooth velocity fields that merely appear "rough" over a long range of scales. More serious questions have been raised concerning the approximation of a white-noise temporal correlation in the Kraichnan model. In a recent detailed comparison of passive scalars in the Kraichnan model and in fluid turbulence, Sreenivasan and Schumacher [2010] have remarked that "It is still unclear in the Kraichnan model as to which qualitative and quantitative differences arise from the finite-time correlation of the advecting flow." This latter paper discussed also some of the challenges in extending results for the Kraichnan model to understanding of the energy cascade in Navier-Stokes turbulence.

The principal contribution of the present chapter is a new approach to the theory of turbulent scalar dissipation based upon an exact *fluctuation-dissipation relation for scalars*. Our new relation expresses an equality between the time-averaged scalar dissipation and the input of scalar variance from the initial data and interior scalar sources, as these are sampled by stochastic Lagrangian trajectories. This relation makes it intuitively clear that scalar dissipation requires non-vanishing Lagrangian stochasticity. In fact, using our new relation, we can prove rigorously the following fact: *Away from boundaries and for any advecting velocity field whatsoever, spontaneous stochasticity of Lagrangian particle trajectories is sufficient for anomalous dissipation of passive scalars, and necessary for anomalous dissipation of both passive and active scalars.* Thus, there is no possible mechanism for a scalar dissipative anomaly in such situations other than spontaneous stochasticity. In this way we completely resolve the controversies on the applicability of the dissipation mechanisms in the Kraichnan model to scalars in hydrodynamic turbulence, at least away from walls. The importance of our exact fluctuation-dissipation relation (FDR) is not limited to analysis of anomalous scalar dissipation and it is valid even when scalar dissipation may vanish as $\nu, \kappa \to 0$. In general our relation gives a new Lagrangian viewpoint on dissipation of scalars, both active and passive. As such, it generalizes some previously derived relations, such as that of Sawford et al. [2005], Buaria et al. [2016] for scalars forced by a mean scalar gradient and the exact balance relations for stochastic scalar sources which are Gaussian white-in-time [Novikov, 1965]. In two submitted papers (Drivas and Eyink [2017b], Eyink and Drivas [2017a]; hereafter denoted II, III), we show how the FDR extends also to wall-bounded domains, with either fixed-scalar (Dirichlet) or fixed-flux (Neumann) conditions for the scalar field, and we apply the FDR to the concrete problem of Nusselt-Rayleigh scaling in turbulent Rayleigh-Bénard convection. In fact, parts of Paper II are presented in the following chapter of this thesis.

The detailed contents of the present chapter are as follows: In section 4.2 we first derive the stochastic representation of scalar advection and our FDR, in case of flows in domains without walls. In section 4.3 we review the notion of spontaneous stochasticity, with numerical verifications from a database of homogeneous, isotropic turbulence. In section 4.4 we establish the connection of spontaneous stochasticity with anomalous scalar dissipation. In the summary and discussion section 5.4 we discuss both the implications for turbulent vortex dynamics and other Lagrangian aspects of turbulence, and also the outstanding challenges, including that of relating spontaneous stochasticity to anomalous dissipation of kinetic energy. Three appendices give further details, including rigorous mathematical proofs of all of the results in the main text. These deal with the connection between spontaneous stochasticity and anomalous dissipation (Appendix 4.A), the relation of our scalar FDR to previous results in the literature (Appendix B of Drivas and Eyink [2017a]), and discussion of numerical methods employed (Appendix C of Drivas and Eyink [2017a]).

4.2 Lagrangian Fluctuation Dissipation Relation

We consider in this chapter turbulent fluid flows in finite domains without walls. A relevant numerical example is DNS of turbulence in a periodic box. A set of examples from Nature is provided by large-scale flows in thin planetary atmospheres, which can be modeled as 2D flows on a sphere. Mathematically speaking, the results in this section apply to fluid flows on any compact Riemannian manifold without boundary, merely replacing the Wiener process with the Brownian motion on the manifold whose infinitesimal generator is the Laplace-Beltrami operator [Ikeda and Watanabe, 1989]. For simplicity of presentation, we derive the relation only for periodic domains.

Scalar fields θ (such as temperature, dye or pollutants) transported by a fluid with velocity $\boldsymbol{u}$ are described by the advection-diffusion equation

$$\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta = \kappa\Delta\theta + S \tag{4.1}$$

with $S(\boldsymbol{x}, t)$ a source field and with $\kappa > 0$ the molecular diffusivity of the scalar. The work of Bernard et al. [1998] employed a stochastic representation of the solutions of this equation which is known as the Feynman-Kac representation in the mathematics literature [Oksendal, 2013] and as a stochastic Lagrangian representation in the turbulence modeling field [Sawford, 2001]. This stochastic approach is the natural extension to dissipative, non-ideal fluid flows of the Lagrangian description developed for smooth, ideal flows. We discuss presently this representation only for domains Ω without boundaries, as assumed also by Bernard et al. [1998], and in the following chapter of this thesis and [Drivas and Eyink, 2017b] we describe the extension to wall-bounded domains. We shall further discuss in chapters only advection by an incompressible fluid satisfying

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.2}$$

so that the ideal advection term formally conserves all integrals of the form $I_h(t) = \int_\Omega d^d x \; h(\theta(\boldsymbol{x}, t))$ for any continuous function $h(\theta)$. Note that the representation applies in any space dimension d, with most immediate physical interest for $d = 2, 3$, of course.

The stochastic representation of non-ideal scalar dynamics involves stochastic Lagrangian flow maps $\widetilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(\boldsymbol{x})$ describing the motion of particles labelled by their positions $\boldsymbol{x}$ at time t to random positions at earlier times $s < t$. The physical relevance of the backward-in-time particle trajectories can be anticipated from the fact that the advection-diffusion equation (5.1) mixes (averages) the values of the scalar field given in the past and not, of course, the future values. Mathematically, the relevant stochastic flows are governed by the backward Itō stochastic differential equations:

$$\hat{\mathrm{d}}_s \widetilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(\boldsymbol{x}) = \boldsymbol{u}^\nu(\widetilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(\boldsymbol{x}), s)\mathrm{d}s + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s, \qquad \widetilde{\boldsymbol{\xi}}_{t,t}^{\nu,\kappa}(\boldsymbol{x}) = \boldsymbol{x}. \tag{4.3}$$

Here $\mathbf{W}_s$ is a standard Brownian motion and $\hat{\mathrm{d}}_s$ denotes the backward Itō stochastic differential in the time s. For detailed discussions of backward Itō equations and stochastic flows, see Friedman [2006], Kunita [1997]. For those who are familiar with the more standard forward Itō equations, the backward equations are simply the time-reverse of the forward ones. Thus, a backward Itō equation in the time variable s is equivalent to a forward Itō equation in the time $\hat{s} = t_r - s$ reflected around a chosen reference time t_r[1]. The noise term involving the Brownian motion in Eq.(4.3) is proportional to the square root of the molecular diffusivity κ. The velocity field $\boldsymbol{u}^\nu$ is assumed to be smooth so long as the parameter $\nu > 0$. In the case of greatest physical interest when $\boldsymbol{u}^\nu$ is a solution of the incompressible Navier-Stokes equation, then ν represents the kinematic viscosity and we assume, for simplicity of presentation, that there is no blow-up in those solutions. (See Rezakhanlou [2014] for weak solutions.) Because equation (4.3) involves both ν and κ, its random solutions $\tilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}$ have statistics

[1] We note that the difference between forward- and backward-Itō equations is not essentially the direction of time in which they are integrated. Rather, the difference has to do with the time-direction in which those equations are *adapted* [Friedman, 2006, Kunita, 1997]. Thus, a forward Itō differential $b(\mathbf{W}_t)\mathrm{d}\mathbf{W}_t$ is discretized in time as $b(\mathbf{W}_{t_n})(\mathbf{W}_{t_{n+1}} - \mathbf{W}_{t_n})$ for $t_{n+1} > t_n$, with the increment $\mathbf{W}_{t_{n+1}} - \mathbf{W}_{t_n}$ statistically independent of $\mathbf{W}_t$ for $t \leq t_n$. Instead, a backward Itō differential $b(\mathbf{W}_t)\hat{\mathrm{d}}\mathbf{W}_t$ is discretized as $b(\mathbf{W}_{t_n})(\mathbf{W}_{t_n} - \mathbf{W}_{t_{n-1}})$ for $t_n > t_{n-1}$, with $\mathbf{W}_{t_n} - \mathbf{W}_{t_{n-1}}$ statistically independent of $\mathbf{W}_t$ for $t \geq t_n$. The distinction only matters when, as in our equation (4.4), the differential of $\mathbf{W}_s$ is multiplied by a stochastic function of $\mathbf{W}$.

which depend upon those parameters, represented by the superscripts. To avoid a too heavy notation, we omit those superscripts and write simply $\tilde{\boldsymbol{\xi}}_{t,s}$ unless it is essential to refer to the dependence upon ν, κ. Note that when $\kappa = 0$ and $\boldsymbol{u}^\nu$ remains smooth, then $\boldsymbol{\xi}_{t,s}^{\nu,0}(\boldsymbol{x})$ is no longer stochastic and gives the usual reverse Lagrangian flow from time t backward to the earlier time $s < t$.

The stochastic representation of the solutions of the advection-diffusion equation follows from the backward differential

$$\begin{aligned} \hat{\mathrm{d}}_s\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) &= [(\partial_s + \boldsymbol{u}^\nu \cdot \nabla - \kappa\Delta)\theta](\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + \sqrt{2\kappa}\hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) \\ &= S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + \sqrt{2\kappa}\hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s), \end{aligned} \tag{4.4}$$

using the backward Itō formula [Friedman, 2006, Kunita, 1997] in the first line and Eq.(5.1) in the second. Integrating over time s from 0 to t, gives

$$\theta(\boldsymbol{x}, t) = \theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s + \sqrt{2\kappa}\int_0^t \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s), \tag{4.5}$$

where θ_0 is the initial data for the scalar at time 0. Because the backward Itō integral term in (4.5) averages to zero, one obtains

$$\theta(\boldsymbol{x}, t) = \mathbb{E}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right] \tag{4.6}$$

where $\mathbb{E}$ denotes the average over the Brownian motion. Eq.(4.6) is the desired stochastic representation of the solution of the advection-diffusion equation (5.1). Note that the reverse statement is also true, that the field $\theta(\boldsymbol{x},t)$ defined *a priori* by Eq.(4.6) is the solution of (5.1) for the initial data θ_0. For a simple proof, see section 4.1 of Eyink and Drivas [2015a] which gives the analogous argument for Burgers equation.

To see that this stochastic representation naturally generalizes the standard Lagrangian description to non-ideal fluids, observe that the scalar values along stochastic Lagrangian trajectories $\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)$ are, for $S\equiv 0$, *martingales* backward in time. This means that

$$\mathbb{E}\left[\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\middle|\,\{\mathbf{W}_\tau, r<\tau<t\}\right]=\theta(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}),r), \quad s<r<t \tag{4.7}$$

where the expectation is conditioned upon knowledge of the Brownian motion over the time interval $[r,t]$. Thus, the conditional average value is the last known value (going backward in time). This is the property for diffusive flow which corresponds to the statement for diffusion-less, smooth advection that θ is conserved along Lagrangian trajectories, or that $\theta(\boldsymbol{\xi}_{t,s}(\boldsymbol{x}),s)$ is constant in s. The proof is obtained by integrating the differential (4.4) over the time-interval $[s,t]$ to obtain

$$\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)=\theta(\boldsymbol{x},t)-\sqrt{2\kappa}\int_s^t \hat{\mathrm{d}}\mathbf{W}_\tau\cdot\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}),\tau)\,\mathrm{d}\tau. \tag{4.8}$$

and then exploiting the corresponding martingale property of the backward Itō integral [Friedman, 2006, Kunita, 1997]. It is important to emphasize that the martingale property like (4.7) does not hold forward in time, which would instead give a solution of the negative-diffusion equation with κ replaced by $-\kappa < 0$. Thus, the backward-in-time martingale property (4.7) expresses the arrow of time arising from the irreversibility of the diffusion process.

The main result of this chapter is a new exact fluctuation-dissipation relation between scalar dissipation due to molecular diffusivity and fluctuations associated to stochastic Lagrangian trajectories. To state the result, we introduce a stochastic scalar field[2]

$$\tilde{\theta}(\boldsymbol{x}, t) \equiv \theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ ds \tag{4.9}$$

which, according to Eq.(4.6), satisfies $\theta(\boldsymbol{x}, t) = \mathbb{E}[\tilde{\theta}(\boldsymbol{x}, t)]$ when averaged over Brownian motions. Thus $\tilde{\theta}(\boldsymbol{x}, t)$ in (4.9) represents the contribution to $\theta(\boldsymbol{x}, t)$ from an individual stochastic Lagrangian trajectory as it samples the initial data θ_0 and scalar source S backward in time. Using this definition and (4.6) we can rewrite (4.5) as

$$\tilde{\theta}(\boldsymbol{x}, t) - \mathbb{E}\left[\tilde{\theta}(\boldsymbol{x}, t)\right] = -\sqrt{2\kappa} \int_0^t \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s). \tag{4.10}$$

Squaring this equation and averaging over the Brownian motion gives

$$\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x}, t)\right] = 2\kappa \int_0^t ds\ \mathbb{E}\left[|\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2\right], \tag{4.11}$$

[2] Note that for all $s < t$ the quantity $\tilde{\theta}(\boldsymbol{x}, t; s) = \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) + \int_s^t S(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}), r)\ \mathrm{d}r$ is a martingale backward in time, by the same argument used previously for $S = 0$.

where "Var" on the lefthand side denotes the stochastic scalar variance in the average over the Brownian motion and on the righthand side we have used the Itō isometry (see Oksendal [2013], section 3.1) to evaluate the mean square of the backward Itō integral. If we now average in $\boldsymbol{x}$ over the flow domain Ω, use the fact that the stochastic flows $\tilde{\boldsymbol{\xi}}_{t,s}$ with condition (5.5) preserve volume, and divide by 1/2 we obtain

$$\frac{1}{2}\left\langle \mathrm{Var}\ \tilde{\theta}(t)\right\rangle_{\Omega} = \kappa \int_0^t ds \left\langle |\nabla\theta(s)|^2 \right\rangle_{\Omega}. \tag{4.12}$$

This is our exact *fluctuation-dissipation relation* (FDR). The quantity on the right is just the volume-averaged and cumulative (time-integrated) scalar dissipation, and the quantity on the left is (half) the stochastic scalar variance. The relation (4.12) thus represents a balance between scalar dissipation and the input of scalar fluctuations from the initial scalar field and the scalar sources, as sampled by stochastic Lagrangian trajectories backward in time.

It is important to emphasize that the origin of statistical fluctuations in our relation (4.12) is *not* that assumed in most traditional discussions of turbulence, i.e. random ensembles of initial scalar fields, of advecting velocity fields, or of stochastic scalar sources. Our FDR (4.12) is valid for fixed realizations of all of those quantities. The fluctuating quantity $\tilde{\theta}(\mathbf{x},t)$ which is defined in (4.9) and that appears in our (4.12) is an entirely different object from the conventional "turbulent" scalar fluctuation $\theta'(\boldsymbol{x},t)$. The latter is usually defined by $\theta' := \theta - \langle\theta\rangle$, where the scalar mean $\langle\theta\rangle$

is taken to be an ensemble- or space/time-average. Instead, the origin of randomness in $\tilde{\theta}(\mathbf{x}, t)$ is the Brownian motion in the stochastic flow equation (4.3). In special cases, e.g. a dye passively advected by a turbulent flow, this mathematical Wiener process has direct significance as the description of a physical Brownian motion of individual dye molecules in the liquid [Saffman, 1960, Buaria et al., 2016]. In general, however, the Wiener process is simply a means to model the effects of diffusion in a Lagrangian framework. For example, for a thermal field there are no "temperature molecules" undergoing physical Brownian motion.

Because our FDR is valid for fixed realizations of initial scalar fields, of advecting velocity fields, or of scalar sources, we are free to average subsequently over random ensembles of these objects. In this manner we recover from (4.12) as special cases some known results. For example, when the scalar source is a random field with zero mean and delta-correlated in time,

$$\langle \tilde{S}(\boldsymbol{x}, t)\tilde{S}(\boldsymbol{x}', t')\rangle = 2C_S(\boldsymbol{x}, \boldsymbol{x}')\delta(t - t'), \tag{4.13}$$

then we recover the steady-state balance equation for the scalar dissipation

$$\langle \kappa|\nabla\theta|^2\rangle_{\Omega,\infty,S} = \frac{1}{V}\int_\Omega d^dx\ C_S(\boldsymbol{x}, \boldsymbol{x}) \tag{4.14}$$

where the average on the left is over space domain Ω, an infinite time-interval, and the random source $\tilde{S}$. This is the standard result usually derived for Gaussian random source fields as an application of the Furutsu-Donsker-Novikov theorem (Frisch [1995], Novikov [1965]). Similar relations hold for freely decaying scalars with no sources but random initial scalar fields. For example, when the initial scalar has a uniform random space-gradient, $\tilde{\theta}_0(\boldsymbol{x}) = \tilde{\mathbf{G}} \cdot \boldsymbol{x}$ with isotropic statistics

$$\langle \tilde{\mathbf{G}} \tilde{\mathbf{G}}^\top \rangle_G = G^2 \mathbf{I}, \tag{4.15}$$

then we recover a relation of Sawford et al. [2005], Buaria et al. [2016]

$$\kappa \int_0^t ds \Big\langle |\nabla \theta(s)|^2 \Big\rangle_{\Omega,\theta_0} = \frac{1}{4} G^2 \ \mathbb{E}^{1,2} \left\langle \left| \tilde{\boldsymbol{\xi}}_{t,0}^{(1)} - \tilde{\boldsymbol{\xi}}_{t,0}^{(2)} \right|^2 \right\rangle_\Omega \tag{4.16}$$

where the $1, 2$ averages are taken over two independent ensembles of Brownian motion. The derivation of these particular consequences can be found in Appendix B of Drivas and Eyink [2017a].

Note, finally, that the result (4.11) provides a spatially *local fluctuation-dissipation relation*, which we may write in the form

$$\frac{1}{2t} \mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x}, t) \right] = \left\langle \mathbb{E}\left[\kappa |\nabla \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2 \right] \right\rangle_t, \tag{4.17}$$

where on the right $\langle \cdot \rangle_t$ denotes an average over s in the time interval $[0, t]$, carried out along stochastic Lagrangian trajectories moving backward-in-time from space-time point $(\boldsymbol{x}, t)$. It follows that at short times the local scalar variance exactly recovers the local scalar dissipation:

$$\lim_{t\to 0}\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right]=\kappa|\nabla\theta(\boldsymbol{x},0)|^2. \tag{4.18}$$

A substantial spatial correlation between $\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right]$ and $\varepsilon_\theta(\boldsymbol{x},t)=\kappa|\nabla\theta(\boldsymbol{x},t)|^2$ should persist for relatively short times t. On the other hand, in the long-time limit the local scalar variance becomes *space-time-independent* and equals

$$\lim_{t\to\infty}\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right]=\left\langle\kappa|\nabla\theta|^2\right\rangle_{\Omega,\infty}\qquad\text{for all } \boldsymbol{x}\in\Omega. \tag{4.19}$$

To see that Eq. (4.19) should be true, note that the random variables $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x})\in\Omega$ for each fixed $\boldsymbol{x}$ are an ergodic random process in the time-variable s for $\kappa>0$. Because of incompressibility of the velocity field and the ergodicity of the stochastic Lagrangian flow, the variables $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x})$ will be nearly uniformly distributed over Ω at times $s\leq t-\tau$, where τ is a characteristic scalar mixing time. This time τ will be at most of the order L^2/κ, where L is the diameter of the domain, and thus finite for $\kappa>0$, but usually much shorter because of advective mixing by the velocity field. For any positive integer n

$$\lim_{t\to\infty}\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right]=\lim_{t\to\infty}\frac{1}{t}\int_0^{t-n\tau}ds\;\mathbb{E}\left[\kappa|\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)|^2\right], \tag{4.20}$$

since the corrections are vanishing as $O(n\tau/t)$. By choosing an n sufficiently large but fixed as $t \to \infty$, we can make the righthand side arbitrarily close to

$$\lim_{t\to\infty} \frac{1}{t-n\tau} \int_0^{t-n\tau} ds \ \left\langle \kappa |\nabla\theta(\boldsymbol{\xi}, s)|^2 \right\rangle_\Omega = \left\langle \kappa |\nabla\theta(\boldsymbol{\xi}, s)|^2 \right\rangle_{\Omega,\infty}, \tag{4.21}$$

where the space-time average $\langle\cdot\rangle_{\Omega,\infty}$ on the right is over $\boldsymbol{\xi} \in \Omega$ and $s \in [0,\infty)$. Since $\lim_{t\to\infty} \frac{1}{2t} \mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x}, t)\right]$ is independent of the choice of n, we obtain (4.19). Of course, here we have assumed all of the various infinite-time averages to exist, as they shall (at least along subsequences of times $t_k \to \infty$) if the space-averaged scalar dissipation remains a bounded function of time.

Note that if the scalar is decaying from bounded initial data θ_0, then the variance on the left-hand-side of Eq. (4.19) is also bounded. In that case, the long-time-averaged scalar dissipation rate tends to zero, which comes as no surprise. In order to have a non-vanishing long-time dissipation, the scalar must be continually supplied to the system so that the variance of $\tilde{\theta}(\boldsymbol{x}, t)$ grows linearly in time. For example, a scalar source $S(\boldsymbol{x}, t)$ within the flow domain can provide the necessary scalar input. In such a case, the variance of $\tilde{\theta}(\mathbf{x}, t)$ grows proportionally to time t at long times because of the cumulative contribution from the scalar source S in the time-integral $\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}, s)\ ds$, and the long-time average scalar dissipation rate matches the mean input rate of the scalar. This result is rigorously demonstrated in Appendix B of Drivas and Eyink [2017a] for a random source field $\tilde{S}(\boldsymbol{x}, t)$, in which case the well-known relation of Novikov [1965] is recovered. The linear t-growth in the variance can be understood more generally from a simple central-limit-theorem argument for the long-time limit of the integral of S.

4.3 Spontaneous Stochasticity of Lagrangian Trajectories

We now specialize in this subsection to the source-less case $S \equiv 0$, in order to make contact with the work of Bernard et al. [1998] on spontaneous stochasticity and anomalous scalar dissipation. The stochastic representation (4.6) simplifies in this case to

$$\theta(\boldsymbol{x}, t) = \mathbb{E}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,0}(\boldsymbol{x}))\right] = \int d^d x_0 \; \theta_0(\boldsymbol{x}_0) \; p^{\nu,\kappa}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) \tag{4.22}$$

where we have introduced the backward-in-time transition probability

$$p^{\nu,\kappa}(\boldsymbol{x}', t'|\boldsymbol{x}, t) = \mathbb{E}\left[\delta^d(\boldsymbol{x}' - \tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,t'}(\boldsymbol{x}))\right] \quad t' < t \tag{4.23}$$

for the stochastic flow. As already noted, the stochastic flow preserves volume when the velocity field is divergence-free. In terms of the transition probability this means that

$$\int p^{\nu,\kappa}(\boldsymbol{x}', t'|\boldsymbol{x}, t) d^d x = 1, \tag{4.24}$$

where $\det(\partial\tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,t'}(\boldsymbol{x})/\partial\boldsymbol{x}) = 1$ is used to write $\delta^d(\boldsymbol{x}' - \tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,t'}(\boldsymbol{x})) = \delta^d(\boldsymbol{x} - (\tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,t'})^{-1}(\boldsymbol{x}'))$ and perform the integral over $\boldsymbol{x}$. Note that if the limit $\kappa \to 0$ is taken with ν fixed (infinite Prandtl number limit), then the stochastic flow (4.23) becomes deterministic and

$$p^{\nu,0}(\boldsymbol{x}', t'|\boldsymbol{x}, t) = \delta^d(\boldsymbol{x}' - \boldsymbol{\xi}^{\nu,0}_{t,t'}(\boldsymbol{x})), \quad t' < t. \tag{4.25}$$

which corresponds to a single deterministic Lagrangian trajectory passing through position $\boldsymbol{x}$ at time t.

In the Kraichnan model of turbulent advection it was shown by Bernard et al. [1998] that the joint limit $\nu, \kappa \to 0$ with $Pr = \nu/\kappa$ fixed is non-deterministic and corresponds to more than one Lagrangian trajectory passing through space-time point $(\boldsymbol{x}, t)$. We remind the reader that the Kraichnan model of turbulent advection replaces the Navier-Stokes solution with a realization drawn from an ensemble of Gaussian random fields $\boldsymbol{u}^\nu$ with mean zero $\langle \boldsymbol{u}^\nu \rangle = \boldsymbol{0}$ and covariance satisfying

$$\langle [u^\nu_i(\boldsymbol{x}+\boldsymbol{r}, t) - u^\nu_i(\boldsymbol{x}, t)][u^\nu_j(\boldsymbol{x}+\boldsymbol{r}, t') - u^\nu_j(\boldsymbol{x}, t')] \rangle = D^\nu_{ij}(\boldsymbol{r})\delta(t-t') \tag{4.26}$$

for a spatial covariance function satisfying $D^\nu_{ij}(\boldsymbol{r}) = D^\nu_{ji}(\boldsymbol{r})$, $\partial D^\nu_{ij}(\boldsymbol{r})/\partial r_j = 0$, and

$$D_{ii}(\boldsymbol{r}) \sim \begin{cases} D_1 r^\xi & \ell_\nu \ll r \ll L \\ D_2 r^2 & r \ll \ell_\nu \end{cases} \tag{4.27}$$

for some $0 < \xi < 2$, with the effective "dissipation length"

$$\ell_\nu = (D_1/D_2)^{1/(2-\xi)}. \tag{4.28}$$

Note that $D_2 \propto \langle |\nabla \boldsymbol{u}^\nu|^2 \rangle$ and in real turbulence would be proportional to ε/ν, where ε is the viscous energy dissipation. Hence, $D_2 \to \infty$ or $\ell_\nu \to 0$ with D_1 fixed is the analogue for the Kraichnan model of the infinite Reynolds-number limit for Navier-Stokes turbulence. In fact, one can introduce a "viscosity" parameter ν for the Kraichnan model with units of $(\text{length})^2/(\text{time})$, so that $\ell_\nu = (\nu/D_1)^{1/\xi}$. For any $\nu > 0$ the velocity realizations are spatially smooth, but in the limit $\nu \to 0$ they are only Hölder continuous in space with exponent $0 < \xi/2 < 1$. It is well-known that for such "rough" limiting velocity fields the solutions of the deterministic initial-value problem

$$d\boldsymbol{\xi}(s)/ds = \boldsymbol{u}(\boldsymbol{\xi}(s), s), \quad \boldsymbol{\xi}(t) = \boldsymbol{x} \tag{4.29}$$

need not be unique and, if not, form a continuum of solutions (e.g. see Hartman [2002]). In the Kraichnan model it has been proved in the double limit with both $\nu \to 0$ and $\kappa \to 0$ that the transition probabilities tend to a limiting form

$$p^*(\boldsymbol{x}', t'|\boldsymbol{x}, t) = \lim_{\nu,\kappa \to 0} p^{\nu,\kappa}(\boldsymbol{x}', t'|\boldsymbol{x}, t). \tag{4.30}$$

It is important to stress here that no average is taken over $\boldsymbol{u}$ in defining these transition probabilities, but only an average over Brownian motions in the stochastic flow equations (4.3), while the velocity realization is held fixed[3]. Most importantly, the limiting transition probabilities for the Kraichnan model are *not* delta-distributions of the form (4.25) but nontrivial probabilities over an ensemble of non-unique solutions of the limiting ODE (4.29)! This remarkable phenomenon is called *spontaneous stochasticity*. See Bernard et al. [1998] and the later papers of Gawędzki and Vergassola [2000], E and Vanden-Eijnden [2000, 2001], Falkovich et al. [2001], Jan and Raimond [2002, 2004].

As shown in those works, spontaneous stochasticity occurs because of the analogue of Richardson [1926] dispersion in the Kraichnan model, which leads to a loss of influence of the molecular diffusivity κ on the separation of the perturbed Lagrangian trajectories after a short time of order $(\kappa^{2-\xi}/D_1)^{1/\xi}$. It is important to emphasize that this result does not mean that randomness in the Lagrangian trajectories suddenly "appears" only for $\nu, \kappa = 0$ but instead that the randomness *persists* even as $\nu, \kappa \to 0$. It is thus a phenomenon that can be observed with sequences of positive values, $\nu, \kappa > 0$, for which the velocity field is smooth. For the case of a divergence-free velocity that we discuss here, it is furthermore known that the result does not depend upon the order of limits $\nu \to 0$ and $\kappa \to 0$ which can be taken in either order or together[4].

[3]It would be less ambiguous to write them as $p_{\boldsymbol{u}}^{\nu,\kappa}(\boldsymbol{x}', t'|\boldsymbol{x}, t)$, with $\boldsymbol{u}$ denoting the fixed flow realization, but this would lead to an even heavier notation.

[4]The only delicate case is when $\kappa \to 0$ first, so that the Prandtl number goes to infinity, and then $\nu \to 0$ subsequently. Since the Brownian motion disappears from the stochastic equation (4.3) while the velocity field remains smooth, the limiting Lagrangian trajectories are deterministic. To observe spontaneous stochasticity in that limit one must additionally allow the *initial condition* to be random, e.g. with $\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{x} + \epsilon\tilde{\boldsymbol{\rho}}$ for a stochastic perturbation $\tilde{\boldsymbol{\rho}}$ drawn from some fixed distribution

See Bernard et al. [1998], Gawędzki and Vergassola [2000], E and Vanden-Eijnden [2000, 2001], Falkovich et al. [2001], Jan and Raimond [2002, 2004] for discussions of this point.

There is empirical evidence for such phenomena also in Navier-Stokes turbulence obtained from numerical studies of 2-particle dispersion. Eyink [2011] studied stochastic Lagrangian particles whose motion is governed by Eq.(4.3) in a 1024^3 DNS at $Re_\lambda = 433$ and found that the mean-square dispersion becomes independent of κ after a short time of order $(\kappa/\varepsilon)^{1/2}$. Bitane et al. [2013] studied dispersion of deterministic Lagrangian trajectories ($\kappa = 0$) in a 2048^3 DNS at $Re_\lambda = 460$ and a 4096^3 DNS at $Re_\lambda = 730$, and found that the mean-square dispersion becomes independent of the initial separation r_0 of particle pairs in a short time of order $r_0^{2/3}/\varepsilon^{1/3}$. The results of these studies provide evidence of Lagrangian spontaneous stochasticity for Navier-Stokes solutions. A significant limitation of those works is that the joint limit $\nu, \kappa \to 0$ could not be studied (although Bitane et al. [2013] did find consistent Richardson-dispersion statistics for the two Reynolds numbers studied there). One should consider together with the limit $\kappa \to 0$ also a limit $\nu \to 0$ so that the Navier-Stokes solution $\boldsymbol{u}^\nu$ converges to a fixed velocity $\boldsymbol{u}$ that is some sort of weak solution of Euler (as always occurs along a suitable subsequence $\nu_k \to 0$; see Lions [1996], section 4.4)[5].

$P(\boldsymbol{\rho})$. In that case, spontaneous stochasticity appears in the double limit with $\epsilon \to 0$ and $\nu \to 0$ together and, for a divergence-free velocity $\boldsymbol{u}$, the limiting transition probabilities are identical to those obtained for the other limits involving $\kappa \to 0$. This infinite-Prandtl case is discussed carefully by Gawędzki and Vergassola [2000] and E and Vanden-Eijnden [2000].

[5] Amusingly, the direct experimental observation of spontaneous stochasticity in such a joint limit

The principal limitation of these previous studies is that they averaged over the release points $\boldsymbol{x}$ of the particles. A particle dispersion averaged over release points which remains non-vanishing in the joint limit $\nu, \kappa \to 0$ is enough to infer spontaneous stochasticity for a set of points $\boldsymbol{x}$ of non-zero volume measure [Bernard et al., 1998]. However, averaging over $\boldsymbol{x}$ removes information about the effects of spatial intermittency and the local fluid environment on the limiting behavior of the particle distributions $p^{\nu,\kappa}(\boldsymbol{x}', t'|\boldsymbol{x}, t)$ for specific release locations $\boldsymbol{x}$. There was some previous study of such spatial intermittency in pair-dispersion by Biferale et al. [2005, 2014] but they studied only deterministic Lagrangian particles at small (Kolmogorov-scale) initial separations, and not the stochastic Lagrangian particles relevant to our FDR. We present here new data obtained from numerical experiments on a high Reynolds-number turbulence simulation in a 2π-periodic box, for a couple of representative release points. We use simulation data from the homogeneous, isotropic dataset in the Johns Hopkins Turbulence Database (Li et al. [2008], Yu et al. [2012]), publicly available online at `http://turbulence.pha.jhu.edu`. It is ideal for our purposes, since the entire time-history of the velocity is stored for a full large-scale eddy-turnover time, allowing us to integrate backward in time the flow equations (4.3). We consider two release points $\boldsymbol{x}$ at time $t_f = 2.048$, the final database time, one chosen in a typical turbulent "background" region and the other in the vicinity of a strong, large-scale vorticity. We study stochastic trajectories with diffusivities κ corresponding to

may be easier in quantum mechanics than in fluid mechanics. See Eyink and Drivas [2015b].

three values of the Prandtl number, $Pr = 0.1$, 1, and 10. See Appendix C of Drivas and Eyink [2017a] for details about the numerical methods employed in our analysis.

In Figure 4.1 the top panels show 30 representative particle trajectories for the two release points and for each of the three Prandtl numbers. To illustrate the local fluid environment, we also plot isosurfaces of the vorticity filtered with a box-filter of width $L/4$ (L the integral scale) at the time $s = (2/3)T_L$ (T_L the large-scale turnover time). The isosurfaces are for magnitudes of filtered vorticity equal to $15/T_L$. The left panel shows the particles released in a typical "background" region with spottier, weaker vortices and the right shows particles released near a strong vortex. The three greyscale shades of the trajectories (black/grey/light grey) represent the three values of the Prandtl numbers $Pr = 0.1, 1, 10$, resp. As one can see by eye, the ensembles of trajectories are quite similar for the three Pr-values. To make this observation more quantitative, we plot in the middle panels of Fig. 4.1 the mean-square dispersion of pairs of stochastic Lagranigian particles with different realizations of the noise, for the two release points and the three Prandtl numbers. The error bars (almost too small to be observed) represent the standard error of the mean (s.e.m.) for averages over N=1024 sample trajectories. For both release points there is an initial period (going backward in time) where the dispersion grows diffusively as $12\kappa\hat{s}$ with $\hat{s} = t_f - s$, but which then crosses over to a regime of super-ballistic separation that is close to the $\hat{s}^3$-growth predicted by Richardson [1926] and is approximately independent of Pr. The two release points shown here illustrate behavior that we have observed also in many

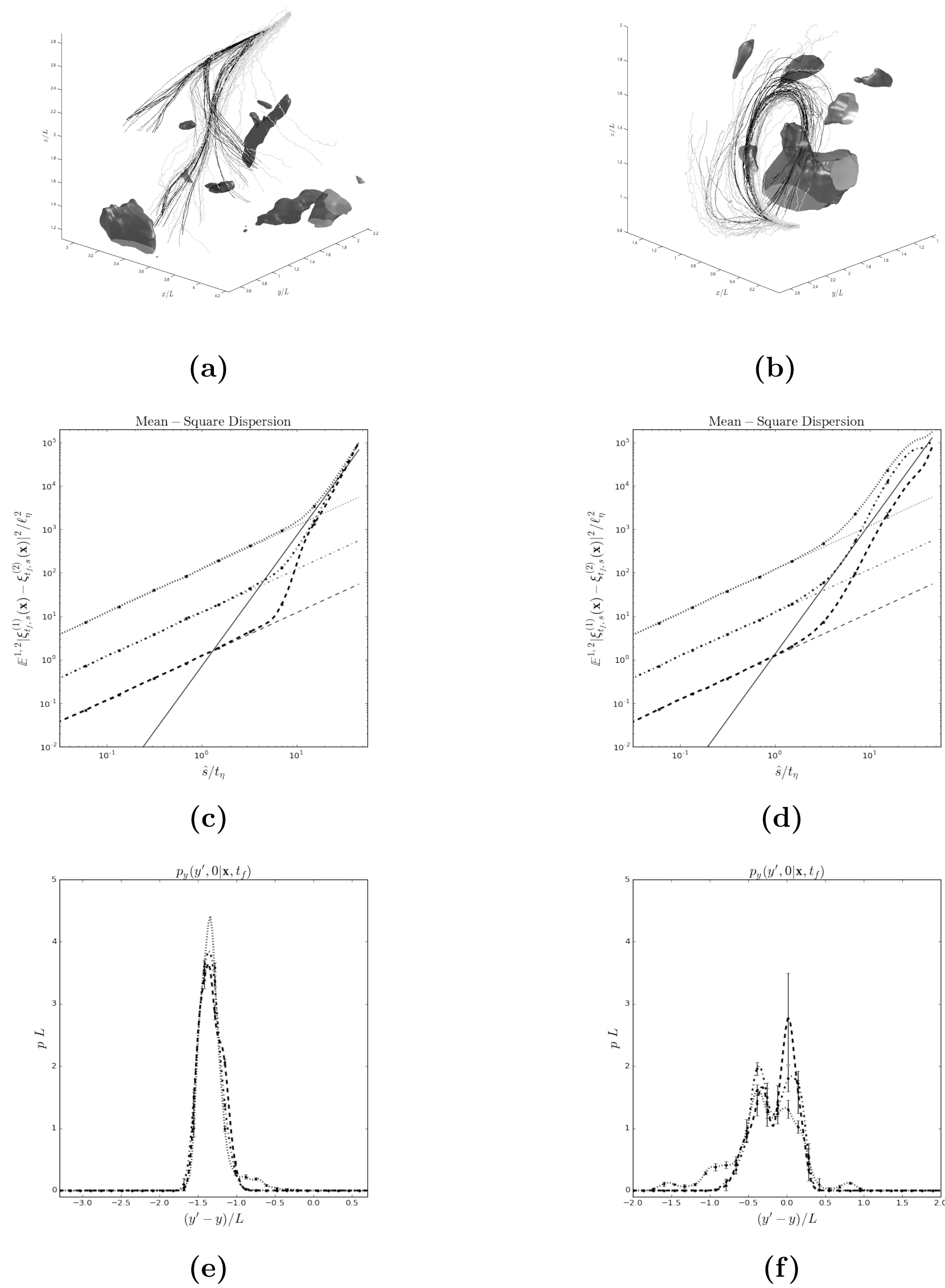

Figure 4.1: *Left panels* are for release at $\boldsymbol{x} = (4.9637, 3.1416, 3.8488)$ in background region, *Right panels* for release at $\boldsymbol{x} = (0.2610, 3.1416, 1.4617)$ near a strong vortex. Top panels (a),(b) plot 30 representative stochastic trajectories for $Pr = 0.1$ (light grey), 1.0 (grey) and 10 (black) together with isosurfaces of coarse-grained vorticity $|\bar{\boldsymbol{\omega}}|T_L = 15$ at time $s = (2/3)T_L$. Middle panels (c),(d) plot particle dispersions (heavy) and short-time results $12\kappa\hat{s}$ (light) for each Pr with $Pr = 0.1$ (dot, $\cdots$), 1.0 (dash-dot, -·-) and 10 (dash, - - -) and a plot in (solid, —) of $g\varepsilon\hat{s}^3$ with $g = 0.7$ (left), $g = 4/3$ (right). The bottom panels (e),(f) plot $p_y(y', 0|\boldsymbol{x}, t_f)$ for the three Pr-values with the same line-styles as (c),(d).

other points of the turbulent fluid, where we find that the Richardson $\hat{s}^3$-law is quite robust, without the necessity of averaging over release points $\boldsymbol{x}$. This is especially so for points $\boldsymbol{x}$ in "background" regions, and is at least approximately observed for $\boldsymbol{x}$ located in more intermittent regions. Finally, we plot in the bottom panels of Fig. 4.1 particle transition probabilities, which provide even further information about the limiting behavior. We plot at time 0, in the approximate Richardson range, the 1-dimensional PDF's of the y-coordinate or

$$p_y^{\nu,\kappa}(y',0|\boldsymbol{x},t_f) = \int dx'dz'\ p^{\nu,\kappa}(\boldsymbol{x}',0|\boldsymbol{x},t_f) \tag{4.31}$$

for each of the two release points $\boldsymbol{x}$ and three Prandtl numbers. We observe very similar behavior also for the x- and z-coordinates. In order to minimize the number of samples required to construct the PDF's numerically, we employed kernel density estimator techniques that gave us good results with only $N = 6144$ samples. See Silverman [1986] and Appendix C of Drivas and Eyink [2017a], where our numerical procedures are completely described. Error bars represent both s.e.m. for the N-sample averages and the effects of variation in the kernel density bandwidth. Consistent with the dispersion plots, we see that the transition PDF's are approximately independent of κ for times in the super-ballistic dispersion range. This is especially true for the release point $\boldsymbol{x}$ in the "background" region, and for the strong vorticity region such independence holds better for the two smallest values of κ (largest Pr).

Recall that in the Kraichnan model with incompressible velocity fields, the limiting densities $p^*(\boldsymbol{x}', 0|\boldsymbol{x}, t_f)$ are independent of the Prandtl number for $\nu, \kappa \to 0$ with Pr fixed [Gawędzki and Vergassola, 2000, E and Vanden-Eijnden, 2000, 2001] and this is approximately observed to be true in our numerical results for incompressible fluid turbulence. These numerical studies illustrate the present quality of direct evidence for Lagrangian spontaneous stochasticity in high-Reynolds-number Navier-Stokes turbulence. As we shall now demonstrate, observations of anomalous scalar dissipation provide further evidence, as the two phenomena are essentially related.

4.4 Spontaneous Stochasticity and Anomalous Dissipation

The phenomenon of spontaneous stochasticity leads to a simple explanation of anomalous dissipation in a turbulent flow, as was first pointed out by Bernard et al. [1998] for decaying scalars (no sources) in the Kraichnan model of random advection. This connection can be understand more directly and more generally using our fluctuation-dissipation relation. In fact, it is intuitively clear from the FDR (4.12) that there can be scalar dissipation non-vanishing in the limit $\kappa \to 0$ only if there is a non-vanishing variance in that same limit, implying that Lagrangian trajectories must remain stochastic. This argument holds in the presence of scalar sources and for a scalar advected by any velocity field $\boldsymbol{u}^\nu$ whatsoever. In particular, the argument holds when $\boldsymbol{u}^\nu$ is a Navier-Stokes solution. Thus, spontaneous stochasticity is the only possible mechanism of anomalous dissipation, for both passive and active scalars, away from walls. Furthermore, we shall show for a passive scalar which does not react back on the flow that spontaneous stochasticity also makes possible anomalous scalar dissipation. Thus, for passive scalars the two phenomena are completely equivalent. In this section, we shall deduce these conclusions, assuming only that the flow domain is compact (closed and bounded) and without any bounding walls.

We first discuss the technically simpler case with $S \equiv 0$ and then show that the same argument extends easily to the case with a non-zero scalar source. When $S \equiv 0$ we can rewrite the lefthand side of the FDR (4.12) using

$$\begin{aligned}\mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x}))\right] = \int d^d x_0 \int d^d x_0'\ \theta_0(\boldsymbol{x}_0)\theta_0(\boldsymbol{x}_0') \\ \times\left[p_2^{\nu,\kappa}(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^{\nu,\kappa}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)p^{\nu,\kappa}(\boldsymbol{x}_0', 0|\boldsymbol{x}, t)\right].\end{aligned} \tag{4.32}$$

where we have introduced the 2-time (backward-in-time) transition probability density

$$p_2^{\nu,\kappa}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) = \mathbb{E}\left[\delta^d(\boldsymbol{y} - \tilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(\boldsymbol{x}))\delta^d(\boldsymbol{y}' - \tilde{\boldsymbol{\xi}}_{t,s'}^{\nu,\kappa}(\boldsymbol{x}))\right], \quad s < t \tag{4.33}$$

which gives the joint probability for the particle to end up at $\boldsymbol{y}$ at time $s < t$ and at $\boldsymbol{y}'$ at time $s' < t$, given that it started at $\boldsymbol{x}'$ at the final time t (moving backward from final to earlier times). At equal times $s = s'$

$$p_2^{\nu,\kappa}(\boldsymbol{y}, s; \boldsymbol{y}', s|\boldsymbol{x}, t) = \delta^d(\boldsymbol{y} - \boldsymbol{y}')p^{\nu,\kappa}(\boldsymbol{y}, s|\boldsymbol{x}, t). \tag{4.34}$$

We now consider the limit $\nu, \kappa \to 0$ so that the transition probabilities approach limiting values $p^*(\boldsymbol{y}, s; \boldsymbol{y}', s|\boldsymbol{x}, t)$, $p^*(\boldsymbol{y}, s|\boldsymbol{x}, t)$. Such limits exist, at least along suitably chosen subsequences $\nu_n, \kappa_n \to 0$, whenever the flow domain is compact. This can be shown using Young measure methods similar to those which have been employed previously to study statistical equilibria for 2D Euler solutions [Robert, 1991, Robert and Sommeria, 1991, Sommeria et al., 1991]. Because the proof of this result is a bit technical, we give it in Appendix 4.A.1. When the Lagrangian particles move according to a deterministic flow $\boldsymbol{\xi}^*_{t,s}$, one easily sees that the 2-time transition probability factorizes as

$$p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) = \delta^d(\boldsymbol{y} - \boldsymbol{\xi}^*_{t,s}(\boldsymbol{x}))\delta^d(\boldsymbol{y}' - \boldsymbol{\xi}^*_{t,s'}(\boldsymbol{x})) = p^*(\boldsymbol{y}, s|\boldsymbol{x}, t)p^*(\boldsymbol{y}', s'|\boldsymbol{x}, t). \tag{4.35}$$

Hence, non-factorization in the limit $\nu, \kappa \to 0$ is an unequivocal sign of spontaneous stochasticity. The variance on the lefthand of the FDR (4.12) can only be non-vanishing in the limit if factorization fails, so that anomalous dissipation clearly requires spontaneous stochasticity. In the other direction, if there is spontaneous stochasticity and thus factorization fails for some positive-measure set of $\boldsymbol{x} \in \Omega$, then the contribution to the volume-integrated variance from that subset must be positive for some suitable smooth choice of θ_0, which implies a positive lower bound to the cumulative, volume-integrated scalar dissipation. In short, anomalous scalar dissipation and Lagrangian spontaneous stochasticity are seen to be equivalent. This argument is given as a formal mathematical proof in the Appendix 4.A.2.

The sufficiency argument works only for a passive scalar. For active scalars, the initial data θ_0 partially determines the velocity field $\boldsymbol{u}$ and so is not free to vary. In order to conclude sufficiency in that case one needs to assume that the resulting velocity field does not "conspire" with the initial scalar to cause the variance to vanish, i.e. for the random trajectories to sample only points on a single level set of θ_0. If this remarkable behavior did happen to occur for some choice of θ_0, then one would not expect it to persist for a small perturbation of θ_0. Thus, it is highly likely also for active scalars that spontaneous stochasticity implies anomalous dissipation, but we have not proved that with the FDR. We can however conclude rigorously both for passive and for active scalars that anomalous dissipation implies spontaneous stochasticity. The above proposition shows that any evidence for anomalous scalar dissipation in the free decay of an active or passive scalar (no sources) obtained from DNS in a periodic box is also evidence for spontaneous stochasticity. The argument in this section is a strong motivation to perform DNS studies to verify anomalous dissipation in the free decay of a scalar, since this would provide additional confirmation of spontaneous stochasticity. All of the DNS cited by Yeung et al. [2005], section 2.1, employed sources (e.g. a mean scalar gradient coupled to the velocity field) that maintained a statistical steady-state for the scalar fluctuations.

Including a non-zero scalar source involves only minor changes to the previous argument. First note that

$$\mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right] = \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x}))\right]$$
$$+2\ \mathrm{Cov}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})), \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right] + \mathrm{Var}\left[\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right]. \tag{4.36}$$

Furthermore, one has for the variance of the time-integrated source sampled along the stochastic particle trajectory that

$$\mathrm{Var}\left[\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right] = \int_0^t ds \int_0^t ds' \int d^d y \int d^d y'\ S(\boldsymbol{y}, s) S(\boldsymbol{y}', s')$$
$$\times\left[p_2^{\nu,\kappa}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) - p^{\nu,\kappa}(\boldsymbol{y}, s|\boldsymbol{x}, t) p^{\nu,\kappa}(\boldsymbol{y}', s'|\boldsymbol{x}, t)\right]. \tag{4.37}$$

and for the covariance between the sampled initial data and integrated source that

$$\mathrm{Cov}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})), \int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s\right] = \int_0^t ds \int d^d x_0 \int d^d y\ \theta_0(\boldsymbol{x}_0) S(\boldsymbol{y}, s)$$
$$\times\left[p_2^{\nu,\kappa}(\boldsymbol{x}_0, 0; \boldsymbol{y}, s|\boldsymbol{x}, t) - p^{\nu,\kappa}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) p^{\nu,\kappa}(\boldsymbol{y}, s|\boldsymbol{x}, t)\right]. \tag{4.38}$$

Clearly, anomalous scalar dissipation requires spontaneous stochasticity. For a passive scalar we can also argue in the other direction. Indeed, we can repeat the previous argument to conclude that, if there is spontaneous stochasticity for a positive measure set of $\boldsymbol{x}$, then not only is there a smooth choice of θ_0 so that the variance associated to the initial condition in (5.33) is positive when integrated over this set of $\boldsymbol{x}$, but also there is a smooth choice of source field S so that the contribution of the variance (4.37) is positive. This is already enough to conclude that there must be anomalous dissipation for the scalar with initial condition 0 and with the chosen source S. We can also conclude that there is anomalous dissipation for the initial condition θ_0 and the source S. Indeed, if the total variance contribution in (4.36) is not positive then it must vanish, which implies that the covariance term in (4.38) provides a negative contribution. In that case, simply take $S \to -S$ to make the contributions of all three terms (5.33),(4.37),(4.38) positive. We thus conclude that, also for the passive scalar rejuvenated by a source, there is equivalence between anomalous scalar dissipation and Lagrangian spontaneous stochasticity. The argument is given more carefully in Appendix 4.A.2.

It has not been generally appreciated that similar conclusions can be reached in the special case of sourceless scalars using the arguments of Bernard et al. [1998], which are not at all restricted to the Kraichnan model. To underline this point and, also, to give additional insight, we here briefly summarize their reasoning. Note that the stochastic representation (4.22) of the advected scalar in the limit $\nu, \kappa \to 0$ becomes, using (4.30),

$$\theta^*(\boldsymbol{x}, t) = \int d^d x_0 \; \theta_0(\boldsymbol{x}_0) \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t). \tag{4.39}$$

It is worth noting that $\theta^*(\boldsymbol{x}, t)$ is a kind of "weak solution" of the ideal advection equation, $\partial_t \theta^* + \boldsymbol{u} \cdot \nabla \theta^* = 0$, although this fact is not needed for the argument. It follows from (4.39) that for any strictly convex function $h(\theta)$, e.g. $h(\theta) = \frac{1}{2}\theta^2$,

$$h(\theta^*(\boldsymbol{x}, t)) \leq \int d^d x_0 \; h(\theta_0(\boldsymbol{x}_0)) \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t), \tag{4.40}$$

and equality holds if and only the transition probability is a delta-distribution of type (4.25). This is the so-called Jensen inequality (e.g. see Itô [1984]). Since the limiting transition probabilities are not delta-distributions in the Kraichnan model, the inequality in (4.40) is strict. Furthermore, the limiting transition probabilities for $\nu, \kappa \to 0$ inherit the volume-preservation property (4.24), so that

$$\int p^*(\boldsymbol{x}', t'|\boldsymbol{x}, t) d^d x = 1. \tag{4.41}$$

In that case, integrating (4.40) over $\boldsymbol{x}$ gives

$$\int h(\theta^*(\boldsymbol{x}, t)) \; d^d x < \int h(\theta_0(\boldsymbol{x}_0)) \; d^d x_0, \tag{4.42}$$

so that the h-integral is decaying (dissipated) even in the limit $\nu, \kappa \to 0$. The anomalous scalar dissipation in the Kraichnan model thus has an elegant Lagrangian mechanism. Essentially, the molecular diffusivity is replaced by a “turbulent diffusivity” associated to the persistent stochasticity of the Lagrangian trajectories, which continues to homogenize the scalar field even as the molecular diffusivity vanishes. We give rigorous details of this argument in Appendix A.3 of Drivas and Eyink [2017a], where, in the absence of sources, we obtain necessary and sufficient conditions for anomalous dissipation identical to those derived from the FDR.

4.5 Summary and Discussion

This chapter has derived a Lagrangian fluctuation-dissipation relation for scalars advected by an incompressible fluid. Our relation expresses an exact balance between molecular dissipation of scalar fluctuations and the input of scalar fluctuations from the initial scalar values and internal sources as these are sampled by stochastic Lagrangian trajectories backward in time. We have exploited this relation to give a simple proof (in domains without walls) that spontaneous stochasticity of Lagrangian trajectories is necessary and sufficient for anomalous dissipation of passive scalars, and necessary (but possibly not sufficient) for anomalous dissipation of active scalars.

An important outstanding question is the extent to which the results of this chapter can be carried over to provide a Lagrangian picture of anomalous energy dissipation in Navier-Stokes turbulence[6]. We briefly comment upon this issue here. The formal extension of our fluctuation-dissipation relation to viscous energy dissipation

[6]The most direct application of our scalar results to Navier-Stokes might appear to be to analyze the viscous dissipation of enstrophy in freely-decaying 2D turbulence, where the vorticity is an active (pseudo)scalar field. Unfortunately, all of our analysis assumes that the initial scalar field is square-integrable or L^2, but it has been shown by Eyink [2001] and Tran and Dritschel [2006] that there can be no anomalous enstrophy dissipation for a freely-decaying 2D Navier-Stokes solution with finite initial enstrophy. It may still be the case that there is anomalous enstrophy dissipation for more singular, infinite-enstrophy initial data and that this dissipation is associated to spontaneous stochasticity (see further discussion in Eyink [2001]). However, we cannot investigate this delicate issue using the fluctuation-dissipation relation of the present chapter.

is straightforward. We can exploit the stochastic Lagrangian representation for the incompressible Navier-Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} \tag{4.43}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.44}$$

recently elaborated by Constantin and Iyer [2008, 2011], which is valid both for flows in domains without boundaries and for wall-bounded flows. Their results can be most simply derived using a backward stochastic particle flow $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x})$ and a corresponding "momentum" $\tilde{\boldsymbol{\pi}}_{t,s}(\boldsymbol{x}) \equiv \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)$ which together satisfy the backward Itō equations

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) = \tilde{\boldsymbol{\pi}}_{t,s}(\boldsymbol{x})\mathrm{d}s + \sqrt{2\nu}\, \hat{\mathrm{d}}\mathbf{W}_s, \tag{4.45}$$

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\pi}}_{t,s}(\boldsymbol{x}) = -\nabla p(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + \sqrt{2\nu}\, \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla \boldsymbol{u}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s). \tag{4.46}$$

These are a stochastic generalization of Hamilton's particle equations, making contact with traditional methods of Hamiltonian fluid mechanics [Salmon, 1988]. See more detailed discussion of Eyink [2010], Rezakhanlou [2014]. By integrating the second of these Hamilton's equations from 0 to t and taking expectations over the Brownian motion, one readily obtains

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbb{E}\left[\boldsymbol{u}_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) - \int_0^t \nabla p(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\mathrm{d}s\right], \tag{4.47}$$

using the fact that the stochastic integral $\sqrt{2\nu}\int_0^t \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla \boldsymbol{u}^\nu(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)$ is a backward martingale and so vanishes under expectation. The formula (4.47) was previously derived by Albeverio and Belopolskaya [2010]. Moreover, by exploiting the same Itō-isometry argument as applied earlier for scalars, one can derive

$$\nu\int_0^t \mathrm{d}s\ \langle|\nabla \boldsymbol{u}(s)|^2\rangle_\Omega = \frac{1}{2}\left\langle \mathrm{Var}\left[\boldsymbol{u}_0(\tilde{\boldsymbol{\xi}}_{t,0}) - \int_0^t \nabla p(\tilde{\boldsymbol{\xi}}_{t,s},s)\mathrm{d}s\right]\right\rangle_\Omega. \tag{4.48}$$

This can be considered a “fluctuation-dissipation relation” for viscous energy dissipation in a Navier-Stokes solution.

Unfortunately this relation does not appear to be particularly useful for analyzing the high-Reynolds number (or inviscid) limit. It has a mixed Eulerian-Lagrangian character, since it involves both the particle trajectories $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x})$ and the Eulerian pressure-gradient field $\nabla p(\boldsymbol{x},t)$. The latter field is furthermore a dissipation-range object, which grows increasingly singular as $\nu \to 0$. For example, using the classical K41 scaling estimates [Obukhov, 1949, Yaglom, 1949, Batchelor, 1951], one expects an rms value of the pressure-gradient $(\nabla p)_{rms} \sim (\varepsilon^3/\nu)^{1/4}$ and intermittency effects will make this field even more singular. Mathematically speaking, the pressure-gradient cannot be expected to exist as an ordinary function in the limit $\nu \to 0$ but only as a distribution. Because of these facts, we cannot derive from (4.48) any relation between anomalous energy dissipation and spontaneous stochasticity for Navier-Stokes turbulence. In particular, even if there were anomalous energy dissipation, the limiting stochastic particle trajectories might become deterministic as $\nu \to 0$. In that case, the variance on the righthand side of (4.48) could remain non-vanishing, because the smaller fluctuations due to vanishing stochasticity could be compensated by the diverging magnitude of the pressure-gradient.

More fundamentally, we believe that (4.48) misses essential physics. Note that this relation holds for freely-decaying Navier-Stokes turbulence both in 2D and in 3D, but in the former case there is certainly no anomalous energy dissipation. Furthermore, in forced, steady-state 2D turbulence there is evidence in the inverse-energy cascade range for Richardson dispersion and Lagrangian spontaneous stochasticity (Boffetta and Sokolov [2002], Faber and Vassilicos [2009]) but this is associated not to small-scale energy dissipation by viscosity but instead to large-scale energy dissipation by Eckman-type damping. A possibly important clue is provided by the fact that Richardson dispersion is faster backward in time for 3D forward energy cascade Sawford et al. [2005], Berg et al. [2006], Eyink [2011], but faster forward in time for 2D inverse energy cascade Faber and Vassilicos [2009]. By a comparison of these observations for 2D and 3D Navier-Stokes turbulence and by means of exact results for Burgers turbulence, Eyink and Drivas [2015a] have argued that anomalous energy dissipation for Navier-Stokes turbulence should be related not simply to presence of spontaneous stochasticity but instead to time-asymmetry of the stochastic Lagrangian trajectories. This is reminiscent of so-called “Fluctuation Theorems” in non-equilibrium statistical mechanics, which imply exponentially asymmetry in the probability of entropy production with positive and negative signs. See Schuster et al. [2013], Gawędzki [2013] for recent reviews. These results are deeply related to traditional fluctuation-dissipation theorems in statistical physics, but we have been unable to discover any connection with our Lagrangian FDR. More recently, a time-asymmetry has been established in the very short-time dispersion of nearby Lagrangian trajectories by Falkovich and Frishman [2013], Jucha et al. [2014]. However, these results hold only for times much less than the Kolmogorov time scale of the flow and therefore cannot explain the long-time Richardson behavior or the observed time-asymmetry therein.

The most important implication of the present work is the additional support provided to the concept of Lagrangian spontaneous stochasticity. Exploiting our Langrangian FDR, we have shown that any empirical evidence for anomalous scalar dissipation, either for passive or for active scalars and away from walls, must be taken as evidence also for spontaneous stochasticity. There are profound implications of this phenomenon for many Lagrangian aspects of turbulent flows. For example, Constantin and Iyer [2008] have shown that the classical Kelvin-Helmoltz theorems for vorticity dynamics in smooth solutions of the incompressible Euler equations generalize within their stochastic framework to solutions of the incompressible Navier-

Stokes equation with a positive viscosity. In fact, similar to the case of the advected scalars discussed in the present work, Constantin and Iyer [2008] proved that circulations around stochastically advected loops are martingales backward in time for the Navier-Stokes solution and also proved that this property completely characterizes those solutions. This “stochastic Kelvin theorem” demonstrates again that the stochastic Lagrangian approach is the natural generalization to non-ideal fluids of the Lagrangian methods for ideal fluids. Furthermore, if there is spontaneous stochasticity, then vortex motion must remain stochastic for arbitrarily high Reynolds numbers. Contrary to the traditional arguments of Taylor and Green [1937], vortex-lines in the ideal limit will not be “frozen-into” the turbulent fluid flow in the usual sense. Similar results holds also for magnetic field-line motion in resistive magnetohydrodynamics [Eyink, 2009], and spontaneous stochasticity then implies the possibility of fast magnetic reconnection in astrophysical plasmas for arbitrarily small electrical conductivity [Eyink et al., 2013a].

Appendix

4.A Mathematical Proofs

4.A.1 Existence of Limiting Transition Probabilities

To make rigorous the arguments in Section 4.4, we note that the transition probabilities $p^{\nu,\kappa}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)$ discussed there are well-defined for any sequence of continuous (or even just bounded) velocity fields $\boldsymbol{u}^\nu$. However, we shall generally assume that these fields are even smooth for $\nu > 0$ and their energies are bounded uniformly in ν. Because of the latter assumption we can always extract a subsequence $\nu_j \to 0$ such that $\boldsymbol{u}^{\nu_j} \to \boldsymbol{u}$, with $\boldsymbol{u}$ a finite energy or $L^2(\Omega \times [0,T])$ velocity field, where convergence is in the weak sense:

$$\lim_{j\to\infty} \int_\Omega d^dx \int_0^T dt\ \boldsymbol{u}^{\nu_j}(\boldsymbol{x},t) \cdot \boldsymbol{w}(\boldsymbol{x},t) = \int_\Omega d^dx \int_0^T dt\ \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{w}(\boldsymbol{x},t) \tag{4.49}$$

for all $\boldsymbol{w} \in L^2(\Omega \times [0,T])$. This is a consequence of the Banach-Alaoglu Theorem [Rudin, 2006]. Thus, we consider limits in which there is a definite, fixed fluid velocity $\boldsymbol{u}$. If the $\boldsymbol{u}^\nu$ are solutions of the incompressible Navier-Stokes equation indexed by viscosity ν, then we can furthermore select the subsequence $\nu_k \to 0$ so that the limiting velocity $\boldsymbol{u}$ is a "dissipative Euler solution" in the sense of Lions [1996], section 4.4.

We must now show that a further subsequence $\nu_k = \nu_{j_k}$ can be selected together with a corresponding subsequence $\kappa_k \to 0$, so that the transition probabilities $p^{\nu_k,\kappa_k}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)$ satisfy the following conditions:

(i) There is a transition density $p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)$ which is measurable in $\boldsymbol{x}$ so that

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega d^d x_0 \int_\Omega d^d x \; f(\boldsymbol{x}_0, \boldsymbol{x}) \; p^{\nu_k,\kappa_k}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) \\ = \int_\Omega d^d x_0 \int_\Omega d^d x \; f(\boldsymbol{x}_0, \boldsymbol{x}) \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t), \end{aligned}$$

for all continuous functions $f \in C(\Omega \times \Omega)$.

(ii) (normalization) $\int_\Omega d^d x_0 \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) = 1$ for a.e. $\boldsymbol{x} \in \Omega$.

(iii) (volume-conservation) $\int_\Omega d^d x_0 \int_\Omega d^d x \; g(x_0) \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) = \int_\Omega d^d x_0 \; g(\boldsymbol{x}_0)$ for all continuous $g \in C(\Omega)$.

To prove the properties (i)-(iii), the key fact we shall use is that the transition probability densities for $\nu, \kappa > 0$ can be regarded as *Young measures*

$$\mu_{\boldsymbol{x}}^{\nu,\kappa,t}(d\boldsymbol{x}_0) = d^d x_0 \; p^{\nu,\kappa}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t), \tag{4.50}$$

that is, as probability measures $\mu_{\boldsymbol{x}}^{\nu,\kappa,t}$ on Ω which are measurably parameterized by $\boldsymbol{x} \in \Omega$. Fluid-dynamicists will be familiar with Young measures from theories of long-time statistical equilibria for two-dimensional fluids [Robert, 1991, Sommeria et al., 1991]. A good introduction are the lectures of Valadier [1994] and a comprehensive treatment can be found in the monograph of Florescu and Godet-Thobie [2012].

Here we briefly review the necessary theory. In the context of our problem, Young measures may be defined as families of probability measures μ_x, defined on a compact set $Y \subseteq \mathbb{R}^m$, measurably parametrized by $x \in X \subset \mathbb{R}^n$, with X also compact. This uniquely defines a positive Radon measure μ over $X \times Y$ given on product sets by

$$\mu(A \times B) = \int_A \mu_x(B) \; \mathrm{d}x. \tag{4.51}$$

By construction, μ satisfies the following identity

$$\langle \mu, f \rangle \equiv \int_{X \times Y} f(x,y)\mu(\mathrm{d}x, \mathrm{d}y) = \int_X \left(\int_Y f(x,y) \; \mu_x(\mathrm{d}y) \right) \mathrm{d}x, \tag{4.52}$$

for any continuous function $f \in C(X \times Y)$. Moreover, for $f \in C(X)$, one has

$$\langle \mu, f \rangle = \int_X f(x)\, \mathrm{d}x, \tag{4.53}$$

that is to say, the projection of μ on X is $\mathrm{d}x$, the Lebesgue measure. One may alternatively take these last two properties as the definition of a Young measure. That is, for any positive Radon measure μ on $X \times Y$ whose projection on X is $\mathrm{d}x$ there is a mapping $x \mapsto \mu_x$ satisfying (4.52). This is the content of the so-called Disintegration Theorems [Jiřina, 1959, Valadier, 1973]. The mapping $x \mapsto \mu_x$ is unique Lebesgue almost everywhere.

Let us denote by $\mathcal{Y}$ the set of Young measures μ on the product set $X \times Y$. This set has the important property that it is a closed subset of the space $M(X \times Y)$ of Radon measures on $X \times Y$ in the topology of narrow convergence. The *narrow topology* is the coarsest topology on $M(X \times Y)$ for which the maps $\mu \mapsto \langle \mu, f \rangle$ are continuous for all $f \in C_b(X \times Y)$, the space of bounded continuous functions. Since $X \times Y$ is compact, this topology coincides with the so-called *vague topology* which is the coarsest for which the maps $\mu \mapsto \langle \mu, f \rangle$ are continuous for all $f \in C_c(X \times Y)$, the space of compactly-supported continuous functions. Furthermore, it coincides with the topology defined by the maps $\mu \mapsto \langle \mu, f \rangle$ for all $f \in C(X \times Y)$. For a detailed discussion of these different topologies, see Florescu and Godet-Thobie [2012]. Here we note only that these make $M(X \times Y)$ into a compact, metrizable topological space for compact X, Y. That $\mathcal{Y}$ is a closed subspace of $M(X \times Y)$ may then easily seen by noting that for any sequence $\mu^n \in \mathcal{Y}$ with $\mu^n \to \mu$ narrowly

$$\int_X f(x)\,\mathrm{d}x = \langle \mu^n, f \rangle \to \langle \mu, f \rangle, \quad \text{for all } f \in C(X) \tag{4.54}$$

so that the projection of μ onto X is $\mathrm{d}x$ and $\mu \in \mathcal{Y}$. A further closed subset $\mathcal{Y}_m \subset \mathcal{Y}$ is the set of *measure-preserving Young measures*, which satisfy the additional condition that

$$\langle \mu, g \rangle = \int_X \left(\int_Y g(y)\mu_x(\mathrm{d}y) \right) \mathrm{d}x = \int_Y g(y)\mathrm{d}y, \quad \text{for all } g \in C(Y) \tag{4.55}$$

which may be stated formally as $\int_X \mathrm{d}x\ \mu_x(\mathrm{d}y) = \mathrm{d}y$. That $\mathcal{Y}_m$ is closed in the narrow topology is shown by an argument exactly like that for $\mathcal{Y}$ above.

From these basic results we can easily derive the consequences (i)-(iii), taking $X = Y = \Omega$, where Ω is the closure of a bounded open set with a smooth boundary. Then with the definition (4.50) one has $\mu^{\nu,\kappa,t} \in \mathcal{Y}_m$ for fixed t and all $\nu, \kappa > 0$. Since $\mathcal{Y}_m$ is a closed subset of the compact, metrizable space $M(X \times Y)$, it is itself (sequentially) compact. Hence, given the subsequence ν_j there is a further subsequence $\nu_k = \nu_{j_k}$ and a corresponding sequence κ_k such that $\mu^{\nu_k \kappa_k, t} \to \mu^{*t} \in \mathcal{Y}_m$ in the narrow topology. Note that the limit μ^{*t} need not be unique and may depend upon the selected subsequence. The narrow convergence $\mu^{\nu_k \kappa_k, t} \to \mu^{*t}$ is equivalent to (i), with the definition

$$d^d x_0 \; p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) = \mu_{\boldsymbol{x}}^{*,t}(d\boldsymbol{x}_0), \tag{4.56}$$

where in general $p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)$ is a distribution in the variable $\boldsymbol{x}_0$ not an ordinary function. Then (ii) is a restatement that $\mu^{*t} \in \mathcal{Y}$ and (iii) is a restatement that $\mu^{*t} \in \mathcal{Y}_m$. These observations complete the proof of properties (i)-(iii) above.

With these results in hand, we now rigorously prove the equivalence of spontaneous stochasticity and anomalous dissipation. We do this by exploiting our general fluctuation dissipation relation. In the submitted paper [Drivas and Eyink, 2017a], we also show how this follows by the original argument of Bernard et al. [1998] for the case of scalars without sources.

4.A.2 Proofs Using the FDR

As in the main text, we first consider the case without a scalar source ($S = 0$). Our starting point is the FDR (4.12), with formula (5.33) for the variance $\mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}^{\nu_k,\kappa_k}(\boldsymbol{x}))\right]$. It follows from (i)-(ii) of Apppendix 4.A.1 that a subsequence $\nu_k = \nu_{j_k}$ can be selected together with a corresponding subsequence $\kappa_k \to 0$, so that the space-averaged variance will satisfy

$$\lim_{k\to\infty} \left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}^{\nu_k,\kappa_k})\right] \right\rangle_\Omega = \int d^d x \int d^d x_0 \int d^d x_0' \; \theta_0(\boldsymbol{x}_0)\theta_0(\boldsymbol{x}_0')$$
$$\times\Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)p^*(\boldsymbol{x}_0', 0|\boldsymbol{x}, t)\Big], \tag{4.57}$$

for all $\theta_0 \in C(\Omega)$, where

$$p_2^*(\boldsymbol{x}_0, 0, \boldsymbol{x}_0', 0|\boldsymbol{x}, t) \equiv \delta^d(\boldsymbol{x}_0 - \boldsymbol{x}_0')p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t). \tag{4.58}$$

Note that p_2^* is a Young measure on $Y = \Omega \times \Omega$ measurably indexed by elements $\boldsymbol{x}$ of $X = \Omega$, since it is a narrow limit of the Young measures $p_2^{\nu_k,\kappa_k}$. We shall not use the property (iii) from Appendix 4.A.1 in our argument, although volume-conservation was, of course, used in the derivation of the FDR (4.12). Since that FDR holds for all $\nu, \kappa > 0$, it follows that the limit of the cumulative global scalar dissipation exists and must coincide with the limiting variance:

$$\begin{aligned} \lim_{k\to\infty} \kappa_k \int_0^t ds \Big\langle |\nabla \theta^{\nu_k,\kappa_k}(s)|^2 \Big\rangle_\Omega = \int d^d x \int d^d x_0 \int d^d x_0' \; \theta_0(\boldsymbol{x}_0)\theta_0(\boldsymbol{x}_0') \\ \times \Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) p^*(\boldsymbol{x}_0', 0|\boldsymbol{x}, t) \Big], \end{aligned} \tag{4.59}$$

for all $\theta_0 \in C(\Omega)$. It follows immediately that anomalous scalar dissipation requires spontaneous stochasticity since, by the exact formula (4.59), a non-vanishing cumulative dissipation necessitates non-factorization on a finite measure set of $\boldsymbol{x}$.

The argument that spontaneous stochasticity implies anomalous dissipation is a bit more involved. We need to show that if non-factorization holds on a finite measure set of $\boldsymbol{x}$, then there exists a smooth choice of θ_0 such that both sides of (4.59) are positive. Thus, assume the opposite, that both sides vanish for all smooth θ_0. The righthand size then also vanishes for all continuous θ_0, since $C^\infty(\Omega)$ is dense in $C(\Omega)$

in the uniform norm. For example, this density follows by the Stone-Weierstrass theorem [Rudin, 2006], since $C^\infty(\Omega)$ is a subalgebra of $C(\Omega)$ containing the constant 1, closed under complex conjugation, and separating points of Ω. Since the integrand with respect to $\boldsymbol{x}$ is a variance, it is non-negative, so that the vanishing of the integral over $\boldsymbol{x}$ implies that there is a subset $\Omega_0 \subset \Omega$ of full measure, such that

$$\int d^d x_0 \int d^d x_0' \; \theta_0(\boldsymbol{x}_0)\theta_0(\boldsymbol{x}_0')\Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)p^*(\boldsymbol{x}_0', 0|\boldsymbol{x}, t)\Big] = 0, \tag{4.60}$$

for all $\boldsymbol{x} \in \Omega_0$ and $\theta_0 \in C(\Omega)$. Note furthermore that the quantity in the square brackets "$[\,\cdot\,]$" in the equation above is symmetric in $\boldsymbol{x}_0$, $\boldsymbol{x}_0'$. Thus, for any pair of functions g, h, one can take $\theta_0 = g + h$ to infer that

$$\int d^d x_0 \int d^d x_0' \; g(\boldsymbol{x}_0)h(\boldsymbol{x}_0')\Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)p^*(\boldsymbol{x}_0', 0|\boldsymbol{x}, t)\Big] = 0 \tag{4.61}$$

for all $\boldsymbol{x} \in \Omega_0$ and $g, h \in C(\Omega)$. Since the product functions $(g \otimes h)(\boldsymbol{x}, \boldsymbol{x}_0') = g(\boldsymbol{x}_0)h(\boldsymbol{x}_0')$ form a subalgebra of $C(\Omega^2)$ that satisfies all of the conditions of the Stone-Weierstrass theorem, we can use this theorem again to extend the equality to

$$\int d^d x_0 \int d^d x_0' \; f(\boldsymbol{x}_0, \boldsymbol{x}_0')\Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}_0', 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t)p^*(\boldsymbol{x}_0', 0|\boldsymbol{x}, t)\Big] = 0 \tag{4.62}$$

for all $\boldsymbol{x} \in \Omega_0$ and $f \in C(\Omega^2)$. The parameterized measure $\nu_{\boldsymbol{x}}$ defined by

$$\nu_{\boldsymbol{x}}(d\boldsymbol{x}_0, d\boldsymbol{x}'_0) = d^d x_0 \; d^d x'_0 \Big[p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}'_0, 0|\boldsymbol{x}, t) - p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) p^*(\boldsymbol{x}'_0, 0|\boldsymbol{x}, t) \Big] \tag{4.63}$$

is a difference of two Young measures and, thus, there is a continuous linear functional on $C(\Omega^2)$ for all $\boldsymbol{x} \in \Omega_0$ also denoted $\nu_{\boldsymbol{x}}$, defined by $\langle \nu_{\boldsymbol{x}}, f \rangle = \int_{\Omega^2} f d\nu_{\boldsymbol{x}}$. Since

$$\langle \nu_{\boldsymbol{x}}, f \rangle = 0, \text{ for all } f \in C(\Omega^2) \text{ and } \boldsymbol{x} \in \Omega_0, \tag{4.64}$$

it follows for all $\boldsymbol{x} \in \Omega_0$ that $\nu_{\boldsymbol{x}} \equiv 0$, as an element of the dual Banach space $C(\Omega^2)^*$. A direct consequence is that

$$p_2^*(\boldsymbol{x}_0, 0; \boldsymbol{x}'_0, 0|\boldsymbol{x}, t) = p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) p^*(\boldsymbol{x}'_0, 0|\boldsymbol{x}, t) \tag{4.65}$$

as distributions in $\boldsymbol{x}_0, \boldsymbol{x}'_0$, for all $\boldsymbol{x} \in \Omega_0$. However, this contradicts our starting assumption that factorization fails on a set of full measure. Hence, there must be a smooth choice of θ_0 which makes the righthand side of (4.59) positive, and thus also the lefthand side.

Let us next consider the case with $\theta_0 \equiv 0$, but with the source S non-vanishing. In this circumstance the FDR (4.12) becomes

$$\kappa \int_0^t ds \Big\langle |\nabla \theta(s)|^2 \Big\rangle_\Omega = \frac{1}{2} \left\langle \mathrm{Var} \left[\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(s) \; \mathrm{d}s \right] \right\rangle_\Omega \tag{4.66}$$

with expression (4.37) for the variance. We show first that there is a suitable subsequence $\nu_k = \nu_{j_k} \to 0$ and $\kappa_k \to 0$ such that

$$\lim_{k\to\infty} \int_\Omega d^d x \ \mathrm{Var}\left[\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}^{\nu_k,\kappa_k}(\boldsymbol{x}), s) \ \mathrm{d}s\right]$$
$$= \int_\Omega d^d x \ \int_0^t ds \int_0^t ds' \int_\Omega d^d y \int_\Omega d^d y' \ S(\boldsymbol{y}, s) S(\boldsymbol{y}', s')$$
$$\times \Big[p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) - p^*(\boldsymbol{y}, s|\boldsymbol{x}, t) p^*(\boldsymbol{y}', s'|\boldsymbol{x}, t)\Big]. \tag{4.67}$$

for any $S \in C(\Omega \times [0,t])$ and for suitable limiting transition probabilities p_2^* and p^*. To show this we note that

$$\mu_{s,s',\boldsymbol{x}}^{\nu,\kappa}(d\boldsymbol{y}, d\boldsymbol{y}') = d^d y \ d^d y' \ p_2^{\nu,\kappa}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) \tag{4.68}$$

defines a set of Young measures on $Y = \Omega \times \Omega$ measurably indexed by elements $(s, s', \boldsymbol{x})$ of $X = [0,t] \times [0,t] \times \Omega$. Since these spaces X and Y are both compact, we can appeal to the general results on Young measures discussed in Appendix 4.A.1 to infer that a subsequence ν_k, κ_k exists so that, for all $f \in C(X \times Y)$,

$$\lim_{k\to\infty} \int_0^t ds \int_0^t ds' \int_\Omega d^d y \int_\Omega d^d y' \ \int_\Omega d^d x \ f(\boldsymbol{y}, s; \boldsymbol{y}', s'; \boldsymbol{x}) \ p_2^{\nu_k,\kappa_k}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t)$$
$$= \int_0^t ds \int_0^t ds' \int_\Omega d^d y \int_\Omega d^d y' \ \int_\Omega d^d x \ f(\boldsymbol{y}, s; \boldsymbol{y}', s'; \boldsymbol{x}) \ p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) \tag{4.69}$$

for some limit Young measure with distributional density p_2^*, which it is easy to show inherits the symmetry of $p_2^{\nu_k,\kappa_k}$ in $(\boldsymbol{y}, s)$ and $(\boldsymbol{y}', s')$. Choosing the function f to be of the form $f(\boldsymbol{y},s;\boldsymbol{y}',s';\boldsymbol{x}) = h(s')g(\boldsymbol{y},s;\boldsymbol{x})$ gives also

$$\begin{aligned}\lim_{k\to\infty}\int_0^t ds\int_\Omega d^dy\ \int_\Omega d^dx\ g(\boldsymbol{y},s;\boldsymbol{x})\ p^{\nu_k,\kappa_k}(\boldsymbol{y},s|\boldsymbol{x},t)\\ =\int_0^t ds\int_\Omega d^dy\ \int_\Omega d^dx\ g(\boldsymbol{y},s;\boldsymbol{x})\ p^*(\boldsymbol{y},s|\boldsymbol{x},t)\end{aligned}\tag{4.70}$$

for all continuous g with

$$p^*(\boldsymbol{y},s|\boldsymbol{x},t)=\int_\Omega d^dy'\ p_2^*(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)\tag{4.71}$$

constant in s' for almost every $s,\boldsymbol{x}$ and defining a consistent 1-time Young measure. We can also establish volume-preserving properties of these limiting Young measures, although that will not be necessary to our argument. From these results (4.67) follows by taking the limit along the subsequence ν_k,κ_k of the formula (4.37) for the variance.

The proof that spontaneous stochasticity is both necessary and sufficient for anomalous scalar dissipation now follows by arguments almost identical to the situation with $\theta_0\neq 0$, $S\equiv 0$ that was first considered in this section. Necessity is immediate from (4.66),(4.67). The proof of sufficiency is very similar to that given before, by showing that vanishing of the space-integrated variance (4.67) for all smooth source fields S implies the factorization

$$p_2^*(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)=p^*(\boldsymbol{y},s|\boldsymbol{x},t)p^*(\boldsymbol{y}',s'|\boldsymbol{x},t)\tag{4.72}$$

for almost every $\boldsymbol{x} \in \Omega$. The non-negativity of the $\boldsymbol{x}$-integrand requires some argument, because it is no longer obviously a variance. However, it is the limit of a variance in the sense that

$$\lim_{k\to\infty} \int_\Omega d^d x\ u(\boldsymbol{x})\ \mathrm{Var}\left[\int_0^t S(\tilde{\boldsymbol{\xi}}_{t,s}^{\nu_k,\kappa_k}(\boldsymbol{x}), s)\ \mathrm{d}s\right]$$
$$= \int_\Omega d^d x\ u(\boldsymbol{x})\ \int_0^t ds \int_0^t ds' \int_\Omega d^d y \int_\Omega d^d y'\ S(\boldsymbol{y}, s) S(\boldsymbol{y}', s')$$
$$\times\Big[p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) - p^*(\boldsymbol{y}, s|\boldsymbol{x}, t) p^*(\boldsymbol{y}', s'|\boldsymbol{x}, t)\Big]. \tag{4.73}$$

for all $u \in C(\Omega)$ and $S \in C(\Omega \times [0, t])$. If also $u \geq 0$, then the lefthand side is non-negative and thus so is the righthand side. This is enough to infer that

$$\int_0^t ds \int_0^t ds' \int_\Omega d^d y \int_\Omega d^d y'\ S(\boldsymbol{y}, s) S(\boldsymbol{y}', s')$$
$$\times\Big[p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) - p^*(\boldsymbol{y}, s|\boldsymbol{x}, t) p^*(\boldsymbol{y}', s'|\boldsymbol{x}, t)\Big] \geq 0 \tag{4.74}$$

for all $\boldsymbol{x} \in \Omega_0$, a set of full measure in Ω. The remainder of the argument uses the same strategy as before, with $\theta_0 \to S$ and the Banach space $C(\Omega^2) \to C((\Omega \times [0, t])^2)$.

The argument when both $\theta_0 \neq 0$ and $S \neq 0$ has already been given in the main text. We only add here the technical detail that a single subsequence may be selected so that one has has narrow convergence both of the 2-time Young measure

$$\mu_{s,s',\boldsymbol{x}}^{\nu_k,\kappa_k}(d\boldsymbol{y}, d\boldsymbol{y}') = d^d y\ d^d y'\ p_2^{\nu_k,\kappa_k}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) \to d^d y\ d^d y'\ p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) \tag{4.75}$$

and also of the 1-time Young measure at time $t_0 = 0$

$$\mu_{\boldsymbol{x}}^{\nu_k,\kappa_k}(d\boldsymbol{x}_0) = d^d x_0 \ p^{\nu_k,\kappa_k}(\boldsymbol{x}_0, 0|\boldsymbol{x}, t) \to d^d x_0 \ p^*(\boldsymbol{x}_0, 0|\boldsymbol{x}, t). \tag{4.76}$$

The second statement does *not* follow from the narrow convergence

$$\mu_{s,\boldsymbol{x}}^{\nu_k,\kappa_k}(d\boldsymbol{y}) = d^d y \ p^{\nu_k,\kappa_k}(\boldsymbol{y}, s|\boldsymbol{x}, t) \to d^d y \ p^*(\boldsymbol{y}, s|\boldsymbol{x}, t) \tag{4.77}$$

because $\{0\}$ is a subset of $[0, t]$ with zero Lebesgue measure. However, after extracting a subsequence for which the 2-time Young measure converges, one can extract a further subsequence so that the 1-time Young measure at time $t_0 = 0$ also converges.

Chapter 5

A Lagrangian Fluctuation-Dissipation Relation for Scalar Turbulence: Wall Bounded Domains

5.1 Introduction

This chapter is a continuation of the work of chapter 4, which initiated a new approach to the theory of turbulent scalar dissipation based upon a *Lagrangian fluctuation-dissipation relation (FDR)*. This relation is derived within a representation of the scalar field by stochastic fluid particles, which naturally extends Lagrangian methods for ideally advected scalars to realistic problems with both advection and diffusion. In chapter 4 (Drivas and Eyink [2016]; hereafter, I) the stochastic representation and FDR were obtained for flows in finite domains without any boundaries. Using the FDR, we resolved an on-going controversy regarding the phenomenon of *Lagrangian spontaneous stochasticity*, or the persistent stochasticity of Lagrangian trajectories and breakdown of uniqueness of particle trajectories for high Reynolds numbers. In ground-breaking work of Bernard et al. [1998] on the Kraichnan [1968] model of turbulent advection, the property of Lagrangian spontaneous stochasticity was first discovered and shown to be a consequence of the classical Richardson [1926] theory of turbulent particle dispersion. Bernard et al. [1998] furthermore showed that anomalous scalar dissipation within the Kraichnan model is due to spontaneous stochasticity. The validity of this concept for scalars advected by a true turbulent flow has, however, been strongly questioned (e.g. Tsinober [2009]). A key result of chapter 4was the demonstration that spontaneous stochasticity is the only possible mechanism of anomalous dissipation for both passive and active scalars with any advecting velocity field whatsoever, away from walls.

Wall-bounded flows are, on the other hand, ubiquitous both in engineering applications and in nature. Nearly every turbulent flow encountered in our daily experience involves solid walls and obstacles that are either stationary or moving, flexible membranes, or free surfaces. Geophysical turbulent flows are most commonly constrained by topography, basin boundaries, or surfaces separating multiple phases (e.g. airwater). These boundaries can profoundly effect the organization of the turbulence and driving from the boundary is very often the origin of the turbulence itself. Thus, any perspective on turbulent scalar dissipation with a claim of generality must be applicable to flows with boundaries. Our aim in this work is to extend the approach and results of chapter 4to the most canonical examples of wall-bounded flows. To be specific, we consider a scalar field θ (such as temperature, dye concentration, etc.) transported by a fluid with velocity $\boldsymbol{u}$, governed by the advection-diffusion equation in the interior of a finite domain Ω:

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \Delta \theta + S \tag{5.1}$$

with appropriate conditions specified on the boundary $\partial\Omega$. Here $\kappa > 0$ is the molecular diffusivity of the scalar and $S(\boldsymbol{x}, t)$ is a bulk source field. Typical situations are those in which scalar field is held fixed on the boundary (Dirichlet conditions), those in which the scalar flux through the wall is fixed (Neumann conditions), or a mixture of these along different parts of the boundary decomposed as $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$. More formally, for a function $\psi(\boldsymbol{x}, t)$ specifying scalar boundary values and a function $g(\boldsymbol{x}, t)$ specifying the fluxes through the wall, one solves Eq. (5.1) subject to the conditions

$$\text{Dirichlet:} \qquad \theta(\boldsymbol{x},t) = \psi(\boldsymbol{x},t), \quad \boldsymbol{x} \in \partial\Omega_D \tag{5.2}$$

$$\text{Neumann:} \qquad -\boldsymbol{\mu}(\boldsymbol{x},t)\cdot\nabla\theta(\boldsymbol{x},t) = g(\boldsymbol{x},t) \quad \boldsymbol{x} \in \partial\Omega_N \tag{5.3}$$

where $\boldsymbol{\mu} = \kappa\mathbf{n}$[1] with $\boldsymbol{n}(\boldsymbol{x})$ being the unit inward normal to the boundary $\partial\Omega$. Additionally, appropriate boundary conditions must be specified for the velocity field at the wall. Throughout this chapter, we consider the most common stick (or no-slip) boundary conditions:

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbf{0}, \quad \boldsymbol{x} \in \partial\Omega. \tag{5.4}$$

We shall further discuss in this chapter advection only by an incompressible fluid satisfying

$$\nabla\cdot\boldsymbol{u} = 0, \tag{5.5}$$

since only then does fluid advection conserve scalar integrals of the form $\int d^dx\, f(\theta(\boldsymbol{x},t))$. However, we place no other general constraints on the velocity $\boldsymbol{u}$, which may obey, for example, an equation of motion involving θ itself. Thus, our considerations apply not only to passive but also to active scalar fields.

For all wall-bounded flows of the above classes we derive Lagrangian fluctuation-

[1] All of our considerations in this chapter apply with straightforward modifications to the more general situation where the scalar flux is given by an anisotropic Fick's law as $\boldsymbol{j}(\boldsymbol{x},t) = -\mathbf{K}(\boldsymbol{x},t)\nabla\theta(\boldsymbol{x},t)$ for a positive-definite, symmetric diffusivity tensor $\mathbf{K}$. In that case, the component of the flux normal to the boundary is $j_n = \boldsymbol{n}\cdot\boldsymbol{j} = -\boldsymbol{\mu}\cdot\nabla\theta$ with the so-called "co-normal vector" $\boldsymbol{\mu} = \mathbf{K}\mathbf{n}$ satisfying $\boldsymbol{\mu}\cdot\boldsymbol{n} = \mathbf{n}^\top\mathbf{K}\mathbf{n} > 0$.

dissipation relations generalizing those in chapter 4 and we exploit them to precisely characterize for wall-bounded flows the connection between Lagrangian spontaneous stochasticity and anomalous scalar dissipation. To our knowledge, both these FDR's and the results derived for wall-bounded flows have never been discussed previously, as all related works (e.g. Buaria et al. [2016]) discuss only flows without boundaries. These FDR's are obtained, however, within known stochastic representations for solutions of the scalar advection equation, with boundary conditions either of Dirichlet, Neumann, or mixed form (e.g. see Soner [2007]). Our discussion of the stochastic representations themselves involves only very modest originality and involves mainly a convenient compilation of existing results from multiple sources. The key concepts from probability theory to derive these representations are the *boundary local-time density* and the *boundary hitting-time*, which are carefully described and illustrated with numerical results from a turbulent channel-flow simulation. The new FDR's derived within these representations express an exact balance between the time-averaged scalar dissipation and the input of stochastic scalar variance from the initial data, interior scalar sources, and boundary conditions as these are sampled by stochastic Lagrangian trajectories backward in time.

With no-flux (zero Neumann) boundary conditions for the scalar, we obtain results identical to those of Bernard et al. [1998] for domains without boundary: namely, spontaneous stochasticity is both necessary and sufficient for anomalous dissipation of a passive scalar, and necessary for anomalous dissipation of an active scalar. However, for general imposed fluxes (Neumann conditions), imposed scalar values at the wall (Dirichlet conditions) or mixed Dirichlet/Neumann conditions, the necessity statement is not generally true. Simple examples, such as pure thermal conduction with imposed heat fluxes at the walls, show that thin scalar boundary layers can in such flows provide an entirely distinct mechanism of non-vanishing scalar dissipation with a vanishing molecular diffusivity. Nevertheless, we obtain from our FDR's a lower bound on the passive scalar dissipation rate, which implies that spontaneous stochasticity is sufficient for anomalous dissipation. With some additional, physically plausible but unproven assumptions, we can extend this sufficiency statement also to active scalars. It thus remains true for all wall-bounded flows that spontaneous stochasticity is a viable source of anomalous scalar dissipation. This is the main theoretical result of the present chapter. The potential of our FDR's is not, however, exhausted by this single result and our chapter sets up the framework for more general applications. In the third submitted paper (Eyink and Drivas [2017a]; hereafter, III) we apply our FDR to turbulent Rayleigh-Bénard convection and obtain a novel Lagrangian formulation of the Nusselt-Rayleigh scaling law.

The detailed contents of the present chapter are as follows: in section 5.2, we derive the FDR in the case of wall-bounded flows with imposed scalar fluxes (§5.2.1) and we relate spontaneous stochasticity and anomalous dissipation (§5.2.3). The next section 5.3 discusses flows with imposed scalar values at the wall and mixed conditions, deriving first an FDR inequality (§5.3.1), using it to relate spontaneous stochasticity to anomalous dissipation (§5.3.3). The numerical results for turbulent channel-flow (§5.2.2,§5.3.2) serve not only to illustrate key probabilistic concepts for a fluid-mechanics audience but also investigate the joint effects of turbulence and walls on scalar mixing rates, which are important in our subsequent study of Rayleigh-Bénard convection (III). In the summary and discussion section 5.4 we discuss some future challenges.

5.2 Imposed Scalar Fluxes at the Wall

We consider in this section a scalar $\theta(\boldsymbol{x},t)$ with pure Neumann boundary conditions ($\partial\Omega_N = \partial\Omega$), satisfying:

$$\begin{aligned} \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta &= \kappa\Delta\theta + S \qquad && \boldsymbol{x}\in\Omega, \\ -\boldsymbol{\mu}(\boldsymbol{x},t)\cdot\nabla\theta(\boldsymbol{x},t) &= g(\boldsymbol{x},t) && \boldsymbol{x}\in\partial\Omega. \end{aligned} \tag{5.6}$$

for given initial data θ_0, a given flux function g, and co-normal vector field $\boldsymbol{\mu}$.

5.2.1 Stochastic Representation and Fluctuation-Dissipation Relation

The stochastic representation of solutions of pure Neumann problems has been discussed in many earlier publications, such as Freĭdlin [1985], Burdzy et al. [2004], Soner [2007] on fundamental mathematical theory and and Mil'shtein [1996], Keanini [2007], Słomiński [2013] for numerical methods. In order to impose Neumann conditions, essentially, one averages over stochastic trajectories which reflect off the boundary of the domain, contributing to the solution a correction related to their sojourn time on the walls. We briefly review the subject here. Our presentation is somewhat different than in any of the above references and, in particular, is based on backward stochastic integration theory. This framework yields, in our view, the simplest and most intuitive derivations. We also specify appropriate physical dimensions for all quantities, whereas the mathematical literature is generally negligent about physical dimensions and studies only the special value of the diffusivity $\kappa = 1/2$, in unspecified units. We have been careful to restore a general value of the diffusivity κ in all positions where it appears.

The fundamental notion of the representation is the *boundary local time density* $\tilde{\ell}_{t,s}(\boldsymbol{x})$ [Stroock and Varadhan, 1971, Lions and Sznitman, 1984, Burdzy et al., 2004], which, for a stochastic Lagrangian particle located at $\boldsymbol{x} \in \Omega$ at time t, is the time within the interval $[s, t]$ which is spent near the boundary $\partial\Omega$ per unit distance. Its physical units are thus (time)/(length) or 1/(velocity). Note that we consider here the backward-in-time process, whereas Burdzy et al. [2004] employ the forward-time process and time-reflection to derive an analogous representation. We thus take the boundary local time density to be a non-positive, non-increasing random process which decreases (backward in time) only when the stochastic particle is on the boundary. It is defined via the “Skorohod problem”, in conjunction with the (backward) stochastic flow $\tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,s}(\boldsymbol{x})$ with reflecting b.c. which satisfies

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) = \boldsymbol{u}^{\nu}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \mathrm{d}s + \sqrt{2\kappa}\ \hat{\mathrm{d}}\mathbf{W}_s - \boldsymbol{\mu}(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \hat{\mathrm{d}}\tilde{\ell}_{t,s}(\boldsymbol{x}) \tag{5.7}$$

with $\tilde{\boldsymbol{\xi}}_{t,t}(\boldsymbol{x}) = \boldsymbol{x}$ as usual. The boundary local time is then given by the formula

$$\tilde{\ell}_{t,s}(\boldsymbol{x}) = \int_t^s dr\ \delta(\mathrm{dist}(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}), \partial\Omega)) \equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^s dr\ \chi_{\partial\Omega_\varepsilon}(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x})), \qquad s < t, \tag{5.8}$$

in terms of an “ε-thickened boundary”

$$\partial\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega : \mathrm{dist}(\boldsymbol{x}, \partial\Omega) < \varepsilon\}. \tag{5.9}$$

See Burdzy et al. [2004], Theorem 2.6[2]. In (5.8) we denote by χ_A the characteristic function of a set defined as

$$\chi_A(\boldsymbol{x}) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{5.10}$$

Lions and Sznitman [1984] have proved existence and uniqueness of strong solutions to this “Skorohod problem” with Lipschitz velocity fields $\boldsymbol{u}$ and sufficient regular normal vectors $\mathbf{n}$ at a smooth boundary. Of some interest for the consideration of non-smooth

[2] Note that the eq.(2.7) of Burdzy et al. [2004] contains an additional factor of $1/2$, because their local time density is our $\kappa\tilde{\ell}_{t,s}(\boldsymbol{x})$ and they consider only the case $\kappa = 1/2$.

velocity fields that might appear in the limit $\nu \to 0$, Stroock and Varadhan [1971] obtain solutions when $\boldsymbol{u}$ is merely bounded, measurable and establish uniqueness in law. The spatial regularity in $\boldsymbol{x}$ of the processes $(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), \tilde{\ell}_{t,s}(\boldsymbol{x}))$ defined jointly as above is the subject of recent works on stochastic flows with reflecting b.c [Pilipenko, 2005, 2014, Burdzy, 2009]. Note that the stochastic particles described by $\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x})$ are reflected in the direction of $\boldsymbol{\mu}$ when they hit the wall, thus staying forever within Ω, and the flow preserves the volume within the domain when $\nabla \cdot \boldsymbol{u}^\nu = 0$, as assumed here. It follows formally from (5.8) that the boundary local-time density can be written as

$$\tilde{\ell}_{t,s}(\boldsymbol{x}) = \int_t^s dr \int_{\partial\Omega} dH^{d-1}(\boldsymbol{z})\delta^d(\boldsymbol{z} - \tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x})), \tag{5.11}$$

where H^{d-1} is the $(d-1)$-dimensional Hausdorff measure (surface area) over the smooth boundary $\partial\Omega$ of the domain. We have not found this intuitive formula anywhere in the probability theory literature, but it should be possible to prove rigorously with suitable smoothness assumptions on the boundary.

By means of the above notions one can obtain a stochastic representation for solutions to the initial-value problem of the system (5.6). The backward Itō formula [Friedman, 2006, Kunita, 1997] applied to $\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)$ gives

$$\begin{aligned} \hat{\mathrm{d}}\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) &= \Big[(\partial_s\theta + \boldsymbol{u}\cdot\nabla\theta - \kappa\Delta\theta)\mathrm{d}s - (\nabla\theta\cdot\boldsymbol{\mu})\hat{\mathrm{d}}\tilde{\ell}_{t,s} + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s\cdot\nabla\theta\Big]_{(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)} \\ &= S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\mathrm{d}s + g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\hat{\mathrm{d}}\tilde{\ell}_{t,s} + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s\cdot\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s) \quad (5.12) \end{aligned}$$

where in the second line we have used the fact that $\theta(\boldsymbol{x},s)$ solves the boundary-value problem (5.6). We find upon integration from 0 to t,

$$\begin{aligned}\theta(\boldsymbol{x},t) &= \theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\ \hat{\mathrm{d}}\tilde{\ell}_{t,s} \\ &\quad + \sqrt{2\kappa}\int_0^t \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) \end{aligned} \tag{5.13}$$

and thus expectation over the Brownian motion yields the representation

$$\theta(\boldsymbol{x},t) = \mathbb{E}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\ \hat{\mathrm{d}}\tilde{\ell}_{t,s}\right]. \tag{5.14}$$

By means of (5.11), this formula can be written equivalently in terms of the transition probability density function $p^{\nu,\kappa}(\boldsymbol{y},s|\boldsymbol{x},t) = \mathbb{E}\left[\delta^d\left(\tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,s}(\boldsymbol{x}) - \boldsymbol{y}\right)\right]$ for the reflected particle to be at position $\boldsymbol{y}$ at time $s<t$, given that it was at position $\boldsymbol{x}$ at time t:

$$\begin{aligned}\theta(\boldsymbol{x},t) &= \int_\Omega d^d\boldsymbol{x}_0\ \theta_0(\boldsymbol{x}_0)\ p(\boldsymbol{x}_0,0|\boldsymbol{x},t) + \int_0^t ds \int_\Omega d^d\boldsymbol{y}\ S(\boldsymbol{y},s)\ p(\boldsymbol{y},s|\boldsymbol{x},t) \\ &\quad + \int_0^t ds \int_{\partial\Omega} dH^{d-1}(\boldsymbol{z})\ g(\boldsymbol{z},s)\ p(\boldsymbol{z},s|\boldsymbol{x},t). \end{aligned} \tag{5.15}$$

Clearly, the scalar value $\theta(\boldsymbol{x},t)$ is an average of the randomly sampled initial data and the scalar inputs from the internal source and the scalar flux through the boundary along a random history backward in time.

Just as for domains without walls in chapter 4, we introduce a stochastic scalar field which represents the contribution for a specific stochastic Lagrangian trajectory

$$\tilde{\theta}(\boldsymbol{x},t) \equiv \theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x})) + \int_0^t \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\hat{\mathrm{d}}\tilde{\ell}_{t,s}, \tag{5.16}$$

that now includes the boundary-flux term, so that $\theta(\boldsymbol{x},t) = \mathbb{E}\left[\tilde{\theta}(\boldsymbol{x},t)\right]$[3]. As in chapter 4, we warn the reader that the quantity $\tilde{\theta}(\boldsymbol{x},t)$ is entirely different from the conventional "turbulent" scalar fluctuation $\theta'(\boldsymbol{x},t)$ defined with respect to ensembles of scalar initial conditions, advecting velocities, or random sources. An application of the Itō-isometry (see Oksendal [2013], section 3.1) exactly as in chapter 4yields for the scalar variance

$$\frac{1}{2}\mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right] = \kappa\ \mathbb{E}\int_0^t \mathrm{d}s\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)|^2, \tag{5.17}$$

which is a local version of our FDR. Integrating over the bounded domain and invoking volume preservation by the reflected flow, we obtain:

$$\frac{1}{2}\left\langle \mathrm{Var}\ \tilde{\theta}(t)\right\rangle_\Omega = \kappa\int_0^t \mathrm{d}s\left\langle|\nabla\theta(s)|^2\right\rangle_\Omega \tag{5.18}$$

This, together with (5.16), is our exact FDR for scalars (either passive or active)

[3]For all $s<t$ the quantity $\tilde{\theta}(\boldsymbol{x},t;s) = \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s) + \int_s^t S(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}),r)\ \mathrm{d}r + \int_s^t g(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}),r)\ \hat{\mathrm{d}}\tilde{\ell}_{t,r}$ is equal to $\theta(\boldsymbol{x},t) - \sqrt{2\kappa}\int_s^t \hat{\mathrm{d}}\mathbf{W}_r\cdot\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,r}(\boldsymbol{x}),r)$, analogous to (5.13). It is thus a martingale back- ward in time, i.e. it is statistically conserved and, on average, equals $\theta(\boldsymbol{x},t)$ for all $s<t$. Clearly, $\tilde{\theta}(\boldsymbol{x},t) = \tilde{\theta}(\boldsymbol{x},t;0)$. This backward martingale property shows that the stochastic representation employed here is a natural generalization to diffusive advection of the standard deterministic Lagrangian representation for non-diffusive, smooth advection.

with general Neumann boundary conditions. Just as for eq.(2.19) of chapter 4 in the case of domains without walls, we obtain for the infinite-time limit of the local scalar variance

$$\lim_{t\to\infty} \frac{1}{2t} \mathrm{Var}\left[\tilde{\theta}(\boldsymbol{x},t)\right] = \left\langle \kappa|\nabla\theta|^2\right\rangle_{\Omega,\infty}, \qquad \text{for all } \boldsymbol{x}\in\Omega, \tag{5.19}$$

when the reflected stochastic particle wanders ergodically over the flow domain Ω. This will generally be true when $\kappa > 0$, and the ergodic average thus coincides with the average over the stationary distribution of the particle. This is uniform (Lebesgue) measure because of incompressibility of the flow. The long-time limit is thus independent of the space point $\boldsymbol{x}$.

5.2.2 Numerical Results

To provide some additional insight into these concepts, we presented in Appendix A of Drivas and Eyink [2017b] an elementary analytical example: pure diffusion on a finite interval with a constant scalar flux imposed at the two ends. We present now some numerical results on stochastic Lagrangian trajectories and boundary local time densities for a proto-typical wall-bounded flow, turbulent channel flow. These numerical results are directly relevant to turbulent thermal transport in a channel with imposed heat fluxes at the walls. They also have relevance to the problem of Rayleigh-Bénard convection, since channel flow provides the simplest example of a turbulent shear flow with a logarithmic law-of-the-wall of the type conjectured to exist at very high Rayleigh numbers in turbulent convection [Kraichnan, 1962]. For the purpose of our study, we use the $Re_\tau = 1000$ channel-flow dataset in the Johns Hopkins Turbulence Database (see Graham et al. [2016] and `http://turbulence.pha.jhu.edu`). We follow the notations in these references and, in particular, the x direction is streamwise, y is wall-normal, and z is spanwise. For the details of our numerical methods, see Appendix D of Drivas and Eyink [2017b]. We plot in Figure 5.1 for three choices of Prandtl number (Pr =0.1, 1, 10) a single realization of $(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), \tilde{\ell}_{t,s}(\boldsymbol{x}))$ for a stochastic particle released at a point $\boldsymbol{x}$ on the lower wall $y = -h$ at the final time t_f in the database, then evolved backward in time. Using Cartesian coordinates $\tilde{\boldsymbol{\xi}}_{t,s} = (\tilde{\xi}_{t,s}, \tilde{\eta}_{t,s}, \tilde{\zeta}_{t,s})$, the left panel of Fig. 5.1 plots the wall-normal particle position $\tilde{\eta}_{t,s}(\boldsymbol{x})$ as height above the wall $\Delta y = y + h$ and the right panel plots the local time density $\tilde{\ell}_{t,s}(\boldsymbol{x})$, both as functions of shifted time $s - t_f$. All quantities are expressed in wall units, in terms of the friction velocity u_τ, the viscous length $\delta_\tau = \nu/u_\tau$, and the viscous time $\tau_\nu = \nu/u_\tau^2$. Important features to observe in Fig. 5.1 are the jumps in the local time density at the instants when the particles are incident upon the wall and reflected. These incidences occur predominantly near the release time and eventually cease (backward in time) as the particles are transported away from the wall. This escape from the wall occurs slower for smaller κ or larger Pr, and the local times at the wall are correspondingly larger magnitude for larger Pr.

We next discuss statistics of the entire ensemble of stochastic particles for two

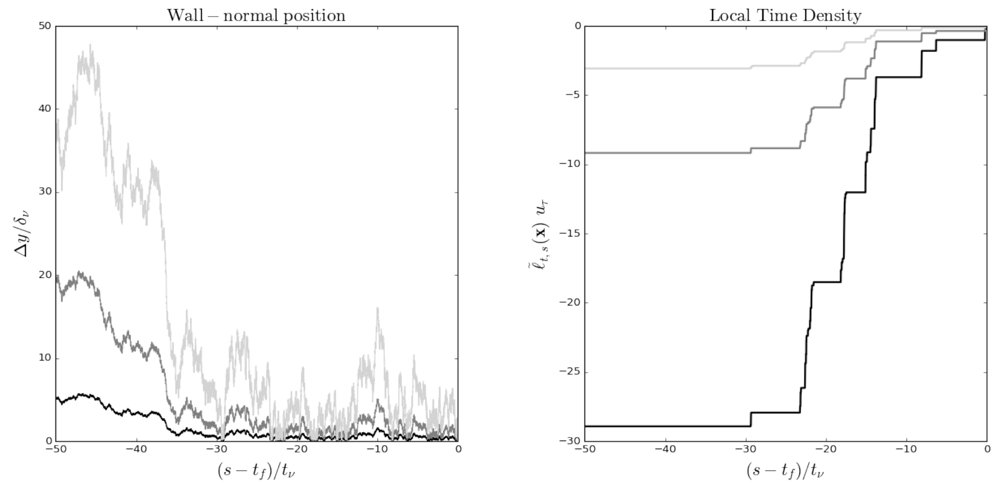

Figure 5.1: Realizations of wall-normal position (left panel) and local time density process (right panel) for a single particle released on the lower wall at $(x, z) = (3543.84\delta_\nu, 1891.56\delta_\nu)$ for Prandtl values $Pr = 0.1$ (light grey), 1.0 (grey), and 10 (black).

specific release points at the wall, one point selected inside a low-speed streak and the other a high-speed streak. We are especially interested here in the role of turbulence in enhancing heat transport away from the wall, compared with pure thermal diffusion. It has sometimes between questioned whether turbulence close to the wall indeed increases thermal transport, because of the restrictions imposed on the vertical motions (e.g. Niemela and Sreenivasan [2003], section 7.2). We thus select two points, one in a low-speed streak with mean motion toward the wall (backward in time) and the other in a high-speed streak with mean motion away from the wall (backward in time). Comparing these two points helps to identify any possible strictures on turbulent transport imposed by the solid wall. The selection is illustrated in Figure 5.2, which plots the streamwise velocity in the buffer layer at distance $\Delta y = 10\delta_\tau$

above the bottom channel wall. The low-speed streaks associated with "ejections" from the wall and high-speed streaks associated with "sweeps" toward the wall seen there are characteristic of turbulent boundary layers [Kline et al., 1967]. The x-z coordinates of the two release points are indicated by the diamonds ($\diamond$) in Fig. 5.2 (and note that the point selected in the low-speed streak is that featured in the previous Fig. 5.1). For each of these two release points and for the three choices of Prandtl number Pr =0.1, 1, 10, we used $N = 1024$ independent solutions of Eqs.(5.7),(5.8) to calculate the dispersion of the stochastic trajectories and the PDF's of the local-time densities backward in time. Error bars in the plots represent s.e.m. for the N-sample averages and in addition, for the PDF's, the effects of variation in kernel density bandwidth.

In Figure 5.3, the top panels show 30 representative particle trajectories for the two release points (left within a high-speed streak, right in a low-speed) and for each of the three Prandtl numbers. By eye, the ensembles of trajectories for the three different Prandtl numbers appear quite similar. In the middle panels of Fig. 5.3, we plot the mean-squared dispersion of the stochastic particles for the two release points and the three Prandtl numbers. There is an initial period (backward in time) where the dispersion grows with $\hat{s} = t_f - s$ as $4\kappa(3 - 2/\pi)\hat{s}$, which is the analytical result for a 3D Brownian motion reflected from a plane[4]. There is then a crossover to a limited regime of super-ballistic separation that is close to $\hat{s}^3$-growth. At still larger

[4]If $\tilde{W}(t)$ is standard 1D Brownian motion with diffusivity κ, then $|\tilde{W}(t)|$ is the Brownian motion reflected at 0 and it is elementary to show that $\mathbb{E}^{1,2}\left[\{|\tilde{W}^{(1)}(t)| - |\tilde{W}^{(2)}(t)|\}^2\right] = 4\kappa(1 - 2/\pi)t$.

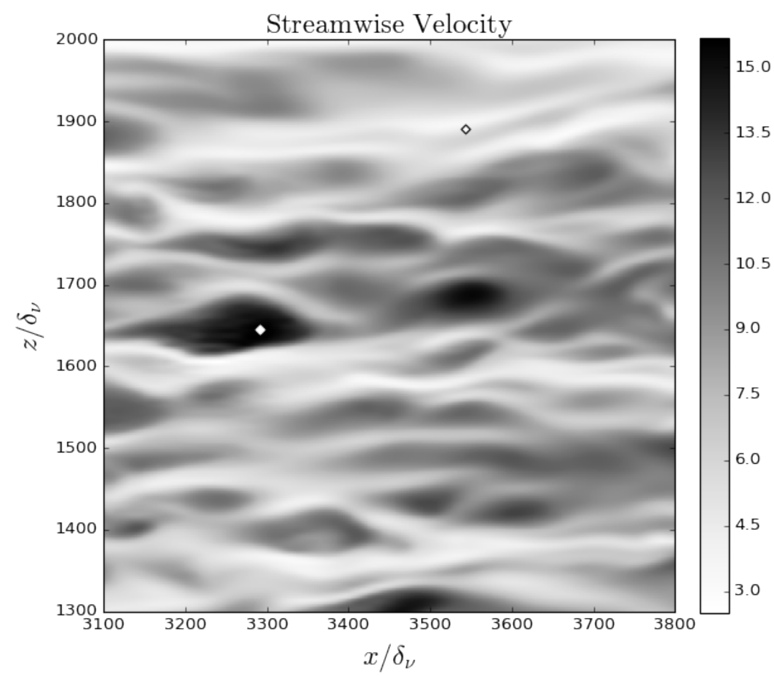

Figure 5.2: Streamwise velocity at normal distance $\Delta y = 10\delta_\nu$ from the lower wall, non-dimensionalized by the friction velocity u_τ. The open diamonds ($\diamond$) mark the (x, z) coordinates, $(3291.73\delta_\nu, 1644.98\delta_\nu)$ and $(3543.84\delta_\nu, 1891.56\delta_\nu)$, of the selected release points at wall-parallel positions of high-speed (dark) and low-speed (light) streaks, respectively.

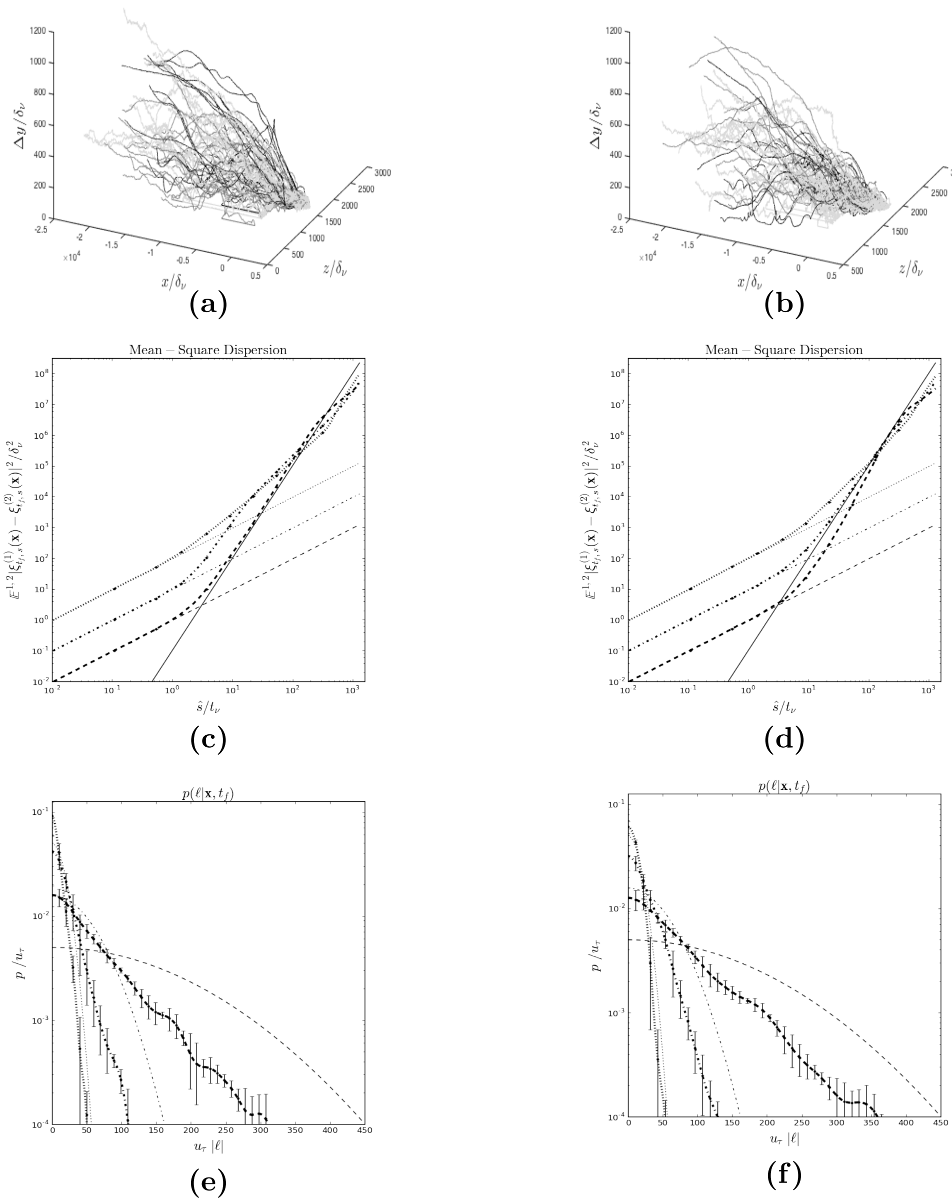

Figure 5.3: *Left panels* are for release at bottom wall near a high-speed streak, *right panels* for release near a low-speed streak. See markers on Figure 5.2. Top panels (a),(b) show 30 representative reflected stochastic trajectories for $Pr = 0.1$ (light grey), 1.0 (grey) and 10 (black). Middle panels (c),(d) plot particle dispersion (heavy) and short-time results $4\kappa(3 - \frac{2}{\pi})\hat{s}$ (light) for each Pr with $Pr = 0.1$ (dot, $\cdots$), 1.0 (dash-dot, $\cdot$-$\cdot$) and 10 (dash, - - -) and a plot in (solid, —) of $g(\nu/t_\nu^2)\hat{s}^3$ with $g = 0.1$. The bottom panels (e),(f) plot PDF's of (absolute values of) boundary local times for channel flow (heavy) and pure diffusion (light) for the three Pr-values with the same line-styles as (c),(d).

$\hat{s}$, the growth clearly slows but remains super-ballistic. Recall from chapter 4that Richardson dispersion with a cubic growth of dispersion is the physical mechanism of spontaneous stochasticity in homogeneous, isotropic turbulence. It would be quite surprising to observe Richardson dispersion in this channel-flow dataset, since the energy spectra published in Graham et al. [2016] and `http://turbulence.pha.jhu.edu` show that the Reynolds number is still too low to support a Kolmogorov-type inertial-range. We believe that the cubic power-law growth observed arises instead from a simple combination of stochastic diffusion and mean shear, a molecular version of the turbulent shear dispersion discussed by Corrsin [1953]. A toy model which illustrates this mechanism is a 2D shear flow with $\boldsymbol{u}(\boldsymbol{x}) = (Sy, 0)$, where

$$d\tilde{\xi}_t = S\tilde{\eta}_t + \sqrt{2\kappa}\, d\tilde{W}_x(t), \quad d\tilde{\eta}_t = \sqrt{2\kappa}\, d\tilde{W}_y(t). \tag{5.20}$$

Integration gives $\tilde{\xi}_t = \sqrt{2\kappa}(S\int_0^t \tilde{W}_y(s)ds + \tilde{W}_x(t))$ and $\tilde{\eta}_t = \sqrt{2\kappa}\, \tilde{W}_y(t)$ when $(\tilde{\xi}_0, \tilde{\eta}_0) = (0, 0)$, and the shear contributes a t^3-term to the x-component of the dispersion:

$$\mathbb{E}^{1,2}|\tilde{\xi}_t^{(1)} - \tilde{\xi}_t^{(2)}|^2 = 4\kappa(t + \frac{1}{3}S^2t^3). \tag{5.21}$$

See Schütz and Bodenschatz [2016] for a very similar discussion of cubic-in-time dispersion in the context of bounded shear flows and weakly turbulent Rayleigh-Bénard convection in 2D. In the toy model (5.20) the y-component of dispersion grows only diffusively. In qualitative agreement with this simple model, we have found that the cubic power-law growth in the channel-flow particle dispersion indeed arises solely from the streamwise component of the particle separation vector. We do not show this here, but the effect can be observed in the particle trajectories plotted in the top panels of Fig. 5.3, where the particles are dispersed farthest along the streamwise direction. The t^3 dispersion in (5.21) differs notably from Richardson dispersion in that it is proportional to κ and it produces no spontaneous stochasticity, since it vanishes as $\kappa \to 0$. In the channel-flow $\hat{s}^3$-regime in Fig. 5.3c,d ($10^1 \lesssim \hat{s}/t_\eta \lesssim 10^2$) there is likewise an observable dependence on the diffusivity. These cautionary results are an admonition against interpreting any cubic or super-ballistic growth of pair-separation whatsoever as evidence of spontaneous stochasticity. Richardson dispersion produces spontaneous stochasticity because of non-smooth relative advection, not simply because of super-ballistic separation.

In the bottom panels of Figure 5.3, we plot PDF's of the local time density $\tilde{\ell}_{t_f,0}(\boldsymbol{x})$ accumulated backward over the entire time interval of the channel-flow database (one flow-through time), for the two release points and the three values of Prandtl number. As could be expected, smaller κ (or larger Pr) correspond to more time at the wall and a PDF supported on larger local time densities. Comparing the results for the two release points, it can be seen that particles starting on the wall inside the high-speed streak tend to spend somewhat less time near the wall than those starting near the low-speed streak. This is easy to understand, if one recalls that low-speed streaks correspond to "ejections" from the wall and high-speed streaks to "sweeps" toward the wall. Backward in time, however, the fluid motions are exactly reversed. Thus, particles starting at the wall inside high-speed streaks are being swept away from the wall by the fluid motion backward in time, and those starting inside low-speed streaks are brought toward the wall. However, the location of the release point has a quite small effect on the PDF of the local-time density accumulated over one flow-through time, and it is plausible that this effect would be reduced by taking t_f even larger.

Finally, we have also plotted in the bottom panels of Fig. 5.3 the analytical results for the local time PDF's of a pure Brownian motion with diffusivity κ that is reflected from a planar wall (see Appendix A of Drivas and Eyink [2017b] and (5.27) below). Compared with these results for Brownian motion, the corresponding channel-flow local-time PDF's exhibit in all cases a substantial reduction of time spent at the wall. These results show that turbulent fluctuations enhance transport away from the wall, even when the local flow initially carries the particles toward the wall (backward in time). Although we show these results only for two release points here, we have observed similar behavior at all other locations on the wall that we have examined. This observation has importance for the problem of turbulent convection, where it was it was argued by Kraichnan [1962] that "...the effect of the small-scale turbulence that arises locally in the shear boundary layer should be to increase heat transport."

5.2.3 Spontaneous Stochasticity and Anomalous Dissipation

As an application of our FDR (5.18) for flows with imposed scalar fluxes at the walls, we now discuss the equivalence between spontaneous stochasticity and anomalous scalar dissipation in this context. As we shall see, the effects of the walls can be profound.

The simplest situation is the case of vanishing wall-fluxes, $g \equiv 0$, which corresponds to *insulating/adiabatic walls* when the scalar is the temperature and to *impermeable walls* when the scalar is the concentration of an advected substance (e.g. a dye, aerosol, etc.). In this case, the flux contribution proportional to the local time density in (5.16) vanishes, and our FDR then becomes simply

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}) + \int_0^t \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}, s)\right]\right\rangle_\Omega = \kappa \int_0^t \mathrm{d}s \left\langle |\nabla\theta(s)|^2 \right\rangle_\Omega . \tag{5.22}$$

This is formally identical to the relation (2.9),(2.19) of Paper I for flows in domains without boundaries, with the simple stochastic flow replaced by a reflected stochastic flow. We can therefore repeat *verbatim* the arguments from chapter 4, section 4 to conclude that, for walls that do not support fluxes, spontaneous stochasticity is equivalent to anomalous dissipation for passive scalars and for active scalars spontaneous stochasticity is (at least) necessary for anomalous scalar dissipation. This result is relevant for many physical situations, such as a dye injected into a turbulent

Taylor-Couette flow. The commonplace example of cream stirred into coffee is also essentially of this type, since the cup walls and surface of the stirrer are impermeable boundaries and there is also no transport of cream across the free fluid surface. The extension of our FDR (5.22) to problems with moving walls and free-fluid surfaces should be relatively straightforward (and, indeed, the main result of the paper of Burdzy et al. [2004] was the construction of reflected Brownian processes for domains enclosed by moving boundaries). In all of these situations, any evidence for anomalous scalar dissipation is also evidence for (requires) spontaneous stochasticity.

The situations with non-vanishing fluxes through the walls are, however, essentially different. From a mathematical point of view, the problem is "loss of compactness." While the trajectories of the reflected diffusion process satisfy uniformly in ν,κ the condition that $\tilde{\boldsymbol{\xi}}_{t,s}^{\nu,\kappa}(\boldsymbol{x})\in\bar{\Omega}$, a closed, bounded domain, the local time densities $\tilde{\ell}_{t,s}^{\nu,\kappa}(\boldsymbol{x})$ may become unboundedly large as $\nu,\kappa\to 0$. This creates a fundamental difficulty for arguments of the type employed previously, where limits as $\nu,\kappa\to 0$ were guaranteed to exist (along subsequences). To understand better the problem, consider how we might try to adapt those arguments to the present context. We can rewrite our FDR (5.18),(5.16) as

$$\begin{aligned}
&\kappa\int_0^t \mathrm{d}s\,\left\langle|\nabla\theta(s)|^2\right\rangle_\Omega=\frac{1}{2}\left\langle\mathrm{Var}\left[\tilde{\theta}^{\nu,\kappa}(t)\right]\right\rangle_\Omega\\
&\quad=\frac{1}{2}\left\langle\int d\psi\int d\psi'\ \psi\psi'\ \left[p_2^{\nu,\kappa}(\psi,\psi'|\cdot,t)-p^{\nu,\kappa}(\psi|\cdot,t)p^{\nu,\kappa}(\psi|\cdot,t)\right]\right\rangle_\Omega, \quad (5.23)
\end{aligned}$$

where

$$p^{\nu,\kappa}(\psi|\boldsymbol{x},t) = \mathbb{E}\Big[\delta(\psi - \tilde{\theta}^{\nu,\kappa}(\boldsymbol{x},t))\Big] \tag{5.24}$$

and

$$p_2^{\nu,\kappa}(\psi,\psi'|\boldsymbol{x},t) = \mathbb{E}\Big[\delta(\psi - \tilde{\theta}^{\nu,\kappa}(\boldsymbol{x},t))\delta(\psi' - \tilde{\theta}^{\nu,\kappa}(\boldsymbol{x},t)\Big] = \delta(\psi-\psi')p^{\nu,\kappa}(\psi|\boldsymbol{x},t), \tag{5.25}$$

with $\tilde{\theta}^{\nu,\kappa}(\boldsymbol{x},t)$ given by (5.16). If weak limits $p^*(\psi|\boldsymbol{x},t) = w\text{-}\lim_{\nu,\kappa\to 0} p^{\nu,\kappa}(\psi|\boldsymbol{x},t)$ existed, then arguments of exactly the type given before would show that anomalous dissipation requires non-factorization of p_2^* into $p^* \cdot p^*$ and, hence, spontaneous stochasticity. The reverse argument that spontaneous stochasticity implies anomalous dissipation for passive scalars would also be essentially the same.

Simple examples show, however, that weak limits $p^*(\psi|\boldsymbol{x},t) = w\text{-}\lim_{\nu,\kappa\to 0} p^{\nu,\kappa}(\psi|\boldsymbol{x},t)$ may not exist when wall-fluxes are not vanishing. Consider the case where $\theta_0 = S \equiv 0$ and the flux into the domain is a space-time constant $g(\boldsymbol{x},s) = J > 0$. In that case

$$\tilde{\theta}(\boldsymbol{x},t) = \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\hat{\mathrm{d}}\tilde{\ell}_{t,s}(\boldsymbol{x}) = -J\tilde{\ell}_{t,0}(\boldsymbol{x}) \geq 0, \tag{5.26}$$

and the stochastic scalar field for one realization of the Brownian process is proportional by a constant to the local time density itself. Consider furthermore the simple case of pure heat-conduction, where the advecting velocity field also vanishes, $\boldsymbol{u}^\nu \equiv \boldsymbol{0}$. In this case, the distribution of $\tilde{\ell}_{t,0}(\boldsymbol{x})$ is known analytically in many cases.

A very simple example which serves to make our point takes the domain to be the semi-infinite one-dimensional interval $\Omega = [0, \infty)$ with boundary at 0. In that case, for $x \geq 0$

$$p^\kappa(\psi|x,t) = \frac{1}{J}\sqrt{\frac{\kappa}{\pi t}} \exp\left[-\frac{(x+\kappa\psi/J)^2}{4\kappa t}\right] \eta(\psi) + \left[2\Phi_{\kappa,t}(x) - 1\right] \delta(\psi) \tag{5.27}$$

where $\eta(\psi)$ is the Heaviside step function and

$$\Phi_{\kappa,t}(x) = \frac{1}{\sqrt{4\kappa\pi t}} \int_{-\infty}^{x} dy \ \exp(-y^2/4\kappa t). \tag{5.28}$$

For details see Appendix A of Drivas and Eyink [2017b]. It is easy to see mathematically from the above expression that weak limits exist for fixed $x > 0$ and, indeed,

$$w\text{-}\lim_{\kappa \to 0} p^\kappa(\psi|x,t) = \delta(\psi), \quad x > 0. \tag{5.29}$$

This is also intuitively obvious, because a stochastic particle released at $x > 0$ never makes it to the boundary at 0 when $\kappa \to 0$. However, *the distribution $p^\kappa(\psi|0,t)$ does not converge weakly as $\kappa \to 0$*, but (5.27) implies that it instead tends to become uniformly spread on the semi-infinite interval. This also makes sense, because a particle released at 0 should tend to stay there as $\kappa \to 0$ and the local time density at 0 will diverge[5].

[5] In fact, if we consider the 1-point compactification $[0, \infty]$ of the range $[0, \infty)$ of possible scalar values, then $\lim_{\kappa \to 0} p^\kappa(\psi|0,t) = \delta_\infty(\psi)$, the delta-function at infinity. However, this compactification of the problem does not yield any result on spontaneous stochasticity.

The physical origin of the divergence in this simple example is a *scalar boundary layer* near $x = 0$. A direct solution of the Neumann problem for the scalar field or integration over the above distribution shows that

$$\theta(x,t) = -J\mathbb{E}[\tilde{\ell}_{t,0}(x)] \sim J\sqrt{\frac{t}{\kappa}} f\left(\frac{x}{\sqrt{\kappa t}}\right) \tag{5.30}$$

for a suitable scaling function f. See Appendix A of Drivas and Eyink [2017b]. There is thus a scalar boundary layer of thickness $\sim \sqrt{\kappa t}$ near 0 where the scalar field diverges as $\theta \sim J\sqrt{t/\kappa}$ as $\kappa \to 0$. Because there is a constant flux J into the domain and diffusive transport into the interior vanishes as $\kappa \to 0$, there is a "pile-up" of the scalar near $x = 0$. The scalar dissipation field due to this boundary layer is

$$\varepsilon_\theta(x,t) = \kappa|\partial_x \theta(x,t)|^2 \sim \frac{J^2}{\kappa}\left| f'\left(\frac{x}{\sqrt{\kappa t}}\right)\right|^2, \tag{5.31}$$

from which one can infer a total scalar dissipation $\propto J^2\sqrt{t/\kappa} \to \infty$ as $\kappa \to 0$. This "dissipative anomaly" occurs even though there is clearly no spontaneous stochasticity in this simple example with $\boldsymbol{u}^\nu \equiv 0$!

The above example shows that the "loss of compactness" is not a mere technical mathematical difficulty, but instead that there may no longer be equivalence of spontaneous stochasticity and non-vanishing dissipation. Scalar boundary layers in wall-bounded flow domains with flux through the walls are a new possible source of scalar dissipation that can be non-zero (or even diverging) as $\nu, \kappa \to 0$, quite distinct from the spontaneous stochasticity mechanism. Another perhaps more elementary way to see the problem posed by wall-fluxes is to exploit formula (5.11) for the boundary local-time to write

$$\mathrm{Var}\left[\int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}),s)\ \hat{\mathrm{d}}\tilde{\ell}_{t,s}(\boldsymbol{x})\right] = \int_0^t ds \int_0^t ds' \int dH^{d-1}(z) \int dH^{d-1}(z')\ g(\boldsymbol{z},s)g(\boldsymbol{z}',s') \\ \times\left[p_2^{\nu,\kappa}(\boldsymbol{z},s;\boldsymbol{z}',s'|\boldsymbol{x},t) - p^{\nu,\kappa}(\boldsymbol{z},s|\boldsymbol{x},t)p^{\nu,\kappa}(\boldsymbol{z}',s'|\boldsymbol{x},t)\right], \tag{5.32}$$

where $p_2^{\nu,\kappa}(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)$ for $s,s' < t$ is the 2-time transition probability density function with respect to Lebesgue measure, as in eq.(4.1) of chapter 4. On the face of it, this appears to be quite similar to the analogous formulas for the scalar variance associated to the initial data or internal sources, e.g.

$$\mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x}))\right] = \int d^dx_0 \int d^dx_0'\ \theta_0(\boldsymbol{x}_0)\theta_0(\boldsymbol{x}_0') \\ \times\left[p_2^{\nu,\kappa}(\boldsymbol{x}_0,0;\boldsymbol{x}_0',0|\boldsymbol{x},t) - p^{\nu,\kappa}(\boldsymbol{x}_0,0|\boldsymbol{x},t)p^{\nu,\kappa}(\boldsymbol{x}_0',0|\boldsymbol{x},t)\right]. \tag{5.33}$$

However, in the variance associated to θ_0 or S, the transition probability densities appear only in the combinations $d^dy\ d^dy'\ p_2^{\nu,\kappa}(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)$ and $d^dy\ p^{\nu,\kappa}(\boldsymbol{y},s|\boldsymbol{x},t)$, and thus only involve the *probability measures* themselves and not their densities. Compactness arguments suffice to show that limits of the probability measures and associated integrals exist (along subsequences) as $\nu,\kappa \to 0$. On the other hand, the formula (5.32) involves the probability density function evaluated on the boundary of the domain. Even if the (weak) limit of the particle probability measures have density functions $p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t)$ and $p^*(\boldsymbol{y}, s|\boldsymbol{x}, t)$, they are undefined at the boundary $\partial\Omega$, which is a Lebesgue zero-measure set. In general, the probability density functions will not have pointwise limits as $\nu, \kappa \to 0$ and, in the example of pure diffusion on the half-line discussed above, they diverge when all three points $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{y}'$ are located on the left boundary at 0!

On the other hand, we can extend all of our results relating spontaneous stochasticity and anomalous scalar dissipation to general choices of wall-fluxes, if we make the additional assumption that pointwise limits of densities exist for all $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{y}' \in \Omega$

$$\lim_{\nu,\kappa\to 0} p_2^{\nu,\kappa}(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) = p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t), \quad \lim_{\nu,\kappa\to 0} p^{\nu,\kappa}(\boldsymbol{y}, s|\boldsymbol{x}, t) = p^*(\boldsymbol{y}, s|\boldsymbol{x}, t), \tag{5.34}$$

so that the limit of the variance in (5.32) exists and is given by the formula

$$\lim_{\nu,\kappa\to 0} \mathrm{Var}\left[\int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\ \hat{\mathrm{d}}\tilde{\ell}_{t,s}(\boldsymbol{x})\right] = \int_0^t ds \int_0^t ds' \int dH^{d-1}(z) \int dH^{d-1}(z')$$
$$\times g(\boldsymbol{z}, s) g(\boldsymbol{z}', s')\ \Big[p_2^*(\boldsymbol{z}, s; \boldsymbol{z}', s'|\boldsymbol{x}, t) - p^*(\boldsymbol{z}, s|\boldsymbol{x}, t) p^*(\boldsymbol{z}', s'|\boldsymbol{x}, t)\Big]. \tag{5.35}$$

This assumption, of course, rules out the presence of scalar boundary layers of the type discussed above. The necessity of spontaneous stochasticity for anomalous scalar dissipation is immediate, because factorization $p_2^*(\boldsymbol{y}, s; \boldsymbol{y}', s'|\boldsymbol{x}, t) = p^*(\boldsymbol{y}, s|\boldsymbol{x}, t) p^*(\boldsymbol{y}', s'|\boldsymbol{x}, t)$ of the limit densities implies that all variances tend to zero. If the scalar is passive, then sufficiency holds as well. Consider, for simplicity, the case of vanishing internal source $S \equiv 0$. As in section 4 and Appendix A of chapter 4 we can make a smooth choice of scalar initial data θ_0 so that the volume-average of the limiting variance in (5.33) is strictly positive, whenever non-factorization holds for a positive measure set of $\boldsymbol{x} \in \Omega$. The limit variance in (5.35) involving the boundary flux g is non-negative when it exists at all. If the covariance term involving both θ_0 and g is negative, then by taking $\theta_0 \to -\theta_0$ the covariance can be made positive without changing the sign of either of the two variances involving θ_0 and g separately. In this manner, for any possible choice of wall-fluxes g, an initial condition θ_0 for the passive scalar advection equation always exists so that scalar dissipation is non-vanishing as $\nu, \kappa \to 0$. The same conclusion holds also for an active scalar, barring a "conspiracy" in which all Lagrangian particles at time 0 originating from a.e. point $\boldsymbol{x} \in \Omega$ at time t are confined to a single isoscalar surface of θ_0 for that $\boldsymbol{x}$, even when varying over all possible θ_0. It is this unlikely coincidence which must be ruled out by a rigorous proof. In this manner we see that all of our previous conclusions for flows without walls or for zero fluxes at the walls, can be extended to the case of general wall-fluxes g, whenever the stringent assumption (5.34) on pointwise limits is valid.

5.3 Imposed Scalar Values at the Wall

In this section we extend our previous results as far as possible to advection-diffusion of scalars in wall-bounded domains with the general mixed Dirichlet-Neumann conditions (5.2),(5.3). It turns out that imposing scalar values (Dirichlet conditions) on even part of the boundary leads to much more essential difficulties than imposing fluxes (Neumann conditions). In particular, the arguments used earlier for deriving a Lagrangian fluctuation-dissipation relation no longer allow us to express the total volume-integrated scalar dissipation in purely Lagrangian terms. Instead, we can derive two distinct FDR's, one relation giving a lower bound on total scalar dissipation in purely Lagrangian terms and another relation of mixed Euler-Lagrangian character for the total scalar dissipation. The inequality relation allows us to deduce that anomalous dissipation can result from spontaneous stochasticity (but not that spontaneous stochasticity is necessary[6]). The equality relation is more useful for physical purposes. We thus derive both relations. We first consider the special case of a scalar $\theta(\boldsymbol{x},t)$ with pure Dirichlet boundary conditions ($\partial\Omega_D = \partial\Omega$):

$$\begin{aligned} \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta &= \kappa\Delta\theta + S \qquad && \text{for} \quad \boldsymbol{x}\in\Omega, \\ \theta(\boldsymbol{x},t) &= \psi(\boldsymbol{x},t) && \text{for} \quad \boldsymbol{x}\in\partial\Omega. \end{aligned} \tag{5.36}$$

for given initial data θ_0 and boundary data ψ. Of course, pure Dirichlet conditions are often encountered in practice, e.g. turbulent channel flow with opposite walls held at two fixed temperatures. This special case already presents the essential new difficulties and, after analyzing it in detail, we shall briefly comment on modifications required for the general mixed case.

[6]In fact, we provide an analytical example of pure heat conduction exhibiting a dissipative anomaly due entirely to thin scalar boundary layers (Appendix B of Drivas and Eyink [2017b]).

5.3.1 Stochastic Representation and Fluctuation-Dissipation Relation

The standard stochastic representation of solutions of the problem (5.36) involves stochastic particles which are stopped at the boundary, contributing a term due to the value maintained at the wall [Keanini, 2007, Soner, 2007, Oksendal, 2013]. The stochastic flow may again be defined with reflection at the boundary, governed by the equation (5.7) involving the boundary local time. The new notion is the *boundary (first-)hitting time* or *stopping time*, which is defined for $\boldsymbol{x} \in \Omega$ by the larger of $\{\sup\{s : \tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) \in \partial\Omega\}$, the first time going in reverse to hit the spatial boundary, and the initial time 0, or

$$\tilde{\tau}(\boldsymbol{x}, t) = \max\left\{\sup\{s : \tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}) \in \partial\Omega\}, 0\right\}, \tag{5.37}$$

This time is the first (going backward) at which the stochastic Lagrangian particle hits the closed set $\mathcal{S}$ which is the union of the "side surface" $\mathcal{S}_s = \partial\Omega \times [0, t)$ and the "base" $\mathcal{S}_b = \Omega \times \{0\}$ of the right cylindrical domain $\mathcal{D} = \Omega \times [0, t)$ in $(d+1)$-dimensional space-time, when starting at a point $(\boldsymbol{x}, t)$ in the "top" $\mathcal{S}_t = \Omega \times \{t\}$. To state conveniently the stochastic representation, we define the function which gives the data on $\mathcal{S}$:

$$\Theta(\boldsymbol{x}, s) = \begin{cases} \psi(\boldsymbol{x}, s), & s > 0 \text{ and } \boldsymbol{x} \in \partial\Omega \\ \\ \theta_0(\boldsymbol{x}), & s = 0 \text{ and } \boldsymbol{x} \in \Omega \end{cases}. \tag{5.38}$$

Using the backward Itō formula (5.12) and integrating from time t down to the hitting time $\tilde{\tau}(\boldsymbol{x}, t)$ gives

$$\begin{aligned} \theta(\boldsymbol{x}, t) = \Theta(\tilde{\boldsymbol{\xi}}_{t,\tilde{\tau}(\boldsymbol{x},t)}(\boldsymbol{x}), \tilde{\tau}(\boldsymbol{x}, t)) + \int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}t' \ S(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}), t') \\ + \sqrt{2\kappa} \int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \hat{\mathrm{d}}\mathbf{W}_{t'} \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,t'}(\boldsymbol{x}), t'), \end{aligned} \tag{5.39}$$

where we used the fact that $\tilde{\ell}_{t,s}(\boldsymbol{x}) \equiv 0$ for all $\tilde{\tau}(\boldsymbol{x}, t) < s < t$ and also used the system (5.36). Taking the expectation over the Brownian motion then yields for solutions of the initial-boundary value problem the following stochastic representation:

$$\theta(\boldsymbol{x}, t) = \mathbb{E}\left[\Theta(\tilde{\boldsymbol{\xi}}_{t,\tilde{\tau}(\boldsymbol{x},t)}(\boldsymbol{x}), \tilde{\tau}(\boldsymbol{x}, t)) + \int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}s \ S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\right] \tag{5.40}$$

See Keanini [2007], Soner [2007], Oksendal [2013] for more details. In particular, the Itō integral in (5.39) is a (backward) martingale by the optional stopping theorem [Oksendal, 2013]. This formula represents the solution $\theta(\boldsymbol{x}, t)$ to the scalar advection-diffusion equation as an average over randomly sampled sources and initial-boundary data.

It is now straightforward to obtain our first version of a fluctuation-dissipation relation by mimicking previous arguments. Applying the Itō isometry valid with the random stopping time as the lower range of the integral (e.g. see Oksendal [2013], Theorem 7.4.1)

$$\mathbb{E}\left|\int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\right|^2 = 2\ \mathbb{E}\left[\int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}s\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2\right] \tag{5.41}$$

Thus we obtain from (5.39), (5.40) that

$$\mathrm{Var}\left[\Theta(\tilde{\boldsymbol{\xi}}_{t,\tilde{\tau}(\boldsymbol{x},t)}(\boldsymbol{x}), \tilde{\tau}(\boldsymbol{x},t)) + \int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)\right] = 2\kappa\ \mathbb{E}\left[\int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}s\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2\right]. \tag{5.42}$$

Finally, averaging over the space domain yields

$$\begin{aligned} &\frac{1}{2}\left\langle \mathrm{Var}\left[\Theta(\tilde{\boldsymbol{\xi}}_{t,\tilde{\tau}(t)}, \tilde{\tau}(t)) + \int_{\tilde{\tau}(t)}^{t} \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}, s)\right]\right\rangle_{\Omega} \\ &\qquad = \kappa\ \left\langle \mathbb{E}\left[\int_{\tilde{\tau}(t)}^{t} \mathrm{d}s\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}, s)|^2\right]\right\rangle_{\Omega}. \end{aligned} \tag{5.43}$$

This exact result is one possible version of a fluctuation-dissipation relation for scalar turbulence with general Dirichlet boundary conditions.

Unlike our previous relations, however, the righthand side of (5.43) above is not the total time-integrated scalar dissipation over the entire domain of the flow. The relation easily yields a lower bound on the total scalar dissipation by simply extending the time integration down to $s = 0$,

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\Theta(\tilde{\boldsymbol{\xi}}_{t,\tilde{\tau}(t)}(\tilde{\tau}(t)) + \int_{\tilde{\tau}(t)}^{t} \mathrm{d}s\ S(\tilde{\boldsymbol{\xi}}_{t,s}, s)\right]\right\rangle_{\Omega} \leq \kappa \left\langle \mathbb{E}\left[\int_0^t \mathrm{d}s\ |\nabla\theta(s)|^2\right]\right\rangle_{\Omega}. \tag{5.44}$$

because the s-integrand is non-negative and the reflected stochastic flow is volume-preserving. The difference between the righthand and lefthand sides is

$$\begin{aligned} \Delta^{\nu,\kappa}(t) &\equiv \kappa \int_0^t \mathrm{d}s\ \frac{1}{|\Omega|}\int_{\Omega} d^d x\ |\nabla\theta(\boldsymbol{x}, s)|^2 - \kappa\ \mathbb{E}\left[\frac{1}{|\Omega|}\int_{\Omega} d^d x \int_{\tilde{\tau}(\boldsymbol{x},t)}^{t} \mathrm{d}s\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2\right] \\ &= \kappa \int_0^t \mathrm{d}s\ \mathbb{E}\left[\frac{1}{|\Omega|}\int_{\tilde{\Omega}_{t,s}} d^d x\ |\nabla\theta(\tilde{\boldsymbol{\xi}}_{t,s}(\boldsymbol{x}), s)|^2\right] \end{aligned} \tag{5.45}$$

where

$$\tilde{\Omega}_{t,s} = \{\boldsymbol{x} \in \Omega :\ \tilde{\tau}(\boldsymbol{x}, t) > s\} \tag{5.46}$$

is the set of positions $\boldsymbol{x}$ for which the stochastic particle has already hit the boundary by time s (going backward). The inequality (5.44) thus fails to be an equality because of the missing contributions to total dissipation at positions of reflected particles. We shall discuss further below the sharpness of the inequality (5.44) and whether it should, in a suitable limit, become equality.

5.3.2 Numerical Results

To gain further insight, we present now some numerical results on PDF's of hitting-times for the channel-flow database. We considered two release points in the buffer layer at height $\Delta y = 10\delta_\tau$ above the bottom wall and at the final time t_f of the database. The wall-parallel positions of the release points are those shown in Figure 5.2, and thus one is in a high-speed streak and the other in a low-speed streak. It turns out that to calculate hitting-time PDF's accurately using N-sample ensembles of particles is quite difficult, because most particles take a very long time to first hit the boundary. For our results presented here, we evolved $N = 14,336$ samples of stochastic particles solving Eq. (5.7) for three Prandtl numbers Pr =0.1, 1, 10 at both release points. Note that hitting-times τ satisfying $t_f - \tau \ll t_\nu$ and $t_f - \tau \gg t_\nu$ are both very rare events which could not be observed even with so many samples. We thus consider here the logarithmic variable $\lambda \equiv \ln((t_f - \tau)/t_\nu)$ which is appropriate to typical values and and calculate PDF's $p(\lambda|\boldsymbol{x}, t_f)$. The numerical procedures are discussed at length in Appendix D of Drivas and Eyink [2017b]. Here we just note that the PDF's are obtained up to the largest available $\lambda = \ln(t_f/t_\nu) \doteq 7.17$ and down to $\lambda = -1$, but the PDF's at the two extremes contain further errors that are not indicated by the error bars (representing both s.e.m. for N-samples averages and variation with kernel bandwidth). The PDF's for $\lambda < 1.6$ are shifted to the right by about 5%, because our time-step $\Delta s \doteq 2 \times 10^{-3}$ cannot fully resolve the smaller hitting times. For $\lambda > 6.5$ the PDF's from kernel-density estimates are too

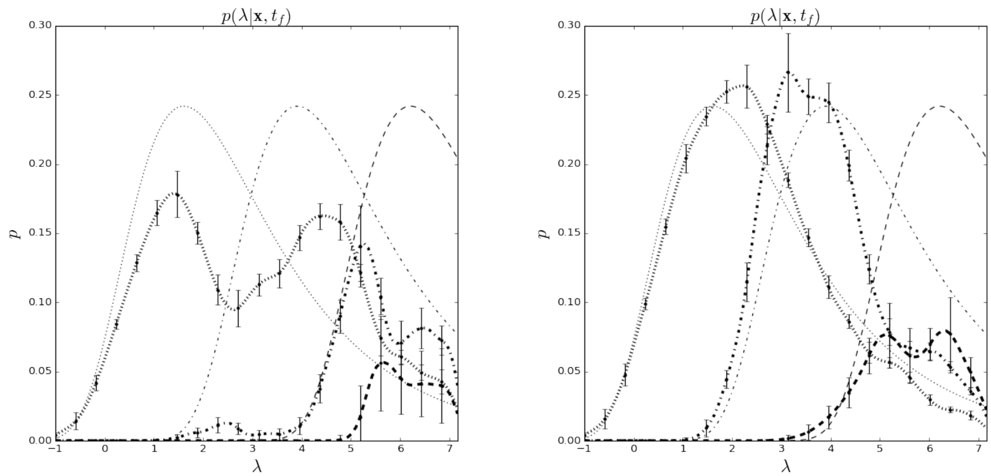

Figure 5.1: PDF's of the logarithmic hitting-time variable $\lambda = \ln((t_f - \tau)/t_\nu)$. *Left panel*: Particles released in a high-speed streak, *Right panel*: particles released in a low-speed streak, for channel flow (heavy) and pure diffusion (light) for Prandtl numbers $Pr = 0.1$ (dot, $\cdots$), 1.0 (dash-dot, -·-) and 10 (dash, - - -).

small, because of the endpoint effect due to unavailability of samples for $\lambda > 7.17$. Our numerical procedures show the same deficiencies when applied to pure Brownian motion, but successfully recover the known analytical results for that case in the range $1.6 < \lambda < 6.5$.

We plot in Fig. 5.1 the PDF's $p(\lambda|\boldsymbol{x}, t_f)$ for the two choices of $\boldsymbol{x}$, the left panel corresponding to particles released in a high-speed streak and the right to particles released in a low-speed streak. We also plot for comparison the analytical results for hitting-time PDF's with pure diffusion (see eqn. D4 of Appendix D of Drivas and Eyink [2017b]). One can see in Fig. 5.1 a very strong effect of the release point on the hitting-time statistics, unlike the situation for the boundary local-times accumulated over one flow-through time. At $Pr = 0.1$ and 1, hitting-times are clearly larger than those of pure diffusion for release in a high-speed streak and smaller for release in a low-speed streak. This effect is easy to understand, because particles released in a low-speed streak are advected toward the wall backward in time, and those released in a high-speed streak swept away from the wall. This effect is not observed for $Pr = 10$, with hitting times for pure diffusion very obviously shorter than those for advected particles started in both high-speed and low-speed streaks. This occurs presumably because advection alone can never bring a particle to the wall (because of the vanishing velocity there) and as diffusivity κ decreases one sees more strongly the effect of fluid advection, which transports the particles away from the wall. One can see in general a strong effect of diffusivity on the hitting times, with smaller values of κ (or larger Pr) leading to larger hitting times backward in time. We shall refer to this property in our discussion below of the sharpness of the inequality Eq. (5.44). However, we first exploit this inequality to prove also for Dirichlet b.c. that spontaneous stochasticity is sufficient for anomalous dissipation of passive scalars.

5.3.3 Spontaneous Stochasticity Implies Anomalous Dissipation

The argument is very similar to those given earlier. To present it first in the simplest context, we consider a decaying scalar with vanishing source ($S = 0$). We first introduce an analogue of the formula (5.33) for the variance which constitutes the lower bound. There is a natural measure on the set $\mathcal{S}$ which is just the d-dimensional Hausdorff measure H^d on subsets of $(d+1)$-dimensional spacetime, restricted to $\mathcal{S}$. This can be described in elementary terms using natural d-dimensional coordinates $\boldsymbol{\sigma}$ on $\mathcal{S}$, where on $\mathcal{S}_b$ we have $\boldsymbol{\sigma} = (\boldsymbol{y}, 0)$ for $\boldsymbol{y} \in \Omega$ and on $\mathcal{S}_s$ we have $\boldsymbol{\sigma} = (\boldsymbol{z}, \tau)$ for $\boldsymbol{z} \in \partial\Omega$, the presumed smooth bounding wall surface, and $\tau \in [0, t]$. Then for $\boldsymbol{\sigma} \in \mathcal{S}$

$$H^d(d\boldsymbol{\sigma}) = \begin{cases} d^d y & \boldsymbol{\sigma} = (\boldsymbol{y}, 0) \in \mathcal{S}_b \\ dH^{d-1}(\boldsymbol{z}) d\tau & \boldsymbol{\sigma} = (\boldsymbol{z}, \tau) \in \mathcal{S}_s \end{cases} \tag{5.47}$$

Here H^{d-1} is the $(d-1)$-dimensional Hausdorff measure on the smooth wall surface $\partial\Omega$ (that is, the usual $(d-1)$-dimensional surface area dS). We can then also define the delta distribution $\delta_{\mathcal{S}}(\boldsymbol{\sigma}', \boldsymbol{\sigma})$ on $\mathcal{S}$ wth respect to measure H^d, so that

$$\int_{\mathcal{S}} H^d(d\boldsymbol{\sigma}') f(\boldsymbol{\sigma}') \delta_{\mathcal{S}}(\boldsymbol{\sigma}', \boldsymbol{\sigma}) = f(\boldsymbol{\sigma}) \tag{5.48}$$

for all smooth functions on $\mathcal{S}$, and we can likewise decompose this delta function as

$$\delta_{\mathcal{S}}(\boldsymbol{\sigma}', \boldsymbol{\sigma}) = \begin{cases} \delta^d(\boldsymbol{y}' - \boldsymbol{y}) & \boldsymbol{\sigma}', \boldsymbol{\sigma} \in \mathcal{S}_b \\ \delta_{\partial\Omega}(\boldsymbol{z}', \boldsymbol{z}) \delta(\tau' - \tau) & \boldsymbol{\sigma}', \boldsymbol{\sigma} \in \mathcal{S}_s \end{cases} \tag{5.49}$$

Denoting by $\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}, t) = (\tilde{\boldsymbol{\xi}}_{t, \tilde{\tau}(\boldsymbol{x}, t)}(\boldsymbol{x}), \tilde{\tau}(\boldsymbol{x}, t))$ the space-time point where the trajectory first hits $\mathcal{S}$ (going backward in time), we can then introduce 1-time transition probability densities for a particle to go backward in time from points $(\boldsymbol{x}, t)$ on the "top" surface $\Omega \times \{t\}$ of the space-time cylinder and to arrive first to $\mathcal{S}$ at the point $\boldsymbol{\sigma} \in \mathcal{S}$:

$$\begin{aligned} q^{\nu,\kappa}(\boldsymbol{\sigma}|\boldsymbol{x},t) &= \mathbb{E}\left[\delta_{\mathcal{S}}\left(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\right)\right] \\ &= \begin{cases} \mathbb{E}\left[\delta^d(\boldsymbol{y}-\tilde{\boldsymbol{\xi}}_{t,0}(\boldsymbol{x}))\right] & \boldsymbol{\sigma}=(\boldsymbol{y},0)\in\mathcal{S}_b \\ \mathbb{E}\left[\delta_{\partial\Omega}(\boldsymbol{z}-\tilde{\boldsymbol{\xi}}_{t,\tau}(\boldsymbol{x}))\delta(\tau-\tilde{\tau}(\boldsymbol{x},t))\right] & \boldsymbol{\sigma}=(\boldsymbol{z},\tau)\in\mathcal{S}_s \end{cases} . \end{aligned} \tag{5.50}$$

This probability density is normalized with respect to H^d on $\mathcal{S}$ so that

$$\int_{\mathcal{S}} H^d(d\boldsymbol{\sigma})q(\boldsymbol{\sigma}|\boldsymbol{x},t) = 1. \tag{5.51}$$

In these terms we obtain a simple formula for the stochastic representation (5.40) in the case of vanishing scalar source:

$$\theta(\boldsymbol{x},t) = \mathbb{E}\left[\Theta(\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t))\right] = \int_{\mathcal{S}} H^d(d\boldsymbol{\sigma})\ \Theta(\boldsymbol{\sigma})\ q(\boldsymbol{\sigma}|\boldsymbol{x},t). \tag{5.52}$$

We should note here that there is another representation for $q(\boldsymbol{\sigma}|\boldsymbol{x},t)$ in terms of a process of stochastic Lagrangian particles which is distinct from that considered so far. Rather than letting the particles reflect off the wall, one can instead kill the particles when they hit the wall (absorbing boundary conditions). If $k(\boldsymbol{a},s|\boldsymbol{x},t)$ for $s<t$ is the transition probability density for this killed process backward in time,

then:

$$q(\boldsymbol{\sigma}|\boldsymbol{x},t) = \begin{cases} k(\boldsymbol{y},0|\boldsymbol{x},t) & \boldsymbol{\sigma} = (\boldsymbol{y},0) \in \mathcal{S}_b \\ \boldsymbol{\mu}(\boldsymbol{z},\tau) \cdot \nabla_{\boldsymbol{a}} k(\boldsymbol{a},\tau|\boldsymbol{x},t)\Big|_{\boldsymbol{a}=\boldsymbol{z}} & \boldsymbol{\sigma} = (\boldsymbol{z},\tau) \in \mathcal{S}_s \end{cases}. \tag{5.53}$$

Here we make use of the general connection between the joint density function of the hitting time and location and between the normal derivative of the transition probability for the killed process; see Freĭdlin [1985], Hsu [1986]. This alternative stochastic interpretation proves useful for calculations in Appendix B of Drivas and Eyink [2017b].

We can obtain a similar formula as (5.52) for the variance, if we introduce also the corresponding 2-fold transition probabilities

$$q_2^{\nu,\kappa}(\boldsymbol{\sigma},\boldsymbol{\sigma}'|\boldsymbol{x},t) = \mathbb{E}\left[\delta_{\mathcal{S}}\left(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\right)\delta_{\mathcal{S}}\left(\boldsymbol{\sigma}',\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\right)\right] = \delta_{\mathcal{S}}(\boldsymbol{\sigma},\boldsymbol{\sigma}')q(\boldsymbol{\sigma}|\boldsymbol{x},t), \tag{5.54}$$

so that

$$\begin{aligned} \mathrm{Var}\left[\Theta(\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t))\right] &= \int_{\mathcal{S}} H^d(\mathrm{d}\boldsymbol{\sigma}) \int_{\mathcal{S}} H^d(\mathrm{d}\boldsymbol{\sigma}')\ \Theta(\boldsymbol{\sigma})\Theta(\boldsymbol{\sigma}') \\ &\quad \times\Big[q_2(\boldsymbol{\sigma},\boldsymbol{\sigma}'|\boldsymbol{x},t) - q(\boldsymbol{\sigma}|\boldsymbol{x},t)q(\boldsymbol{\sigma}'|\boldsymbol{x},t)\Big]. \end{aligned} \tag{5.55}$$

These results are formally identical to those for the source-less ($S = 0$) scalar in domains without boundary or with zero-flux Neumann conditions at the wall. Furthermore, $\mathcal{S}$ is a compact subset of space-time. Thus, we can exactly mimic our previous arguments with no change. Limits q^*, q_2^* as $\nu_j, \kappa_j \to 0$ always exist along suitable subsequences ν_j, κ_j. If the limits were deterministic, then one would have

$$q_2^*(\boldsymbol{\sigma}, \boldsymbol{\sigma'}|\boldsymbol{x}, t) = q^*(\boldsymbol{\sigma}|\boldsymbol{x}, t)q^*(\boldsymbol{\sigma'}|\boldsymbol{x}, t) \tag{5.56}$$

so that q_2^* would factorize in a product of two q^*'s. However, if the limits are non-deterministic so that q_2^* does not factorize for a positive-measure set of $\boldsymbol{x}$ at time t, then there must be some smooth choice of $\Theta = (\theta_0, \psi)$ that makes the non-negative space-averaged variance in (5.55) in fact non-zero. For such choice of initial condition θ_0 and boundary conditions ψ, the space-average and time-integrated scalar dissipation will then be non-zero taking the limit $\nu_j, \kappa_j \to 0$. As before, this argument works rigorously only for a passive scalar, not an active one. An additional limitation discussed further in Appendix C of Drivas and Eyink [2017b] is that we cannot use this argument to prove, for a fixed choice of the boundary data ψ, that spontaneous stochasticity implies there exists a smooth initial datum θ_0 for which the scalar dissipation rate is positive[7]. Minor changes to these arguments are needed to incorporate a bulk source S and we leave details also to Appendix C of Drivas and Eyink [2017b].

The previous arguments do not imply that spontaneous stochasticity is required

[7]These comments have implications for the problem of Rayleigh-Bénard convection with fixed temperatures at top/bottom plates, which is discussed in the paper Eyink & Drivas, [2017c]. There is some evidence for Richardson dispersion of Lagrangian particles in numerical simulations of turbulent convection [Schumacher, 2008]. However, even if there were compelling evidence for spontaneous stochasticity associated to this effect, we could not rigorously conclude from our results that there is anomalous thermal dissipation. First, the temperature is an active scalar in that case and, second, our present arguments do not work with fixed boundary data.

for anomalous scalar dissipation with imposed scalar values at the wall. Even if the factorization (5.56) holds and there is no spontaneous stochasticity, the variance in (5.55) provides only a lower bound to the scalar dissipation. Hence, the scalar dissipation rate can tend to a non-zero limit even when the variance vanishes! This is not just a failure of the proof, but also an indication that other physical mechanisms are available to produce spontaneous stochasticity with Dirichlet boundary conditions. The most obvious alternative mechanism is a narrow scalar boundary layer at the walls. One can imagine a flow with extremely rapid advective mixing in the interior and with opposite scalar values $\pm\psi_0$ imposed at two ends, so that large scalar gradients of the order $\nabla\theta \sim O(\psi_0/\kappa)$ are achieved in a boundary layer of thickness $\sim O(\kappa)$ and a near-zero scalar amplitude $\theta \simeq 0$ occurs over the bulk of the domain. Such a flow would be the simplest that provides anomalous scalar dissipation from a scalar boundary-layer mechanism. For example, one might consider a fluid undergoing solid-body rotation with frequency ω in the interior of the domain and an imposed inhomogeneous temperature distribution on the boundary varying between $-\psi_0$ and $+\psi_0$. In the rotating frame of reference, this would correspond to a motionless fluid (pure heat conduction) but with a time-dependent boundary temperature oscillating with frequency ω between the extreme values $\pm\psi_0$.

As the simplest example of this type, consider pure diffusion on the interval $[0, H]$ with $\theta_0 = S \equiv 0$ and boundary temperatures specified as $\psi(0,t) = \psi_0 \sin(\omega t)$ and

$\psi(H, t) = 0$. In this case, the stochastic scalar field is simply

$$\tilde{\theta}(\boldsymbol{x}, t) = \psi_0 \sin(\omega \tilde{\tau}(x, t)). \tag{5.57}$$

The scalar field $\theta(x, t) = \mathbb{E}[\tilde{\theta}(\boldsymbol{x}, t)]$ can be constructed with the knowledge of the hitting-time distribution at zero for Brownian motion on the half-line, which is:

$$p^\kappa(\tau|x, t) = \begin{cases} \frac{x}{\sqrt{4\pi\kappa(t-\tau)^3}} \exp\left(-\frac{x^2}{4\kappa(t-\tau)}\right) & \tau < t \\ 0 & \tau > t \end{cases}. \tag{5.58}$$

See Appendix B of Drivas and Eyink [2017b]. The solution $\theta(x, t)$ of the conduction problem can be obtained by integration over the above distribution, and scalar gradients can then be explicitly computed. Temperature fluctuations are found to propagate with velocity $(\kappa\omega)^{1/2}$ and travel to distances $(\kappa/\omega)^{1/2}$ away from the wall. In this small boundary layer region, the scalar field gradients diverge as $\partial_x \theta \sim \psi_0 (\omega/\kappa)^{1/2}$ as $\kappa \to 0$. We obtain:

$$\int_0^L dx\ \kappa |\partial_x \theta(x, t)|^2 \propto (\kappa/\omega)^{1/2} \cdot \kappa \cdot \psi_0^2 (\omega/\kappa) \sim \psi_0^2 \sqrt{\kappa\omega}. \tag{5.59}$$

See Appendix B of Drivas and Eyink [2017b] for a more precise statement. The scalar field in this simple example exhibits anomalous dissipation if ω is very large for $\kappa \to 0$, e.g. with $\omega \sim \kappa^{-\alpha}$ for $\alpha \geq 1$. In particular, for $\alpha = 1$ the limiting scalar dissipation rate is non-zero and finite as $\kappa \to 0$, with scalar gradients and boundary layer thicknesses as suggested in the previous paragraph. Thus, just as for flux b.c., thin scalar boundary layers can be a source of non-zero (possibly diverging) dissipation as $\nu, \kappa \to 0$.

5.4 Summary and Discussion

This chapter has extended to flows in wall-bounded domains the Lagrangian fluctuation-dissipation relation introduced in chapter 4for scalars advected by an incompressible fluid. This relation expresses an exact balance between molecular dissipation of the scalar and input of scalar variance from the initial values, boundary fluxes, and internal sources as these are sampled by stochastic Lagrangian trajectories backward in time. We have exploited this relation to prove, for domains with no scalar flux through the wall, that spontaneous stochasticity of Lagrangian trajectories is necessary and sufficient for anomalous dissipation of passive scalars, and necessary (but possibly not sufficient) for anomalous dissipation of active scalars. Cream stirred into coffee is an everyday example of this type, as is any scalar advected by a fluid in a container with impermeable walls. For more general mixed boundary conditions on the scalar, with imposed values at the wall or imposed non-zero fluxes, simple examples show that thin scalar boundary layers provide a distinct mechanism for non-vanishing scalar dissipation. Nevertheless, we can still show rigorously for general scalar boundary conditions that spontaneous stochasticity is sufficient for anomalous dissipation of passive scalars, and this result plausibly extends to active scalars as well. Thus, in addition to scalar boundary layers, Lagrangian spontaneous stochasticity is shown here to be another possible source of anomalous scalar dissipation in wall-bounded flows.

An interesting issue for further study is whether Lagrangian spontaneous stochasticity plays any role in anomalous dissipation of kinetic energy in wall-bounded flows. There is some evidence from experimental measurements of turbulence in closed containers that, while the kinetic energy dissipation due to viscous boundary layers decreases very slowly with increasing Reynolds number, the energy dissipation in the bulk of the turbulent flow is very nearly Reynolds-number independent [Cadot et al., 1997]. It appears possible that such "anomalous dissipation" in the bulk is produced by very similar mechanisms as energy dissipation in homogeneous, isotropic turbulence and thus may be related to spontaneous stochasticity in the bulk, as discussed in chapter 4. Note that the stochastic Lagrangian representation of incompressible Navier-Stokes solutions discussed in chapter 4has been successfully extended to wall-bounded flows with stick boundary conditions for the velocity at the wall [Constantin and Iyer, 2011]. The Lagrangian dynamics of vorticity in this stochastic formulation generalizes the classical Helmholtz theorem for ideal Euler solutions. Indeed, similar to the case of scalar fields with Dirichlet conditions discussed in section 5.3, the vorticity field at a point is an average of "frozen-in" vorticity vectors that are transported along an ensemble of stochastic Lagrangian trajectories backward in time. Those trajectories that hit the wall transport the vorticity that they encounter there, while those that never hit the wall before time 0 transport the initial vorticity. See Constantin and Iyer [2011] for detailed discussion and proofs. These results make it possible to study in detail the contribution of Lagrangian vorticity dynamics, e.g. stretching of vortex filaments, to turbulent energy dissipation in wall-bounded flows. A crucial difference from homogeneous, isotropic turbulence, however, is that random stretching dynamics is insufficient to explain energy dissipation in the presence of walls, but instead vortex lines generated at the wall must undergo organized motion away from the wall [Taylor, 1932, Huggins, 1970, Eyink, 2008]. We therefore expect that the effects of walls on Lagrangian mechanisms of turbulent energy dissipation are quite profound.

The Lagrangian fluctuation-dissipation relation proved in the present chapter is valid in wall-bounded flows for all scalars, passive or active, whether either Lagrangian spontaneous stochasticity or scalar anomalous dissipation occur or not in a particular flow. In general, the FDR provides a novel Lagrangian view of turbulent scalar dissipation. In the submitted paper, Eyink & Drivas, [2017c], we apply the FDR derived here to describe the thermal dissipation rate in turbulent Rayleigh-Bénard convection, where the mean thermal dissipation rate and the mean kinetic energy dissipation rate are closely related. This concrete application thus provides a wider window into the Lagrangian mechanisms of turbulent dissipation.

Chapter 6

An Onsager Singularity Theorem for Turbulent Solutions of Compressible Euler Equations

6.1 Introduction

In a 1949 paper on turbulence in incompressible fluids L. Onsager announced a result that spatial Hölder exponents $\leq 1/3$ are required of the velocity field for anomalous turbulent dissipation (that is, energy dissipation non-vanishing in the limit of zero viscosity). His sketched argument involved the idea that the velocity field in the limit of infinite Reynolds number is a weak (distributional) solution of the incompressible Euler equations. Onsager never published a detailed proof of his singularity theorem, but works of Eyink [1994], Constantin et al. [1994], and Duchon and Robert [2000], among others later, proved Onsager's claimed result and even more precise results. Onsager's own unpublished argument was essentially the same as that given in Duchon and Robert [2000], according to the historical evidence (Eyink and Sreenivasan [2006a]). More recent mathematical work has established existence of dissipative weak Euler solutions of the type conjectured by Onsager, beginning with pioneering work of De Lellis & Székelyhidi, Jr. 2010, 2012 on the convex integration approach, that has since culminated in constructions of solutions with the critical 1/3 regularity. None of these theorems establish that dissipative Euler solutions exist as the zero-viscosity limits of incompressible Navier-Stokes solutions, necessary to rigorously found Onsager's theory for fluid turbulence from first principles.

In this paper, we prove an Onsager singularity theorem for weak solutions of the compressible Euler equations in arbitrary space-dimension $d \geq 1$. The Euler system consists of $d+2$ dynamical equations for the basic state variables: the mass density $\varrho := \varrho(\boldsymbol{x}, t)$, fluid velocity $\boldsymbol{u} := \boldsymbol{u}(\boldsymbol{x}, t)$ and internal energy density $u := u(\boldsymbol{x}, t)$ (related to the specific internal energy u_m by $u = \varrho u_m$), where $E := \frac{1}{2}\varrho|\boldsymbol{u}|^2 + u$ is the total energy density. Explicitly,

$$\partial_t \varrho + \nabla_x \cdot (\varrho \boldsymbol{u}) = 0, \tag{6.1}$$

$$\partial_t (\varrho \boldsymbol{u}) + \nabla_x \cdot (\varrho \boldsymbol{u}\boldsymbol{u} + p\mathbf{I}) = 0, \tag{6.2}$$

$$\partial_t E + \nabla_x \cdot ((p + E)\boldsymbol{u}) = 0. \tag{6.3}$$

We use the "dyadic product" notation $\boldsymbol{u}\boldsymbol{u}$ of J. W. Gibbs for the tensor product $\boldsymbol{u} \otimes \boldsymbol{u}$ of space-vectors, which is convenient in this paper. The pressure is given by a *thermodynamic equation of state* $p := p(u, \varrho)$ as a function of u and ϱ. A previous paper Feireisl et al. [2017] has studied a similar problem, but under the assumption of a barotropic equation of state, with pressure $p = p(\varrho)$ a function only of mass density and with no independent equation for the total energy density E. Our results are valid for a general equation of state $p(u, \varrho)$, assuming only that the fluid undergoes no phase transitions during its evolution (see Assumption 6.1.2 for a more precise statement). We also consider strong limits of solutions of the compressible Navier-Stokes equations for Reynolds and Péclet numbers tending to infinity. As we shall show, such strong limits are weak solutions of the compressible Euler system (6.1)–(6.3). This is a subclass of all Euler solutions, but arguably the one most relevant to compressible fluid turbulence.

In order to state precisely our results, recall that the Navier-Stokes-Fourier system (or, simply, the compressible Navier-Stokes equations) for a viscous, heat-conducting fluid takes the form:

$$\partial_t \varrho + \nabla_x \cdot (\varrho \boldsymbol{u}) = 0, \tag{6.4}$$

$$\partial_t (\varrho \boldsymbol{u}) + \nabla_x \cdot (\varrho \boldsymbol{u}\boldsymbol{u} + p\mathbf{I} + \mathbf{T}) = 0, \tag{6.5}$$

$$\partial_t E + \nabla_x \cdot ((p + E)\boldsymbol{u} + \mathbf{T} \cdot \boldsymbol{u} + \boldsymbol{q}) = 0. \tag{6.6}$$

The *viscous stress tensor* $\mathbf{T}$ is given by *Newton's rheological law*:

$$\mathbf{T} := -2\eta\mathbf{S} - \zeta\Theta\mathbf{I} \quad \text{with} \quad \mathbf{S} := \frac{1}{2}\left(\nabla_x\boldsymbol{u} + (\nabla_x\boldsymbol{u})^\top - \frac{2}{d}\Theta\mathbf{I}\right) \quad \text{and} \quad \Theta := \mathrm{div}_x\boldsymbol{u},$$

where $\eta := \eta(u,\varrho) > 0$ and $\zeta := \zeta(u,\varrho) > 0$ represent the shear and bulk viscosity, respectively. The *heat flux* $\boldsymbol{q}$ is given by *Fourier's law*:

$$\boldsymbol{q} := -\kappa\nabla_x T, \tag{6.7}$$

with thermal conductivity $\kappa := \kappa(u,\varrho) > 0$, where $T := T(u,\varrho)$ is the *temperature* of the fluid. For this system, see standard physics texts such as Landau and Lifshitz [1987] (§49) or de Groot and Mazur [1984], (Ch. XII, §1), and, in the mathematics literature, Gallavotti [2013] (§1.1), Feireisl [2004], Feireisl and Novotnỳ [2013] or Lions [1998].

Balance equations of kinetic energy, internal energy density, and entropy density follow straightforwardly for smooth solutions of the system (6.4)–(6.6). The equations for kinetic and internal energy are:

$$\partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right) + \nabla_x \cdot \left(\left(p + \frac{1}{2}\varrho|\boldsymbol{u}|^2\right)\boldsymbol{u} + \mathbf{T}\cdot\boldsymbol{u}\right) = p\,\Theta - Q, \tag{6.8}$$

$$\partial_t u + \nabla_x \cdot (u\boldsymbol{u} + \boldsymbol{q}) = Q - p\,\Theta, \tag{6.9}$$

where the rate of viscous heating of the fluid is explicitly:

$$Q := -\mathbf{T} : \nabla_x \boldsymbol{u} = 2\eta|\mathbf{S}|^2 + \zeta\Theta^2. \tag{6.10}$$

The entropy density $s := s(u, \varrho)$ (related to the specific entropy s_m by $s = \varrho s_m$) is thermodynamically related to u and ϱ through the first law of thermodynamics in the form:

$$T\mathrm{d}s = \mathrm{d}u - \mu\mathrm{d}\varrho. \tag{6.11}$$

with the *chemical potential* $\mu := \mu(u, \varrho)$. From the system (6.4)–(6.6) and the thermodynamic relation (6.11) follows the balance equation for the entropy density:

$$\partial_t s + \nabla_x \cdot \left(s\boldsymbol{u} + \frac{\boldsymbol{q}}{T}\right) = \frac{Q}{T} + \Sigma_\kappa. \tag{6.12}$$

The entropy production rate $\Sigma := Q/T + \Sigma_\kappa$ involves a viscous heating contribution with Q again given by (6.10), and a term due to thermal conduction:

$$\Sigma_\kappa := -\frac{\boldsymbol{q} \cdot \nabla_x T}{T^2} = \kappa\frac{|\nabla_x T|^2}{T^2}. \tag{6.13}$$

In accord with second law of thermodynamics, entropy is globally increased since:

$$\Sigma := \frac{Q}{T} + \Sigma_\kappa = 2\frac{\eta}{T}|\mathbf{S}|^2 + \frac{\zeta}{T}|\Theta|^2 + \kappa\frac{|\nabla_x T|^2}{T^2} \geq 0. \tag{6.14}$$

For these standard results, see Landau and Lifshitz [1987], de Groot and Mazur [1984].

Smooth solutions of the compressible Euler system satisfy the same balance equations as (6.8), (6.9), and (6.12), but with $\zeta, \eta, \kappa \equiv 0$ so all of the non-ideal terms vanish, i.e. $\mathbf{T}, \boldsymbol{q} = 0$ and $Q, \Sigma \equiv 0$. This need not be true, of course, for weak solutions. An important class of weak solutions that we consider are those arising from limits of solutions $\varrho^\varepsilon, u^\varepsilon, \boldsymbol{u}^\varepsilon$ of the Navier-Stokes system with transport coefficients scaled as $\eta^\varepsilon = \varepsilon\eta$, $\zeta^\varepsilon = \varepsilon\zeta$, $\kappa^\varepsilon = \varepsilon\kappa$, for $\varepsilon \to 0$. Essentially, $1/\varepsilon$ represents the Reynolds and Péclet numbers of the fluid. To avoid purely technical issues associated with unbounded domains, we shall consider here flows on the d-torus $\mathbb{T}^d$ and over a finite time interval $[0, T]$. The main mathematical simplification of this case is that $L^\infty(\mathbb{T}^d \times [0, T]) \subset L^p(\mathbb{T}^d \times [0, T])$ for all $p \geq 1$. We then make the following specific assumptions:

Assumption 6.1.1 *Given $\varepsilon > 0$, we assume that there exists a unique smooth solution $\varrho^\varepsilon, e^\varepsilon, \boldsymbol{u}^\varepsilon$ of the compressible Navier-Stokes system (6.4)–(6.6) on $\mathbb{T}^d \times [0, T]$ for a given equation of state. In fact, most of our analysis will apply to suitable weak Navier-Stokes solutions. We assume $\varrho^\varepsilon, u^\varepsilon, \boldsymbol{u}^\varepsilon \in L^\infty(\mathbb{T}^d \times [0, T])$ uniformly bounded for $\varepsilon < \varepsilon_0$ and that for some $1 \leq p < \infty$ strong limits exist*

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u}, \quad \varrho^\varepsilon \to \varrho, \quad u^\varepsilon \to u \ \text{ in } \ L^p(\mathbb{T}^d \times [0, T]). \tag{6.15}$$

Thus the convergence is pointwise almost everywhere for a subsequence $\varepsilon_k \to 0$ and

$\varrho, u, \boldsymbol{u} \in L^\infty(\mathbb{T}^d \times [0,T])$. *The mode of convergence (6.15) permits limiting fields with jump discontinuities. We also assume* $\varrho^\varepsilon \geq \varrho_0$ *for some* $\varrho_0 > 0$ *and* $\varepsilon < \varepsilon_0$, *so that the fluid nowhere approaches a vacuum state with zero density.*

Assumption 6.1.2 *We assume that the solutions involve thermodynamic states* (u, ϱ) *strictly away from phase transitions, so that all thermodynamic functions* $h = p$, T, μ, s, η, ζ, κ, *etc. are smooth in* u, ϱ. *The set of states attained by any solution is the* essential range *over space-time,* $\mathcal{R} = \text{ess.ran}(u, \varrho)$ *and* $\mathcal{R}^\varepsilon = \text{ess.ran}(u^\varepsilon, \varrho^\varepsilon)$ *for* $\varepsilon > 0$, *which are compact sets in* $\mathbb{R}^2$ *Rudin [1987]. The uniform boundedness in* $L^\infty(\mathbb{T}^d \times [0,T])$ *of* u^ε, ϱ^ε *for* $\varepsilon < \varepsilon_0$ *implies that there exists a compact set* $K \subset \mathbb{R}^2$ *such that the closed convex hull*

$$\text{conv}[\mathcal{R}^\varepsilon \cup \mathcal{R}] \subseteq K, \quad \forall \varepsilon < \varepsilon_0. \tag{6.16}$$

We then assume for h *that there is an open set* $U \subset \mathbb{R}^2$, *with* $K \subset U$ *and* $h \in C^M(U)$ *with smoothness exponent* $M \geq 2$.

Assumption 6.1.3 *Assume that the dissipation terms defined in equations (6.10) and (6.14) converge as* $\varepsilon \to 0$ *in the sense of distributions:*

$$Q^\varepsilon_\eta := 2\eta^\varepsilon |\mathbf{S}^\varepsilon|^2, \quad Q^\varepsilon_\zeta := \zeta^\varepsilon (\Theta^\varepsilon)^2, \quad Q^\varepsilon := Q^\varepsilon_\eta + Q^\varepsilon_\zeta \xrightarrow{\mathcal{D}} Q,$$

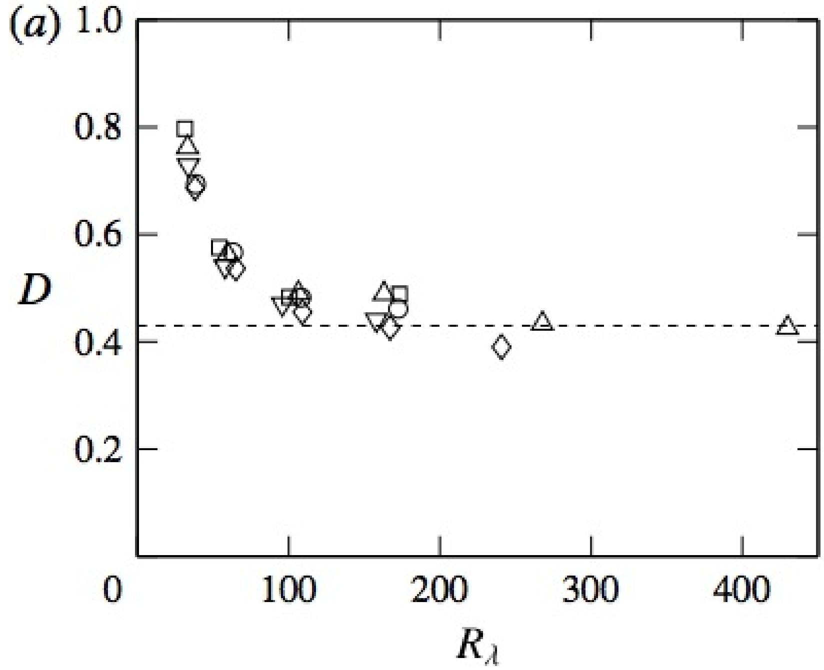

Figure 6.1: Evidence for Assumption 6.1.3, i.e. the "zeroth law": direct numerical simulation data from Jagannathan and Donzis [2016] of solenoidally-forced compressible turbulence with ideal gas equation of state. Results show the dissipation rate, Q^ε, tending to a constant as $\varepsilon \to 0$.

and

$$\Sigma^\varepsilon_\eta := \frac{Q^\varepsilon_\eta}{T^\varepsilon}, \quad \Sigma^\varepsilon_\zeta := \frac{Q^\varepsilon_\eta}{T^\varepsilon}, \quad \Sigma^\varepsilon_\kappa := \kappa^\varepsilon \left| \frac{\nabla_x T^\varepsilon}{T^\varepsilon} \right|^2, \quad \Sigma^\varepsilon := \Sigma^\varepsilon_\eta + \Sigma^\varepsilon_\zeta + \Sigma^\varepsilon_\kappa \xrightarrow{\mathcal{D}} \Sigma.$$

The limit distributions are obviously non-negative, and thus Radon measures. For experimental evidence for dissipative anomaly $Q > 0$ in compressible turbulence, see Fig. 6.1 and the paper [Jagannathan and Donzis, 2016].

Remark The set of compressible Navier-Stokes solutions satisfying these three assumptions is non-empty and includes, in particular, shock solutions. See examples in Johnson [2014] and Eyink and Drivas [2017b] (albeit in Euclidean space $\mathbb{R}^d$ rather than the torus $\mathbb{T}^d$.) Numerical simulations of compressible turbulence with the system (6.4)–(6.6) show that small-scale shocks (or "shocklets") naturally develop. There is also some evidence, however, that at sufficiently high Mach numbers the limiting mass density ϱ as $\varepsilon \to 0$ may exist only as a measure and not as a bounded function Kim and Ryu [2005]. There is thus empirical motivation to weaken Assumption 6.1.1 in future work.

We now state our main theorems. First, we establish the balance equations of energy and entropy for general bounded weak Euler solutions:

Theorem 6.1.4 *Let $u, \varrho, \boldsymbol{u} \in L^\infty(\mathbb{T}^d \times [0,T])$ be any weak solution of the compressible Euler system (6.1)–(6.3) satisfying $\varrho \geq \varrho_0 > 0$ and Assumption 6.1.2. Let Q_{flux} be the "energy flux" defined by (6.62) below and $\Sigma_\ell^{\text{inert}*}$ the "entropy flux" defined by (6.87). Assuming that the distributional limit of Q_ℓ^{flux} exists,*

$$Q_{\text{flux}} = \mathcal{D}\text{-}\lim_{\ell \to 0} Q_\ell^{\text{flux}} \tag{6.17}$$

then local energy and entropy balance equations hold in the sense of distributions on

$\mathbb{T}^d \times (0,T)$:

$$\partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right)+\nabla_x\cdot\left(\left(p+\frac{1}{2}\varrho|\boldsymbol{u}|^2\right)\boldsymbol{u}\right) = p\circ\Theta - Q_{\rm flux}, \tag{6.18}$$

$$\partial_t u+\nabla_x\cdot(u\boldsymbol{u}) = Q_{\rm flux}-p\circ\Theta, \tag{6.19}$$

$$\partial_t s+\nabla_x\cdot(s\boldsymbol{u}) = \Sigma_{\rm flux}. \tag{6.20}$$

where $\Sigma_{\rm flux}$ and $p\circ\Theta$ necessarily exist and are defined by the distributional limits

$$\Sigma_{\rm flux}=\mathcal{D}\text{-}\lim_{\ell\to 0}\Sigma_\ell^{\rm inert*},\qquad p\circ\Theta=\mathcal{D}\text{-}\lim_{\ell\to 0}(p*\mathcal{G}_\ell)(\Theta*\mathcal{G}_\ell), \tag{6.21}$$

with $\mathcal{G}_\ell$, $\ell>0$ a space-time mollifying sequence.

Remark This result is analogous to Proposition 2 of Duchon and Robert [2000] for weak solutions of incompressible Euler with $\boldsymbol{u}\in L^3([0,T]\times\mathbb{T}^d)$. In their theorem, the assumption on the existence of $Q_{\rm flux}$ was unnecessary. We need to add this as an additional hypothesis, because of the new term $p\circ\Theta$ that appears in the energy balance equations. Note $p\circ\Theta=0$ assuming incompressibility.

Remark Note that the second equation in (6.21) for $p\circ\Theta$ is a standard definition of a generalized distributional product of p and Θ Oberguggenberger [1992]. This standard definition requires that the limit be independent of the chosen mollifier $\mathcal{G}$. We note that for the purposes of Theorem 6.1.4, one could alternatively assume existence of $p\circ\Theta$ and then deduce it for $Q_{\rm flux}$. The combination $p\circ\Theta-Q_{\rm flux}$ always exists.

Our next results concern the strong limits of Navier-Stokes solutions satisfying Assumptions 6.1.1 – 6.1.3. First, we prove that these limits are necessarily weak solutions of the Euler equations, even if the limit dissipation measures in Assumption 6.1.3 remain positive: $Q > 0$ and $\Sigma > 0$. Moreover, we show that such solutions satisfy weak energy and entropy balance laws which include possible anomalies:

Theorem 6.1.5 *The strong limits $u, \varrho, \boldsymbol{u}$ of compressible Navier-Stokes solutions under Assumptions 6.1.1 – 6.1.3 are weak solutions of the compressible Euler system (6.1)–(6.3) on $\mathbb{T}^d \times [0,T]$. Furthermore, the following local energy and entropy equations hold in the sense of distributions on $\mathbb{T}^d \times (0,T)$:*

$$\begin{aligned} \partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right) + \nabla_x\cdot\left(\left(p+\frac{1}{2}\varrho|\boldsymbol{u}|^2\right)\boldsymbol{u}\right) &= p*\Theta - Q, && (6.22)\\ \partial_t u + \nabla_x\cdot(u\boldsymbol{u}) &= Q - p*\Theta, && (6.23)\\ \partial_t s + \nabla_x\cdot(s\boldsymbol{u}) &= \Sigma, && (6.24)\end{aligned}$$

with $Q \geq 0$ and $\Sigma \geq 0$ given by Assumption 6.1.3 and with

$$p*\Theta := \mathcal{D}\text{-}\lim_{\varepsilon\to 0} p^\varepsilon\Theta^\varepsilon, \qquad (6.25)$$

where this distributional limit necessarily exists.

Remark Theorem 6.1.5 is analogous to Proposition 4 of Duchon and Robert [2000] for the strong limits of solutions of the incompressible Navier-Stokes equation with viscosity tending to zero. Again, in their theorem, the analogue of our Assumption 6.1.3 was unnecessary, whereas we needed to add this as additional hypothesis because of the new term $p * \Theta$ defined by (6.25) that appears in the energy balance equations.

Remark Euler solutions obtained from Theorem 6.1.5 for vanishing viscosity necessarily satisfy Theorem 6.1.4 for general weak Euler solutions. It follows that:

$$\Sigma_{\text{flux}} = \Sigma \geq 0 \qquad \text{and} \qquad Q_{inert} := Q_{\text{flux}} + \tau(p, \Theta) = Q \geq 0, \tag{6.26}$$

where $\tau(p, \Theta)$ is the "pressure-dilatation defect" defined by

$$\tau(p, \Theta) = p * \Theta - p \circ \Theta. \tag{6.27}$$

The lefthand sides in (6.26) are "inertial-range" expressions for Q and Σ, analogous to those established in Proposition 1 and Section 5 of Duchon and Robert [2000] for incompressible fluids. In particular, Σ_{flux} and Q_{flux} describe "cascade" and can be expressed in terms of increments of the variables u, ϱ, $\boldsymbol{u}$ by analogues of the Kolmogorov "4/5th-law" for compressible turbulence. Whereas Σ_{flux}, Q_{flux} can have any signs for general weak Euler solutions, they are constrained by (6.26) for zero-viscosity solutions. The pressure-dilation defect defined in (6.27) is an additional source of anomalous energy dissipation that has no analogue for incompressible fluids.

Remark Shock solutions as discussed in Johnson [2014] and Eyink and Drivas [2017b] provide examples for which $Q > 0$ and $\Sigma > 0$ in (6.22)–(6.24). It is of some interest to note that for stationary, planar shocks with an ideal-gas equation of state, $Q = \tau(p, \Theta) > 0$, so that the entire contribution to Q is from the pressure-dilatation defect. See Eyink and Drivas [2017b] for this result. Although shock solutions with discontinuous state variables u, ϱ, $\boldsymbol{u}$ provide the simplest examples of weak Euler solutions with Q, Σ positive, presumably positive anomalies can occur even with continuous solutions.

We now state an analogue of the Onsager singularity theorem. We prove necessary conditions for anomalous dissipation involving Besov space exponents, as in the improvement by Constantin et al. [1994] of Onsager's Hölder-space statement. Here we note that the Besov space $B_p^{\sigma,\infty}(\Lambda)$ for a general bounded, Lipschitz domain $\Lambda \subset \mathbb{R}^D$ is made up of measurable functions $f : \Lambda \to \mathbb{R}$ which are finite in the norm:

$$\|f\|_{B_p^{\sigma,\infty}(\Lambda)} := \|f\|_{L^p(\Lambda)} + \sup_{h\in\mathbb{R}^D, |h|<1} \frac{\|f(\cdot + h) - f\|_{L^p(\Lambda\cap(\Lambda-h))}}{|h|^\sigma}, \tag{6.28}$$

for $p \geq 1$ and $\sigma \in (0, 1)$. Here the set $(\Lambda - h) := \{x - h : x \in \Lambda\}$. See Feireisl et al. [2017] and, for a general discussion, Triebel [2006], §1.11.9.

Theorem 6.1.6 *Let $u, \varrho, \boldsymbol{u} \in L^\infty(\mathbb{T}^d \times [0, T])$ be any weak solution of the compressible*

Euler system (6.1)–(6.3) satisfying $\varrho \geq \varrho_0 > 0$*, Assumption 6.1.2, and additionally*

$$u \in B_p^{\sigma_p^u,\infty}(\mathbb{T}^d \times [0,T]), \quad \varrho \in B_p^{\sigma_p^\varrho,\infty}(\mathbb{T}^d \times [0,T]), \quad \boldsymbol{v} \in B_p^{\sigma_p^v,\infty}(\mathbb{T}^d \times [0,T]),$$

with all three of the following conditions satisfied

$$2\min\{\sigma_p^u, \sigma_p^\varrho\} + \sigma_p^v \;>\; 1, \tag{6.29}$$

$$\min\{\sigma_p^u, \sigma_p^\varrho\} + 2\sigma_p^v \;>\; 1, \tag{6.30}$$

$$3\sigma_p^v \;>\; 1, \tag{6.31}$$

for any $p \geq 3$*. Then* Q_{flux}*,* Σ_{flux} *necessarily exist and equal zero. Further, inviscid limit solutions from Theorem 6.1.5 satisfying exponent conditions (6.29)-(6.31) have*

$$Q = \Sigma = 0 \quad \text{and} \quad p * \Theta = p \circ \Theta.$$

Thus, it is only possible that $Q > 0$ *or* $\Sigma > 0$ *if at least one of (6.29)–(6.31) fails to hold for each* $p \geq 3$*.*

Remark Our proof of Theorem 6.1.6 generalizes the argument of Constantin et al. [1994], which employed a simple mollification of the weak Euler solution. In fact, this idea can be exploited to give a new notion of "coarse-grained Euler solution" which we introduce in section 6.2 and show there to be equivalent to the standard notion of "weak solution," not only for compressible Euler equations but for very general balance relations. As discussed in Eyink and Drivas [2017b], the concept of "coarse-grained solution" makes connection with renormalization-group methods in physics. We employ this notion to prove both our Theorems 6.1.5 and 6.1.6. Our analysis of compressible Navier-Stokes and Euler solutions was directly motivated by the earlier work of Aluie Aluie [2013]. Our theorems also generalize previous results in Leslie and Shvydkoy [2016] for variable density incompressible fluids and in Feireisl et al. [2017] for barotropic compressible flow. It is worth noting that all of the results in this paper generalize readily to relativistic Euler equations in Minkowski spacetime, following the discussion in Eyink and Drivas [2017c].

Remark Our Theorem 6.1.6 is formulated in terms of space-time regularity, whereas the original statement of Onsager and most following works have given necessary conditions for anomalous dissipation in terms of space-regularity only. Note that our proof of Theorem 6.1.6 requires mollification/coarse-graining in time as well as space, and thus space-time regularity is natural for the proof (and also in the relativistic setting). However, we obtain conditions involving space-regularity only from the next theorem.

Theorem 6.1.7 *Let $u, \varrho, \boldsymbol{u}$ be any weak Euler solution satisfying $\varrho \geq \varrho_0 > 0$ and $\varrho, u, \boldsymbol{u} \in L^\infty(\mathbb{T}^d \times [0,T])$ together with:*

$$u \in L^\infty([0,T]; B_p^{\sigma_p^u,\infty}(\mathbb{T}^d)), \;\; \varrho \in L^\infty([0,T]; B_p^{\sigma_p^\varrho,\infty}(\mathbb{T}^d)), \;\; \boldsymbol{u} \in L^\infty([0,T]; B_p^{\sigma_p^v,\infty}(\mathbb{T}^d)),$$

for Besov exponents $0 \leq \sigma_p^u, \sigma_q^\varrho, \sigma_q^v \leq 1$. *Then the solutions are also Besov regular in space-time:*

$$u \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^v, \sigma_p^u\}, \infty}(\mathbb{T}^d \times [0,T]), \tag{6.32}$$

$$\varrho \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^u\}, \infty}(\mathbb{T}^d \times [0,T]), \tag{6.33}$$

$$\boldsymbol{u} \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^v, \sigma_p^u\}, \infty}(\mathbb{T}^d \times [0,T]). \tag{6.34}$$

Remark This result is very similar to that obtained in recent work of P. Isett for Hölder-continuous weak solutions of incompressible Euler Isett [2013], and the proof is almost the same. In fact, we shall derive Theorem 6.1.7 as a consequence of a more general result which derives time-regularity from space-regularity for a wide class of weak balance equations.

Remark It is interesting to know how sharp are the necessary conditions for anomalous dissipation following from Theorems 6.1.6 and 6.1.7. While answering this question for the incompressible case has required more sophisticated tools Isett [2012], Isett and Oh [2016a], Buckmaster [2015], we have a very cheap argument showing that our conditions are sharp for $p = 3$. In fact, shock solutions with $\varrho, u, \boldsymbol{u} \in BV \cap L^\infty$ provide a simple example of dissipative Euler solutions saturating our bounds. As noted in Feireisl et al. [2017], $BV(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d) \subset B_p^{1/p,\infty}(\mathbb{T}^d)$. For $p = 3$ this means that we may take $\sigma_3^u = \sigma_3^\varrho = \sigma_3^v = 1/3$ and then (6.29)–(6.31) are satisfied as equalities. For $p > 3$ the question remains open. Note that a standard Besov embedding gives

$B^{\sigma,\infty}_p(\mathbb{T}^d) \subset C^{\sigma-d/p}(\mathbb{T}^d)$ and $B^{\sigma,\infty}_p(\mathbb{T}^d \times [0,T]) \subset C^{\sigma-(d+1)/p}(\mathbb{T}^d \times [0,T])$ (see Triebel [2006], §1.11.1). Thus, if our necessary conditions are sharp, then dissipative solutions at the critical values for sufficiently large p must be Hölder-continuous.

The detailed contents of the present paper are as follows: In section 6.2 we introduce the space-time coarse-graining operation and prove the equivalence of distributional and coarse-grained solutions. In section 6.3 we derive balance equations for the coarse-grained compressible Navier-Stokes system. In section 6.4 we establish auxiliary commutator estimates necessary for our main theorems. In sections 6.5–6.8 we prove Theorems 6.1.4–6.1.7.

6.2 Coarse-Grained Solutions and Weak Solutions

We are concerned in this section with general balance equations of the form

$$\partial_t \boldsymbol{v} + \nabla_x \cdot \mathbf{F} = \mathbf{0} \tag{6.35}$$

on a space-time domain $\Omega \times \mathbb{R}$ where either $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$, for simplicity, and $\boldsymbol{v} \in \mathbb{R}^m$ and $\mathbf{F} \in \mathbb{R}^{d\times m}$. As usual, one defines $(\boldsymbol{v}, \mathbf{F})$ to be a *weak/distributional solution* of (6.35) iff

$$\langle \partial_t \varphi, \boldsymbol{v}\rangle + \langle \nabla_x \varphi; \mathbf{F}\rangle = \mathbf{0}, \quad \forall \varphi \in D(\Omega \times \mathbb{R}), \tag{6.36}$$

where the space $D(\Omega \times \mathbb{R}) = C_c^\infty(\Omega \times \mathbb{R})$ of test functions consists of C^∞ functions φ compactly supported in space-time, provided the topology defined by uniform convergence of functions and all their derivatives on compact sets containing all the supports. Components u_a, F_{ia} belong to the space $D'(\Omega \times \mathbb{R})$ of continuous linear functionals on $D(\Omega\times\mathbb{R})$, with $\langle \partial_t\varphi, \boldsymbol{v}\rangle_a = \langle \partial_t\varphi, u_a\rangle$ and $\langle \nabla_x\varphi; \mathbf{F}\rangle_a = \sum_{i=1}^d \langle \nabla_{x_i}\varphi, F_{ia}\rangle$ for $a = 1, \ldots, m$. For these standard notions, e.g. see Showalter [2011], Rudin [2006]. We offer here a slightly different point of view on these topics.

Let $\mathcal{G}$ be a standard space-time *mollifier*, with $\mathcal{G} \in D(\Omega \times \mathbb{R})$, $\mathcal{G} \geq 0$, and also $\int_\Omega d^dr \int_\mathbb{R} \mathrm{d}\tau\, \mathcal{G}(\boldsymbol{r}, \tau) = 1$. To simplify certain estimates we also assume, without loss of generality, that $\mathrm{supp}(\mathcal{G})$ is contained in the Euclidean unit ball in $(d+1)$ dimensions. Define the dilatation $\mathcal{G}_\ell(\boldsymbol{r}, \tau) = \ell^{-(d+1)}\mathcal{G}(\boldsymbol{r}/\ell, \tau/\ell)$ and space-time reflection $\check{\mathcal{G}}(\boldsymbol{r}, \tau) = \mathcal{G}(-\boldsymbol{r}, -\tau)$. For any $\boldsymbol{v} \in D'(\Omega \times \mathbb{R})$ we define its *coarse-graining at scale* ℓ by

$$\bar{\boldsymbol{v}}_\ell = \check{\mathcal{G}}_\ell * \boldsymbol{v} \in C^\infty(\Omega \times \mathbb{R}). \tag{6.37}$$

Here $*$ denotes the convolution defined by

$$(\check{\mathcal{G}}_\ell * \boldsymbol{v})(\boldsymbol{x}, t) = \langle S_{\boldsymbol{x},t}\mathcal{G}_\ell, \boldsymbol{v}\rangle \tag{6.38}$$

for shift operator $(S_{\boldsymbol{x},t}\mathcal{G}_\ell)(\boldsymbol{r}, \tau) = \mathcal{G}_\ell(\boldsymbol{r} - \boldsymbol{x}, \tau - t)$ or, equivalently, by

$$\langle \varphi, \check{\mathcal{G}}_\ell * \boldsymbol{v}\rangle = \langle \varphi * \mathcal{G}_\ell, \boldsymbol{v}\rangle \tag{6.39}$$

for all test functions $\varphi \in D(\Omega \times \mathbb{R})$. See Rudin [2006]. We say that $(\boldsymbol{v}, \mathbf{F})$ are a *(space-time) coarse-grained solution* of (6.35) iff

$$\partial_t \bar{\boldsymbol{v}}_\ell + \nabla_x \cdot \bar{\mathbf{F}}_\ell = \mathbf{0} \tag{6.40}$$

holds pointwise in space-time for all $\ell > 0$. We then have:

Proposition 6.2.1 $(\boldsymbol{v}, \mathbf{F})$ *are a distributional solution of (6.35) on* $\Omega \times \mathbb{R}$ *iff* $(\boldsymbol{v}, \mathbf{F})$ *are a coarse-grained solution of (6.35) on* $\Omega \times \mathbb{R}$

Proof If $(\boldsymbol{v}, \mathbf{F})$ satisfy (6.35) weakly, then taking $\varphi = S_{\boldsymbol{x},t}\mathcal{G}_\ell$ in (6.36) for any space-time point $(\boldsymbol{x}, t)$ implies (6.40) by the definition (6.38) of the convolution.

On the other hand, suppose that $(\boldsymbol{v}, \mathbf{F})$ are a coarse-grained solution of (6.35). Smearing (6.40) with an arbitrary test function $\varphi \in D(\Omega \times \mathbb{R})$, then gives by the second definition (6.39) of convolution that

$$\langle (\partial_t \varphi) * \mathcal{G}_\ell, \boldsymbol{v} \rangle + \langle (\nabla_x \varphi) * \mathcal{G}_\ell; \mathbf{F} \rangle = 0. \tag{6.41}$$

However, in the limit $\ell \to 0$, then $(\partial_t \varphi) * \mathcal{G}_\ell \to \partial_t \varphi$ and $(\nabla_x \varphi) * \mathcal{G}_\ell \to \nabla_x \varphi$ in the standard Fréchet topology on test functions. Since $\boldsymbol{v}$, $\mathbf{F} \in D'(\Omega \times \mathbb{R})$ are, by definition, continuous functionals on $D(\Omega \times \mathbb{R})$, the equation (6.36) of the standard weak formulation immediately follows. $\square$

This equivalence extends to solutions with prescribed initial-data. The standard approach to define weak solutions $(\boldsymbol{v}, \mathbf{F})$ of (6.35) on space-time domain $\Omega \times [0, \infty)$ with initial data $\boldsymbol{v}_0 \in D'(\Omega)$ is to require that

$$\langle \partial_t \varphi, \boldsymbol{v} \rangle + \langle \nabla_x \varphi; \mathbf{F} \rangle + \langle \varphi(\cdot, 0), \boldsymbol{v}_0 \rangle = \mathbf{0}, \quad \forall \varphi \in D(\Omega \times [0, \infty)), \tag{6.42}$$

where test functions $\varphi \in D(\Omega \times [0, +\infty)$ are *causal*, with $\varphi(\boldsymbol{x}, t) = 0$ for $t < 0$. In order to make this definition meaningful, one must require weak-$*$ continuity of the distribution $\boldsymbol{v}$ in time, so that $t \mapsto \langle \psi, \boldsymbol{v}(\cdot, t) \rangle$ is continuous for all $\psi \in D(\Omega)$. In that case, initial data is achieved in the sense that

$$\lim_{t \to 0+} \langle \psi, \boldsymbol{v}(\cdot, t) \rangle = \langle \psi, \boldsymbol{v}_0 \rangle, \quad \forall \psi \in D(\Omega). \tag{6.43}$$

The coarse-graining approach can be also carried over with only minor changes. The mollifier $\mathcal{G}$ must now be chosen to be *strictly causal*, with $\mathcal{G}(\boldsymbol{r}, \tau) \equiv 0$ for $\tau \leq 0$. The definition (6.37) of coarse-graining still applies, noting that the convolution in time is $(\chi_1 * \chi_2)(t) = \int_0^t \mathrm{d}s\ \chi_1(s)\chi_2(t-s)$ for causal functions χ_1, χ_2. We can again define $(\boldsymbol{v}, \mathbf{F})$ to be a coarse-grained solution of (6.35) if (6.40) holds pointwise in space-time for all $\ell > 0$. Since $\bar{\boldsymbol{v}}_\ell \in C^\infty(\Omega \times [0, \infty))$, the functions $\bar{\boldsymbol{v}}_\ell(\cdot, 0) \in C^\infty(\Omega)$ are well-defined and the coarse-grained solution is naturally said to take on initial data $\boldsymbol{v}_0 \in D'(\Omega)$ when

$$\mathcal{D}\text{-}\lim_{\ell \to 0} \bar{\boldsymbol{v}}_\ell(\cdot, 0) = \boldsymbol{v}_0. \tag{6.44}$$

It is straightforward to see for all $\psi \in D(\Omega)$ that

$$\langle \psi, \bar{\boldsymbol{v}}_\ell \rangle = \int d^d r \int_0^\infty d\tau \; \mathcal{G}_\ell(\boldsymbol{r},\tau)\Psi(\boldsymbol{r},t), \quad \Psi(\boldsymbol{r},\tau) := \langle S_{\boldsymbol{r}}\psi, \boldsymbol{v}(\cdot,\tau)\rangle. \tag{6.45}$$

Suppose that one requires not only weak-$*$ continuity of $\boldsymbol{v}$ in time, but also the stronger statement that $\Psi(\boldsymbol{r},\tau)$ defined in (6.45) is jointly continuous in $(\boldsymbol{r},\tau)$ for all $\psi \in D(\Omega)$. The initial data prescribed by (6.43) and (6.44) are then the same.

This leads to:

Proposition 6.2.2 *If $(\boldsymbol{v}, \mathbf{F})$ is a coarse-grained solution of (6.35) on $\Omega \times [0,\infty)$ with initial data $\boldsymbol{v}_0$, then it is a distributional solution with the same initial data. If also $\langle S_{\boldsymbol{r}}\psi, \boldsymbol{v}(\cdot,\tau)\rangle$ is jointly continuous in $(\boldsymbol{r},\tau)$ for all $\psi \in D(\Omega)$, then a distributional solution $(\boldsymbol{v}, \mathbf{F})$ of (6.35) on $\Omega \times [0,\infty)$ with initial data $\boldsymbol{v}_0$ is a coarse-grained solution with the same initial data.*

Proof To prove the first statement, smear the coarse-grained equation (6.40) with an arbitrary $\varphi \in D(\Omega \times [0,\infty)$. An integration-by-parts in time gives that

$$\langle (\partial_t \varphi) * \mathcal{G}_\ell, \boldsymbol{v}\rangle + \langle (\nabla_x \varphi) * \mathcal{G}_\ell; \mathbf{F}\rangle + \langle \varphi(\cdot,0), \bar{\boldsymbol{v}}_\ell\rangle = 0.$$

Taking the limit $\ell \to 0$ and using (6.44) recovers (6.42).

For the second statement, take $\varphi = S_{\boldsymbol{x},t}\mathcal{G}_\ell \in D(\Omega \times [0,\infty))$ for any $\boldsymbol{x} \in \Omega$ and $t \geq 0$. We see that φ is strictly causal, i.e. $\varphi(\cdot,0) = 0$. The equation (6.42) of the weak formulation thus yields the coarse-grained equation (6.40) for that choice of $(\boldsymbol{x}, t)$ and ℓ. Furthermore, because of (6.45) and the joint continuity of $\langle S_{\boldsymbol{r}}\psi, \boldsymbol{v}(\cdot,\tau)\rangle$ in $(\boldsymbol{r},\tau)$, $\bar{\boldsymbol{v}}_\ell(\cdot,0)\xrightarrow{\mathcal{D}}\boldsymbol{v}_0$ holds for the same $\boldsymbol{v}_0$ given by (6.43). $\square$

Remark If $\boldsymbol{v}\in C([0,\infty);L_p(\Omega))$ with continuity in the strong L_p-norm topology for some $p\geq 1$, then the joint continuity follows from the obvious continuity of $\Psi(\boldsymbol{r},\tau)$ in $\boldsymbol{r}$ for each τ and the Hölder inequality

$$|\Psi(\boldsymbol{r},\tau)-\Psi(\boldsymbol{r},\tau')|\leq \|\psi\|_q\|\boldsymbol{v}(\cdot,\tau)-\boldsymbol{v}(\cdot,\tau')\|_p, \quad q=p/(p-1),$$

which implies continuity of $\Psi(\boldsymbol{r},\tau)$ in τ uniform in $\boldsymbol{r}\in\Omega$.

Remark In Lemma 8 of [De Lellis & Szekelyhidi, 2010] it was proved that, if $(\boldsymbol{v},\mathbf{F})$ is a weak solution with $\boldsymbol{v}\in L^\infty([0,\infty),L^2(\Omega))$ and $\mathbf{F}\in L^1_{\rm loc}(\Omega\times[0,\infty)$, then $\boldsymbol{v}$ can always be altered on a zero measure set of times so that $\boldsymbol{v}\in C_w([0,\infty),L^2(\Omega))$, with continuity in the weak topology of $L^2(\Omega)$. In that case, $\Psi(\boldsymbol{r},\tau)$ defined for any $\psi\in D(\Omega)$ by (6.45) is continuous in τ for each $\boldsymbol{r}\in\Omega$. By Cauchy-Schwartz,

$$|\nabla_r\Psi(\boldsymbol{r},\tau)|\leq\|\nabla\psi\|_2\|\boldsymbol{v}\|_{L^\infty([0,\infty);L^2(\Omega))},$$

so that $\Psi(\boldsymbol{r},\tau)$ is also (Lipschitz) continuous in $\boldsymbol{r}$ uniformly in τ, and thus is jointly continuous in $(\boldsymbol{r},\tau)$ under the same assumptions as in [De Lellis & Szekelyhidi, 2010].

Remark The above results hold with only minor modifications for solutions on $\Omega\times$

$[0, T)$ with $0 < T < \infty$. Coarse-grained solutions are required now to satisfy equations (6.40) only for $\boldsymbol{x}, t$ and ℓ such that $S_{\boldsymbol{x},t}\mathcal{G}_\ell \in D(\Omega \times [0, T))$. On the other hand, for any $\varphi \in D(\Omega \times [0, T))$, then $T_\varphi = \max\{t : (\boldsymbol{x}, t) \in \mathrm{supp}(\varphi)\} < T$. Since $\mathrm{supp}(\mathcal{G})$ is contained in the unit ball, then $S_{\boldsymbol{x},t}\mathcal{G}_\ell \in D(\Omega \times [0, T))$ for any $\ell < T - T_\varphi$ and $(\boldsymbol{x}, t) \in \mathrm{supp}(\varphi)$ and our previous arguments on equivalence of the two notions of solution can be repeated without change.

Remark In the paper Constantin et al. [1994], only space mollification was employed. One can also define a space coarse-graining with a standard mollifier $G_\ell(\boldsymbol{r}) = \ell^{-d}G(\boldsymbol{r}/\ell)$, that is, $\hat{\boldsymbol{v}}_\ell = \check{G}_\ell * \boldsymbol{v}$. This is a smooth function of space but only a distribution in time. In that case, we say that $(\boldsymbol{v}, \mathbf{F})$ are a *(space) coarse-grained solution* of the balance relation (6.35) iff

$$\partial_t \hat{\boldsymbol{v}}_\ell + \nabla_x \cdot \hat{\mathbf{F}}_\ell = \mathbf{0} \tag{6.46}$$

holds pointwise in space and distributionally in time for all $\ell > 0$. This is also equivalent to the standard notion of weak solution, as can be seen by arguments very similar to those given above. If furthermore $\boldsymbol{v}$, $\mathbf{F} \in L^1_{loc}(\Omega \times [0, T])$, then standard approximation arguments show that the time-derivative in (6.46) can be taken to be a classical derivative at Lebesgue almost all times.

In many applications, including those considered in this paper, $\boldsymbol{v}$ is not merely a distribution but a measurable function of space-time, and $\mathbf{F} := \mathbf{F}(\boldsymbol{v})$ is a pointwise nonlinear function of $\boldsymbol{v}$. A key aspect of the coarse-graining operation is that coarse-graining nonlinear functions of fields generally gives a result different from evaluating the function at the coarse-grained fields, i.e. the operations of coarse-graining and function-evaluation do not commute. For simple products of the form $f_1 f_2 \cdots f_n$, this non-commutation can be measured by *coarse-graining cumulants*, which are defined iteratively in n by $\tau_\ell(f) = \bar{f}_\ell$ and

$$\overline{(f_1 \cdots f_n)}_\ell = \sum_\Pi \prod_{p=1}^{|\Pi|} \bar{\tau}_\ell(f_{i_1^{(p)}}, \ldots, f_{i_{n_p}^{(p)}}), \tag{6.47}$$

where the sum is over all partitions Π of the set $\{1, 2, \ldots, n\}$ into $|\Pi|$ disjoint subsets $\{i_1^{(p)}, \ldots, i_{n_p}^{(p)}\}$, $p = 1, \ldots, |\Pi|$. For example, for $n = 2$

$$\overline{(fg)}_\ell = \bar{f}_\ell \bar{g}_\ell + \bar{\tau}_\ell(f, g) \quad \text{or} \quad \bar{\tau}_\ell(f, g) = \overline{(fg)}_\ell - \bar{f}_\ell \bar{g}_\ell. \tag{6.48}$$

For general composed functions $h = h(f_1, \cdots, f_n)$ with h a smooth nonlinear function on $\mathbb{R}^n$, the non-commutation is measured by the quantity

$$\Delta_\ell h := \overline{h(f_1, \cdots, f_n)}_\ell - h(\overline{(f_1)}_\ell, \cdots, \overline{(f_n)}_\ell). \tag{6.49}$$

To simplify the writing of various expressions, we shall often use an "under-bar" notation to indicate the function evaluated at coarse-grained fields:

$$\underline{h}_\ell := h(\overline{(f_1)}_\ell, \cdots, \overline{(f_n)}_\ell), \tag{6.50}$$

whereas $\bar{h}_\ell = \overline{h(f_1, \cdots, f_n)}_\ell$.

6.3 Coarse-Grained Navier-Stokes and Balance Equations

We now discuss the results of coarse-graining the solutions of the compressible Navier-Stokes system. None of the results in this section depend upon the particular type of coarse-graining and are valid whether coarse-graining is in space, time, space-time or using some other averaging procedure (such as as weighted coarse-graining). We set $\varepsilon = 1$ in this section to simplify notations.

The coarse-grained Navier-Stokes equations for mass density ϱ, momentum density $\boldsymbol{j} = \varrho \boldsymbol{u}$, and energy density E are

$$\begin{aligned} \partial_t \bar{\varrho}_\ell \quad &+ \quad \nabla_x \cdot \bar{\mathbf{j}}_\ell = 0, && (6.51)\\ \partial_t \bar{\mathbf{j}}_\ell \quad &+ \quad \nabla_x \cdot \left(\overline{(\boldsymbol{j}\boldsymbol{u})}_\ell + \bar{p}_\ell \mathbf{I} + \overline{\mathbf{T}}_\ell \right) = \mathbf{0}, && (6.52)\\ \partial_t \overline{E}_\ell \quad &+ \quad \nabla_x \cdot \left(\overline{((E+p)\boldsymbol{u})}_\ell + \overline{(\mathbf{T}\cdot\boldsymbol{u})}_\ell + \bar{\boldsymbol{q}}_\ell \right) = 0. && (6.53) \end{aligned}$$

It is useful to rewrite the equations (6.51) and (6.52) employing the *Favre (density-weighted) averaging*:

$$\tilde{f}_\ell = \overline{(\varrho f)}_\ell / \bar{\varrho}_\ell. \tag{6.54}$$

One may likewise define cumulants $\tilde{\tau}_\ell(f_i, \ldots, f_n)$ with respect to this Favre filtering. See Favre [1969], Aluie [2013]. With this new averaging, (6.51)–(6.52) may be rewritten:

$$\partial_t \bar{\varrho}_\ell \quad + \quad \nabla_x \cdot (\bar{\varrho}_\ell \tilde{\boldsymbol{u}}_\ell) = 0, \tag{6.55}$$

$$\bar{\varrho}_\ell(\partial_t + \tilde{\boldsymbol{u}}_\ell \cdot \nabla_x)\tilde{\boldsymbol{u}}_\ell \quad + \quad \nabla_x \cdot \left(\tilde{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u}) + \bar{p}_\ell \mathbf{I} + \overline{\mathbf{T}}_\ell\right) = 0. \tag{6.56}$$

We emphasize that our use of Favre coarse-graining is mathematically only a matter of convenience, in order to reduce the number of terms in our coarse-grained equations (and to provide them with simple physical interpretations Aluie [2013], Eyink and Drivas [2017b]). Favre cumulants of $f_1, \ldots, f_n$ may always be rewritten in terms of unweighted cumulants of $f_1, \ldots, f_n$ and ϱ. For example Aluie [2013], Eyink [2015b]:

$$\tilde{f}_\ell = \bar{f}_\ell + \frac{1}{\bar{\varrho}} \bar{\tau}_\ell(\varrho, f), \tag{6.57}$$

$$\tilde{\tau}_\ell(f, g) = \bar{\tau}_\ell(f, g) + \frac{1}{\bar{\varrho}_\ell} \bar{\tau}_\ell(\varrho, f, g) - \frac{1}{\bar{\varrho}_\ell^2} \bar{\tau}_\ell(\varrho, f) \bar{\tau}_\ell(\varrho, g), \tag{6.58}$$

$$\begin{aligned}\tilde{\tau}_\ell(f, g, h) &= \bar{\tau}_\ell(f, g, h) + \frac{1}{\bar{\varrho}_\ell} \bar{\tau}_\ell(\varrho, f, g, h) \\ &- \frac{1}{\bar{\varrho}_\ell^2} [\bar{\tau}_\ell(\varrho, f) \bar{\tau}_\ell(\varrho, g, h) + \text{cyc. perm. } f, g, h] + \frac{2}{\bar{\varrho}_\ell^3} \bar{\tau}_\ell(\varrho, f) \bar{\tau}_\ell(\varrho, g) \bar{\tau}_\ell(\varrho, h).\end{aligned} \tag{6.59}$$

We next derive various balance equations for the coarse-grained fields.

Resolved Kinetic Energy: Following Aluie [2013], we consider a resolved kinetic energy $\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}|^2 = |\bar{\mathbf{j}}|_\ell^2/2\bar{\varrho}_\ell$. Using (6.55) and (6.56) one can derive its balance equation:

$$\partial_t\left(\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2\right) + \nabla_x \cdot \mathbf{J}_\ell^v = \bar{p}_\ell\overline{\Theta}_\ell - Q_\ell^{\text{flux}} - D_\ell^v, \tag{6.60}$$

where the various terms are defined by:

$$\mathbf{J}_\ell^v := \left(\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2 + \bar{p}_\ell\right)\tilde{\boldsymbol{u}}_\ell + \bar{\varrho}\tilde{\boldsymbol{u}}_\ell \cdot \tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}) - \frac{\bar{p}_\ell}{\bar{\varrho}_\ell}\bar{\tau}_\ell(\varrho,\boldsymbol{u}) + \tilde{\boldsymbol{u}}_\ell \cdot \overline{\mathbf{T}}_\ell, \tag{6.61}$$

$$Q_\ell^{\text{flux}} := \frac{\nabla_x\bar{p}_\ell}{\bar{\varrho}_\ell} \cdot \bar{\tau}_\ell(\varrho,\boldsymbol{u}) - \bar{\varrho}_\ell\nabla_x\tilde{\boldsymbol{u}}_\ell : \tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}), \tag{6.62}$$

$$D_\ell^v := -\nabla_x\tilde{\boldsymbol{u}}_\ell : \overline{\mathbf{T}}_\ell. \tag{6.63}$$

Equation (6.60) may be rewritten as

$$\partial_t\left(\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2\right) + \nabla_x \cdot \mathbf{J}_\ell^v = \overline{(p\Theta)}_\ell - Q_\ell^{\text{inert}} - D_\ell^v, \tag{6.64}$$

where the "inertial dissipation" is defined by

$$Q_\ell^{\text{inert}} := Q_\ell^{\text{flux}} + \bar{\tau}_\ell(p,\Theta). \tag{6.65}$$

Unresolved Kinetic Energy. We define this quantity (with summation over repeated

i indices) as

$$k_\ell := \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i). \tag{6.66}$$

A straightforward calculation yields

$$\partial_t k_\ell + \nabla \cdot \mathbf{J}_\ell^k = (\bar{\tau}_\ell(p,\Theta) - \overline{Q}_\ell) + Q_\ell^{\text{flux}} + D_\ell^k, \tag{6.67}$$

where

$$\begin{aligned} \mathbf{J}_\ell^k :&= \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i)\tilde{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(p, \boldsymbol{u}) + \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i, \boldsymbol{u}) && (6.68) \\ &\quad + \overline{(\mathbf{T} \cdot \boldsymbol{u})}_\ell - \overline{\mathbf{T}}_\ell \cdot \tilde{\boldsymbol{u}}_\ell, \\ D_\ell^k :&= -\overline{\mathbf{T}}_\ell : \nabla_x \tilde{\boldsymbol{u}}_\ell. && (6.69) \end{aligned}$$

Resolved Internal Energy: Directly coarse-graining equation (6.9), one finds the following balance equation for the resolved internal energy:

$$\partial_t \bar{u}_\ell + \nabla_x \cdot \mathbf{J}_\ell^u = \overline{Q}_\ell - \overline{(p\Theta)}_\ell, \tag{6.70}$$

where

$$\mathbf{J}_\ell^u = \overline{(u\boldsymbol{u})}_\ell + \bar{\boldsymbol{q}}_\ell = \bar{u}_\ell \overline{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(u, \boldsymbol{u}) + \bar{\boldsymbol{q}}_\ell. \tag{6.71}$$

A more important quantity for our analysis is

$$\bar{u}^*_\ell := \bar{u}_\ell + k_\ell, \tag{6.72}$$

which we term the "*intrinsic resolved internal energy*". Noting the straightforward identity $\overline{E}_\ell = \frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2 + \bar{u}_\ell + k_\ell$, one derives a balance equation for this intrinsic internal energy by subtracting the resolved kinetic energy balance (6.60) from the coarse-grained total energy equation (6.53):

$$\partial_t \bar{u}^*_\ell + \nabla_x \cdot \mathbf{J}^{u*}_\ell = Q^{\text{flux}}_\ell - \bar{p}_\ell \overline{\Theta}_\ell + D^k_\ell, \tag{6.73}$$

where D^k_ℓ is defined by equation (6.69) and

$$\begin{aligned} \mathbf{J}^{u*}_\ell \;\; &= \;\; \bar{u}_\ell \bar{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(h, \boldsymbol{u}) + \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i)\tilde{\boldsymbol{u}}_\ell + \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i, \boldsymbol{u}) \\ &\quad + \bar{\boldsymbol{q}}_\ell + \overline{(\mathbf{T}\cdot\boldsymbol{u})}_\ell - \overline{\mathbf{T}}_\ell \cdot \tilde{\boldsymbol{u}}_\ell, \end{aligned} \tag{6.74}$$

with $h := u + p$ defining the standard thermodynamic enthalpy.

Resolved Entropy: We derive an equation for $\underline{s}_\ell := s(\bar{u}_\ell, \bar{\varrho}_\ell)$ using (6.70), also (6.51) rewritten as

$$\partial_t \bar{\varrho}_\ell + \nabla_x \cdot (\bar{\varrho}_\ell \bar{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(\varrho, \boldsymbol{u})) = 0, \tag{6.75}$$

and the first law of thermodynamics:

$$\underline{T}_\ell \overline{\mathcal{D}}_t \underline{s}_\ell = \overline{\mathcal{D}}_t \bar{u}_\ell - \underline{\mu} \overline{\mathcal{D}}_t \bar{\varrho}_\ell, \tag{6.76}$$

with $\overline{\mathcal{D}}_t = \partial_t + \bar{\boldsymbol{u}}_\ell \cdot \nabla$ being the material derivative along the smoothed flow. One then finds that the resolved entropy satisfies:

$$\partial_t \underline{s}_\ell + \nabla_x \cdot \mathbf{J}_\ell^s = \frac{\overline{Q}_\ell - \bar{\tau}_\ell(p, \Theta)}{\underline{T}_\ell} - I_\ell^{\text{flux}} + \Sigma_\ell^{\text{flux}} + D_\ell^s, \tag{6.77}$$

where

$$\begin{aligned}
\mathbf{J}_\ell^s &:= \underline{s}_\ell \bar{\boldsymbol{u}}_\ell + \underline{\beta}_\ell \left(\bar{\tau}_\ell(u, \boldsymbol{u}) + \bar{\boldsymbol{q}}_\ell \right) - \underline{\lambda}_\ell \bar{\tau}_\ell(\varrho, \boldsymbol{u}), && (6.78) \\
I_\ell^{\text{flux}} &:= \underline{\beta}_\ell (\bar{p}_\ell - \underline{p}_\ell) \overline{\Theta}_\ell, && (6.79) \\
\Sigma_\ell^{\text{flux}} &:= \nabla_x \underline{\beta}_\ell \cdot \bar{\tau}_\ell(u, \boldsymbol{u}) - \nabla_x \underline{\lambda}_\ell \cdot \bar{\tau}_\ell(\varrho, \boldsymbol{u}), && (6.80) \\
D_\ell^s &:= -\frac{\bar{\boldsymbol{q}}_\ell \cdot \nabla_x \underline{T}_\ell}{\underline{T}_\ell^2}, && (6.81)
\end{aligned}$$

with $\beta := 1/T$ and $\lambda := \mu/T$. Considering the source terms on the righthand side of (6.77), we shall see that all of the terms marked "flux" satisfy simple bounds, and the direct dissipation term D_ℓ^s will be seen to vanish as $\varepsilon \to 0$, but the quantity $\overline{Q}_\ell - \bar{\tau}_\ell(p, \Theta)$ is more difficult to estimate. Fortunately, the same term appears in the balance equation for "unresolved kinetic energy."

Intrinsic Resolved Entropy: In order to cancel the difficult term $\overline{Q}_\ell - \bar{\tau}_\ell(p, \Theta)$, we introduce an "intrinsic resolved entropy density" by

$$\underline{s}_\ell^* := s(\bar{u}_\ell, \bar{\varrho}_\ell) + \underline{\beta}_\ell k_\ell. \tag{6.82}$$

From the homogenous Gibbs relation, $\underline{s}_\ell = \underline{\beta}_\ell(\bar{u}_\ell + \underline{p}_\ell) - \underline{\lambda}_\ell \bar{\varrho}_\ell$, it follows that $\underline{s}_\ell^* = \underline{\beta}_\ell(\bar{u}_\ell^* + \underline{p}_\ell) - \underline{\lambda}_\ell \bar{\varrho}_\ell$ where $\bar{u}_\ell^*$ is the intrinsic resolved internal energy defined in (6.72). Using Eq. (6.73) together with the standard thermodynamic relation

$$\overline{\mathcal{D}}_t(\underline{\beta}_\ell \underline{p}_\ell) = \bar{\varrho}_\ell \overline{\mathcal{D}}_t \underline{\lambda}_\ell - \bar{u}_\ell \overline{\mathcal{D}}_t \underline{\beta}_\ell, \tag{6.83}$$

it is straightforward to derive the balance equation for $\underline{s}_\ell^*$:

$$\partial_t \underline{s}_\ell^* + \nabla_x \cdot \mathbf{J}_\ell^{s*} = -I_\ell^{\mathrm{flux}} + \Sigma_\ell^{\mathrm{flux}*} + D_\ell^s + \underline{\beta}_\ell D_\ell^k \tag{6.84}$$

with

$$\mathbf{J}_\ell^{s*} := \mathbf{J}_\ell^s + \underline{\beta}_\ell \mathbf{J}_\ell^k, \tag{6.85}$$

$$\Sigma_\ell^{\mathrm{flux}*} := \Sigma_\ell^{\mathrm{flux}} + \underline{\beta}_\ell Q_\ell^{\mathrm{flux}} + \partial_t \underline{\beta}_\ell \, k_\ell + \nabla_x \underline{\beta}_\ell \cdot \mathbf{J}_\ell^k. \tag{6.86}$$

We also then write

$$\Sigma_\ell^{\mathrm{inert}*} = -I_\ell^{\mathrm{flux}} + \Sigma_\ell^{\mathrm{flux}*} \tag{6.87}$$

for the net "inertial" production of the intrinsic entropy. The balance equation (6.84) of the intrinsic entropy turns out to be the key identity for the proof of Theorem 6.1.6. On the righthand side, the direct dissipation terms will be shown to vanish as $\varepsilon \to 0$ and the remaining terms are "flux-like" and depend only upon increments of the basic variables u, ϱ, $\boldsymbol{u}$. This latter result follows from commutator estimates of Section 6.4.

Note that the balance equations (6.60) for resolved kinetic energy, (6.73) for intrinsic resolved internal energy and (6.84) for intrinsic resolved entropy are valid for general weak Euler solutions after setting $\mathbf{T} = \mathbf{q} = \mathbf{0}$, without the need for considering the viscous regularization with $\varepsilon > 0$ and taking $\varepsilon \to 0$. On the other hand, the balance equations (6.67) for unresolved kinetic energy, (6.70) for resolved internal energy, and (6.77) for resolved entropy are valid with $\mathbf{T} = \mathbf{q} = \mathbf{0}$ only for weak Euler solutions obtained from the inviscid limit. In fact, the latter equations contain the quantities $\overline{Q}_\ell$ and $\bar{\tau}_\ell(p, \Theta)$ which are *a priori* undefined for general weak Euler solutions.

6.4 Commutator Estimates

The estimates that we derive in this section are valid for coarse-graining in space, time, or space-time. We state them here for the space-time coarse-graining that we use in our proofs of Theorems 6.1.4–6.1.7. The need for coarse-graining in time as well as in space is due to the time-derivative term in expression (6.86) for $\Sigma_\ell^{\mathrm{flux}*}$. In order to present the estimates, it is useful to employ a "space-time vector" notation, with $X = (\boldsymbol{x}, ct)$, $R = (\boldsymbol{r}, c\tau)$ where c is a constant with dimensions of velocity which is fixed independent of ϵ and ℓ. For example, we may take c to be the speed of sound (or, in the relativistic case, the speed of light). We correspondingly take the $(d+1)$-dimensional compact domain $\Gamma = \mathbb{T}^d \times [0, T]$ and consider coarse-graining of functions $f_i \in L^\infty(\Gamma)$, $i = 1, 2, 3, \ldots$ with a non-negative, standard mollifier $\mathcal{G} \in C^\infty(\Gamma)$. We assume, for convenience, that $\mathrm{supp}(\mathcal{G})$ is contained in the Euclidean unit ball.

A basic result is the following:

Lemma 6.4.1 *For $n > 1$, the coarse-graining cumulants are related to cumulants of the difference fields $\delta f(R; X) := f(X + R) - f(X)$ as follows:*

$$\tau_\ell(f_1, \ldots, f_n) = \langle \delta f_1, \ldots, \delta f_n \rangle_\ell^c, \tag{6.88}$$

where $\langle \cdot \rangle_\ell$ denotes average over the displacement vector R with density $\mathcal{G}_\ell(R)$ and the superscript c indicates the cumulant with respect to this average.

This result is proved in Constantin et al. [1994] for $n = 2$ and, in the more general form quoted here, in Eyink [2015c] or Eyink [2015b], Appendix B. The proof is any easy application of the invariance of cumulants of "random variables" to shifts of those variables by "non-random" constants. A direct consequence of Lemma 6.4.1 is:

Proposition 6.4.2 (cumulant estimates) *With $\| \cdot \|_p$ the standard norms in $L_p(\Gamma)$*

for $p \in [1, \infty]$ *and* $n > 1$

$$\|\tau_\ell(f_1, \ldots, f_n)\|_p = O\left(\prod_{i=1}^{n} \|\delta f_i(\ell)\|_{p_i}\right) \quad \text{with} \quad \frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}, \tag{6.89}$$

where $\|\delta f(\ell)\|_p := \sup_{|R|<\ell} \|\delta f(R)\|_p$. *Assuming* $f_i \in B^{\sigma_i,\infty}_{p_i}(\Gamma)$ *with* $0 < \sigma_i \leq 1$ *for* $i = 1, \ldots, n$*:*

$$\|\tau_\ell(f_1, \ldots, f_n)\|_p = O\left(\ell^{\sum_{i=1}^{n} \sigma_i}\right), \tag{6.90}$$

If only $f_i \in L^\infty(\Gamma)$, *then at least*

$$\lim_{\ell \to 0} \|\tau_\ell(f_1, \ldots, f_n)\|_p = 0, \quad 1 \leq p < \infty, \tag{6.91}$$

but without an estimate of the rate.

Here "big-O" notation, as usual, means inequality up to a constant independent of ℓ, which in this case depends on the details of the mollifier $\mathcal{G}$. The final statement is a consequence of the bound (6.89) and the strong continuity of the shift operators $(S_{-\boldsymbol{r}} f)(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{r})$ in the L^p-norm, a standard fact which follows from a simple density argument.

We also need bounds on space-time derivatives of the cumulants. This can be accomplished using the fact that all derivatives of cumulants with respect to X can be transferred to space-derivatives of the filter kernels $\mathcal{G}_\ell(R)$ with respect to R. This is another consequence of the invariance of cumulants to constant shifts; see Eyink

[2015c] or Eyink [2015b]. For example, with

$$
\begin{aligned}
\frac{\partial}{\partial X_k}\bar{\tau}_\ell(f_i) &= \frac{\partial\overline{(f_i)}_\ell}{\partial X_k} = -\frac{1}{\ell}\int \mathrm{d}^{d+1}R\ \left(\frac{\partial \mathcal{G}}{\partial R_k}\right)_\ell (R)\delta f_i(R), &\quad (6.92)\\
\frac{\partial}{\partial X_k}\bar{\tau}_\ell(f_i,f_j) &= -\frac{1}{\ell}\left\{\int \mathrm{d}^{d+1}R\ \left(\frac{\partial \mathcal{G}}{\partial R_k}\right)_\ell (R)\delta f_i(R)\delta f_j(R)\right.\\
&\quad -\int \mathrm{d}^{d+1}R\ \left(\frac{\partial \mathcal{G}}{\partial R_k}\right)_\ell (R)\delta f_i(R)\int \mathrm{d}R'\mathcal{G}_\ell(r')\delta f_j(R')\\
&\quad \left.-\int \mathrm{d}^{d+1}R\ \mathcal{G}_\ell(R)\delta f_i(R)\int \mathrm{d}R'\left(\frac{\partial \mathcal{G}}{\partial R'_k}\right)_\ell (R')\delta f_j(R')\right\}, &\quad (6.93)
\end{aligned}
$$

and so forth. Using expressions of this type, one obtains bounds of the form:

Proposition 6.4.3 (cumulant-derivative estimates) *For $n \geq 1$ and $\partial_k = \partial/\partial X_k$*

$$
\|\partial_{k_1}\cdots\partial_{k_m}\tau_\ell(f_1,\ldots,f_n)\|_p = O\left(\ell^{-m}\prod_{i=1}^{n}\|\delta f_i(\ell)\|_{p_i}\right) \quad \text{with} \quad \frac{1}{p} = \sum_{i=1}^{n}\frac{1}{p_i}. \tag{6.94}
$$

Assuming $f_i \in B^{\sigma_i,\infty}_{p_i}(\Gamma)$ with $0 < \sigma_i \leq 1$ for $i = 1,\ldots,n$:

$$
\|\partial_{k_1}\cdots\partial_{k_m}\tau_\ell(f_1,\ldots,f_n)\|_p = O\left(\ell^{-m+\sum_{i=1}^{n}\sigma_i}\right). \tag{6.95}
$$

For the "unresolved" or "fluctuation" part of a field $f'_\ell := f - \overline{f}_\ell$, we have the simple formula

$$
f'_\ell(X) = -\int \mathrm{d}^{d+1}R\ \mathcal{G}_\ell(R)\delta f(R;X), \tag{6.96}
$$

which gives

Proposition 6.4.4 (fluctuation estimates) *For $p \in [1,\infty]$, $\|f'_\ell\|_p = O\left(\|\delta f(\ell)\|_p\right)$ and*

$\|f'_\ell\|_p = O(\ell^\sigma)$ *when also* $f \in B^\sigma_p(\Gamma)$ *for* $0 < \sigma \leq 1$.

Finally, we will also require estimates on $\Delta_\ell h = \bar{h}_\ell - \underline{h}_\ell$ for composite functions of the form $h(f,g)$, where $f,g \in L^\infty(\Gamma)$ and h is a smooth function of two variables. We have the following Lemma:

Lemma 6.4.5 *Let* $f \in (B^{\sigma^f_p,\infty}_p \cap L^\infty)(\Gamma)$ *and* $g \in (B^{\sigma^g_p,\infty}_p \cap L^\infty)(\Gamma)$. *Let* $U \subset \mathbb{R}^2$ *be open and containing the closed convex hull of* $\mathcal{R} = \text{ess.ran}(f,g)$*, the essential range of the measurable function* $(f,g) \in L^\infty(\Gamma,\mathbb{R}^2)$. *Consider* $H := h(f,g)$ *with* $h \in C^1(U,\mathbb{R})$. *Then* $H \in (B^{\min\{\sigma^f_p,\sigma^g_p\},\infty}_p \cap L^\infty)(\Gamma)$, $p \geq 1$.

Proof Clearly, $H \in L^\infty(\Gamma)$. Since $h \in C^1(U,\mathbb{R})$, the mean value theorem gives:

$$\begin{aligned} \delta H(R;X) &:= h(f(X+R),g(X+R)) - h(f(X),g(X)) \\ &= (\delta f(R;X),\delta g(R;X)) \cdot \vec{\partial} h(f_*,g_*) \end{aligned} \tag{6.97}$$

for (f_*,g_*) on the line segment joining $(f(X),g(X))$, $(f(X+R),g(X+R))$. We have used the notation $\vec{\partial} = (\partial/\partial f, \partial/\partial g)$. Since $\mathcal{R} \subset U$ is compact, then so also is its closed convex hull $\text{conv}(\mathcal{R}) \subset U$ and $\vec{\partial}h$ is bounded on $\text{conv}(\mathcal{R})$. It follows that for any $p \geq 1$, $\|\delta h(R)\|_p = O\left(|R|^{\min\{\sigma^f_p,\sigma^g_p\}}\right)$. $\square$

Corollary 6.4.6 *Let* f,g *be as in Lemma 6.4.5. Then* $fg \in (B^{\min\{\sigma^f_p,\sigma^g_p\},\infty}_p \cap L^\infty)(\Gamma)$.

The estimate on $\Delta_\ell h = \bar{h}_\ell - \underline{h}_\ell$ is as follows:

Proposition 6.4.7 *Let $h \in C^2(U)$ with f, g and U as in Lemma 6.4.5. Then*

$$\|\Delta_\ell h\|_{p/2} = O\left(\ell^{2\min\{\sigma_p^f, \sigma_p^g\}}\right), \quad p \geq 2. \tag{6.98}$$

Assuming only $f, g \in L^\infty(\Gamma)$, then at least

$$\lim_{\ell \to 0} \|\Delta_\ell h\|_{p/2} = 0, \quad 2 \leq p < \infty, \tag{6.99}$$

but without an estimate of the rate.

Proof Using the notation $\vec{\partial} = (\partial/\partial f, \partial/\partial g)$, we have:

$$\begin{aligned} \Delta_\ell h \quad &:= \quad \overline{h(f,g)}_\ell - h(\bar{f}_\ell, \bar{g}_\ell) \\ &= \quad \left(\overline{h(f,g)}_\ell - h(f,g) + (f'_\ell, g'_\ell) \cdot \vec{\partial} h(f,g)\right) \\ &\qquad + \left(h(f,g) - h(\bar{f}_\ell, \bar{g}_\ell) - (f'_\ell, g'_\ell) \cdot \vec{\partial} h(f,g)\right). \end{aligned}$$

The first term can be rewritten as

$$\begin{aligned} &\overline{h(f,g)}_\ell - h(f,g) + (f'_\ell, g'_\ell) \cdot \vec{\partial} h(f,g) \\ &= \int_\Gamma d^{d+1}R\ \mathcal{G}_\ell(R)\ \Big(h(f(X+R), g(X+R)) - h(f(X), g(X)) \\ &\qquad\qquad - (\delta f(R;X), \delta g(R;X)) \cdot \vec{\partial} h(f(X), g(X))\Big) \\ &= \int_\Gamma d^{d+1}R\ \mathcal{G}_\ell(R)\ (\vec{\partial}\vec{\partial})h\Big|_{(f_*, g_*)} : (\delta f(R;X), \delta g(R;X))(\delta f(R;X), \delta g(R;X)), \end{aligned}$$

where in the second equality the Taylor theorem with remainder was employed and (f_*, g_*) is defined similarly as in Lemma 6.4.5. Likewise, using $f = \overline{f}_\ell + f'_\ell$, the second term can be rewritten as

$$h(f,g) - h(\overline{f}_\ell, \bar{g}_\ell) - (f'_\ell, g'_\ell)\cdot\vec{\partial}h(f,g)$$
$$= (\vec{\partial}\vec{\partial})h\Big|_{(f_\star,g_\star)} : (f'_\ell, g'_\ell)(f'_\ell, g'_\ell),$$

and $(f_\star, g_\star)$ is a point on the line segment connecting $(\overline{f}_\ell(X), \bar{g}_\ell(X))$,$(f(X), g(X))$. Note that $(\overline{f}_\ell(X), \bar{g}_\ell(X)) \in \mathrm{conv}(\mathcal{R})$ because the coarse-grained field with a non-negative mollifier $\mathcal{G}_\ell$ is a limit of averages of values in ess.ran.(f, g). Thus, $(\vec{\partial}\vec{\partial})h\Big|_{(f_\star,g_\star)}$ is uniformly bounded, since $(\vec{\partial}\vec{\partial})h$ is bounded on $\mathrm{conv}(\mathcal{R})$. It follows from the above formulas, the Hölder inequality, and Proposition 6.4.4 that

$$\|\Delta_\ell h\|_{p/2} = O\left(\max\{\|\delta f(\ell)\|_p, \|\delta g(\ell)\|_p\}^2\right). \tag{6.100}$$

The above estimate immediately yields $\|\Delta_\ell h\|_{p/2} = O\left(\ell^{2\min\{\sigma_p^f,\sigma_p^g\}}\right)$ assuming the appropriate Besov regularity.

The final statement of the proposition is obtained from the estimate (6.100) and the strong continuity of the shift operators in the L^p-norm. □

One last estimate will be needed:

Proposition 6.4.8 *Let $h \in C^1(U)$ with f, g and U as in Lemma 6.4.5. Then*

$$\|\nabla_x \underline{h}_\ell\|_p = O\left(\ell^{\min\{\sigma_p^f, \sigma_p^g\}-1}\right), \quad p \geq 1. \tag{6.101}$$

Proof By the chain rule, $\nabla_x \underline{h} = \vec{\partial} h(\bar{f}_\ell, \bar{g}_\ell) \cdot (\nabla_x \bar{f}_\ell, \nabla_x \bar{g}_\ell)$. Since $(\bar{f}_\ell, \bar{g}_\ell)$ is in the closed convex hull of $\mathcal{R}$, one immediately obtains from Proposition 6.4.3 that

$$\|\nabla_x \underline{h}_\ell\|_p = O\left(\frac{1}{\ell} \max\{\|\delta f(\ell)\|_p, \|\delta g(\ell)\|_p\}\right), \tag{6.102}$$

which gives the claimed estimate for the assumed Besov regularity. $\square$

6.5 Proof of Theorem 6.1.4

We consider in order the three balance equations (6.18)–(6.20) in Theorem 1.

Kinetic Energy: Setting $\varepsilon = 0$, the coarse-grained kinetic energy balance (6.64) for compressible Navier-Stokes simplifies, because terms involving $\mathbf{T}^\varepsilon$ vanish:

$$\partial_t \left(\frac{1}{2} \bar{\varrho}_\ell |\tilde{\boldsymbol{u}}_\ell|^2\right) + \nabla_x \cdot \mathbf{J}_\ell^v = \bar{p}_\ell \overline{\Theta}_\ell - Q_\ell^{\text{flux}}, \tag{6.103}$$

where the various terms are defined by:

$$\mathbf{J}_\ell^v := \left(\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2 + \bar{p}_\ell\right)\tilde{\boldsymbol{u}}_\ell + \bar{\varrho}_\ell\tilde{\boldsymbol{u}}_\ell \cdot \tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}) - \frac{\bar{p}_\ell}{\bar{\varrho}_\ell}\bar{\tau}_\ell(\varrho,\boldsymbol{u}), \tag{6.104}$$

$$Q_\ell^{\text{flux}} := \frac{\nabla_x\bar{p}_\ell}{\bar{\varrho}_\ell} \cdot \bar{\tau}_\ell(\varrho,\boldsymbol{u}) - \bar{\varrho}_\ell\nabla_x\tilde{\boldsymbol{u}}_\ell : \tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}). \tag{6.105}$$

We now consider the limit as $\ell \to 0$ of the equation (6.103). Of course, by standard results, $\bar{u}_\ell$, $\bar{\varrho}_\ell$, $\overline{\boldsymbol{u}}_\ell$, $\bar{p}_\ell \to u$, ϱ, $\boldsymbol{u}$, p strong in L^p for any $1 \leq p < \infty$. As a special case of (6.57)

$$\tilde{\boldsymbol{u}}_\ell = \overline{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(\varrho,\boldsymbol{u})/\bar{\varrho}_\ell, \tag{6.106}$$

which implies for any $p \geq 1$ that

$$\|\tilde{\boldsymbol{u}}_\ell - \boldsymbol{u}\|_p \leq \|\overline{\boldsymbol{u}}_\ell - \boldsymbol{u}\|_p + \|1/\varrho\|_\infty \|\bar{\tau}_\ell(\varrho,\boldsymbol{u})\|_p,$$

so that $\tilde{\boldsymbol{u}}_\ell \to \boldsymbol{u}$ strongly as well. Here (6.91) of Proposition 6.4.2 was used. We infer that $\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2$ converges to $\frac{1}{2}\varrho|\boldsymbol{u}|^2$ strong in L^p for any $p \geq 1$, and thus

$$\partial_t\left(\frac{1}{2}\bar{\varrho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2\right) \xrightarrow{\mathcal{D}} \partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right) \tag{6.107}$$

as $\ell \to 0$. Using the special case of (6.58)

$$\tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}) = \bar{\tau}_\ell(\boldsymbol{u},\boldsymbol{u}) + \frac{1}{\bar{\varrho}_\ell}\bar{\tau}_\ell(\varrho,\boldsymbol{u},\boldsymbol{u}) - \frac{1}{\bar{\varrho}_\ell^2}\bar{\tau}_\ell(\varrho,\boldsymbol{u})\bar{\tau}_\ell(\varrho,\boldsymbol{u}), \tag{6.108}$$

one obtains by exactly similar arguments with Proposition 6.4.2 that

$$\nabla_x \cdot \mathbf{J}_\ell^v \xrightarrow{\mathcal{D}} \nabla_x \left((\frac{1}{2}\varrho|\boldsymbol{u}|^2 + p)\boldsymbol{u} \right). \tag{6.109}$$

Also, under our assumptions, Q_ℓ^{flux} has a distributional limit:

$$Q_\ell^{\text{flux}} \xrightarrow{\mathcal{D}} Q_{\text{flux}}. \tag{6.110}$$

Thus, all of the terms in (6.103) except $\bar{p}_\ell\overline{\Theta}_\ell$ have been proved to have distributional limits as $\ell \to 0$. It follows that the limit of $\bar{p}_\ell\overline{\Theta}_\ell$ also exists and equals $-Q_{\text{flux}} - \partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right) - \nabla_x\left((\frac{1}{2}\varrho|\boldsymbol{u}|^2 + p)\boldsymbol{u}\right)$, independent of choice of $\mathcal{G}$. Thus,

$$\bar{p}_\ell\overline{\Theta}_\ell \xrightarrow{\mathcal{D}} p \circ \Theta \tag{6.111}$$

which completes the derivation of the kinetic energy balance (6.18).

Internal Energy: Note that the internal energy constructed as $u = E - \frac{1}{2}\varrho|\boldsymbol{u}|^2$, satisfies (6.19) distributionally. This could be alternatively deduced by considering the $\ell \to 0$ limit of the intrinsic resolved internal energy balance (6.73) with $\varepsilon = 0$.

Entropy: Setting $\varepsilon = 0$ in the intrinsic resolve entropy equation (6.84), we obtain

$$\partial_t \underline{s}_\ell^* + \nabla_x \cdot \mathbf{J}_\ell^{s*} = \Sigma_\ell^{\text{inert}*}, \tag{6.112}$$

for

$$\mathbf{J}_\ell^{s*} := \mathbf{J}_\ell^s + \underline{\beta}_\ell \mathbf{J}_\ell^k, \tag{6.113}$$

$$\mathbf{J}_\ell^s := \underline{s}_\ell \bar{\boldsymbol{u}}_\ell + \underline{\beta}_\ell \bar{\tau}_\ell(u, \boldsymbol{u}) - \underline{\lambda}_\ell \bar{\tau}_\ell(\varrho, \boldsymbol{u}), \tag{6.114}$$

$$\mathbf{J}_\ell^k := \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i)\tilde{\boldsymbol{u}}_\ell + \bar{\tau}_\ell(p, \boldsymbol{u}) + \frac{1}{2}\bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i, \boldsymbol{u}), \tag{6.115}$$

and, with $\Sigma_\ell^{\text{inert}*} = -I_\ell^{\text{flux}} + \Sigma_\ell^{\text{flux}*}$, for

$$I_\ell^{\text{flux}} := \underline{\beta}_\ell(\bar{p}_\ell - \underline{p}_\ell)\overline{\Theta}_\ell, \tag{6.116}$$

$$\Sigma_\ell^{\text{flux}*} := \Sigma_\ell^{\text{flux}} + \underline{\beta}_\ell Q_\ell^{\text{flux}} + \partial_t \underline{\beta}_\ell \ k_\ell + \nabla_x \underline{\beta}_\ell \cdot \mathbf{J}_\ell^k, \tag{6.117}$$

$$\Sigma_\ell^{\text{flux}} := \nabla_x \underline{\beta}_\ell \cdot \bar{\tau}_\ell(u, \boldsymbol{u}) - \nabla_x \underline{\lambda}_\ell \cdot \bar{\tau}_\ell(\varrho, \boldsymbol{u}). \tag{6.118}$$

We next show that $\partial_t \underline{s}_\ell^* + \nabla_x \cdot \mathbf{J}_\ell^{s*} \xrightarrow{\mathcal{D}} \partial_t s + \nabla_x \cdot (s\boldsymbol{u})$ as $\ell \to 0$. Note that

$$\|s(\bar{u}_\ell, \bar{\varrho}_\ell) - s(u, \varrho)\|_p \le \|\overline{s(u, \varrho)}_\ell - s(u, \varrho)\|_p + \|\overline{s(u, \varrho)}_\ell - s(\bar{u}_\ell, \bar{\varrho}_\ell)\|_p.$$

Obviously $\bar{s}_\ell \to s$ strong in L^p for $p \ge 1$, but also $\|\Delta_\ell s\|_p \to 0$ by (6.99) of Proposition 6.4.7. Thus, $\underline{s}_\ell \to s$ strong in L^p. Also, $\|\underline{\beta}_\ell k_\ell\|_p \to 0$ by (6.91) of Proposition 6.4.2. It follows that $\bar{s}_\ell^* \to s$ strong in L^p for $p \ge 1$ and thus

$$\partial_t \underline{s}_\ell^* \xrightarrow{\mathcal{D}} \partial_t s(u, \varrho).$$

Using the formula (6.108) for $\tilde{\tau}_\ell(\boldsymbol{v},\boldsymbol{v})$ and the similar formula for $\tilde{\tau}_\ell(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v})$ that follows from (6.59), then similar arguments with Propositions 6.4.2 and 6.4.7 show that $\mathbf{J}_\ell^{s*}\xrightarrow{\mathcal{D}} s\boldsymbol{u}$ strong in L^p for $p \geq 1$ and thus

$$\nabla_x \cdot \mathbf{J}_\ell^{s*} \xrightarrow{\mathcal{D}} \nabla_x \cdot (s(u,\varrho)\boldsymbol{u})\,.$$

We infer from (6.112) that the distributional limit of $\Sigma_\ell^{\text{inert}*}$ as $\ell \to 0$ exists and is equal to $\Sigma_{\text{flux}} := \partial_t s + \nabla_x \cdot (s\boldsymbol{u})$. Thus, entropy balance (6.20) holds, with

$$\Sigma_\ell^{\text{inert}*} \xrightarrow{\mathcal{D}} \Sigma_{\text{flux}}. \tag{6.119}$$

This completes the proof of Theorem 6.1.4. □

6.6 Proof of Theorem 6.1.5

To prove that the strong limits of u^ε, ϱ^ε, $\boldsymbol{u}^\varepsilon$ in $L^p(\Gamma)$ for some $1 \leq p < \infty$ as $\varepsilon \to 0$ satisfy the Euler equations weakly, we use the concept of "coarse-grained solution" discussed in section 6.2. The coarse-grained Navier-Stokes system with transport coefficients scaled by ε appears the same as (6.51)–(6.53) except that there is now a factor ε implicitly contained in the terms $\mathbf{T}^\varepsilon$ and $\boldsymbol{q}^\varepsilon$ wherever they appear. Our strategy shall be to show that, pointwise in space-time, these terms indeed vanish as $\varepsilon \to 0$, while all of the other terms in the coarse-grained Navier-Stokes equation converge pointwise as $\varepsilon \to 0$ to the corresponding terms in the coarse-grained Euler equations for the limiting fields u, ϱ, $\boldsymbol{u}$.

We first note that the properties that (i) $\|f^\varepsilon\|_\infty$ is bounded uniformly in ε and (ii) the $L^p(\Gamma)$ norms $\|f^\varepsilon - f\|_p \to 0$ for some $1 \leq p < \infty$ as $\varepsilon \to 0$ for the basic fields $f^\varepsilon = u^\varepsilon$, ϱ^ε, $\boldsymbol{u}^\varepsilon$ immediately implies that the same is true for simple product functions such as $\boldsymbol{j}^\varepsilon = \varrho^\varepsilon \boldsymbol{u}^\varepsilon$, $\varrho^\varepsilon|\boldsymbol{u}^\varepsilon|^2$, $\varrho^\varepsilon|\boldsymbol{u}^\varepsilon|^2\boldsymbol{u}^\varepsilon$, etc. For compositions $h^\varepsilon := h(u^\varepsilon, \varrho^\varepsilon)$ with thermodynamic functions such as $h = T,\ p,\ \mu,\ \eta,\ \zeta,\ \kappa$ we need the precise Assumption 6.1.2 on smoothness of h with $M = 1$. Of course, $\mathcal{R}^\varepsilon, \mathcal{R} \subset K$ for $\varepsilon < \varepsilon_0$, so that $\|h^\varepsilon\|_\infty$ is bounded uniformly for $\varepsilon < \varepsilon_0$ and $\|h\|_\infty$ satisfies the same bound. Furthermore, we can write

$$\begin{aligned} h(u^\varepsilon(X), \varrho^\varepsilon(X)) &- h(u(X), \varrho(X)) \\ &= \vec{\partial} h(u_*, \varrho_*) \cdot (u^\varepsilon(X) - u(X), \varrho^\varepsilon(X) - \varrho(X)), \end{aligned} \tag{6.120}$$

where (u_*, ϱ_*) is on the line segment between $(u^\varepsilon(X), \varrho^\varepsilon(X))$ and $(u(X), \varrho(X))$. Since $(u_*, \varrho_*) \in K$, then, by Assumption 6.1.2, the 2-vector ℓ_q-norm $|\vec{\partial} h(u_*, \varrho_*)|_q$ with $q = p/(p-1)$ is bounded by the maximum value $C_{h,q}$ of $|\vec{\partial} h|_q$ on K. It thus follows easily that

$$\|h(u^\varepsilon, \varrho^\varepsilon) - h(u, \varrho)\|_p \leq C_{h,q}[\|u^\varepsilon - u\|_p^p + \|\varrho^\varepsilon - \varrho\|_p^p]^{1/p}, \tag{6.121}$$

so that $h^\varepsilon = h(u^\varepsilon, \varrho^\varepsilon)$ also satisfies $\|h^\varepsilon - h\|_p \to 0$ for the same p as $\varepsilon \to 0$.

Next note from the identity (6.92) that

$$\frac{\partial}{\partial X_k}\overline{(f^\varepsilon - f)}_\ell(X) = -\frac{1}{\ell}\int \mathrm{d}^{d+1}R\ \left(\frac{\partial \mathcal{G}}{\partial R_k}\right)_\ell (R-X)(f^\varepsilon(R) - f(R)), \tag{6.122}$$

Hence, for each X,

$$|\partial_k\overline{(f^\varepsilon - f)}_\ell(X)| \leq (c_{\ell,p}/\ell)\|f^\varepsilon - f\|_p \tag{6.123}$$

with $c_{\ell,p} = \|(\partial\mathcal{G})_\ell\|_q$ for $q = p/(p-1)$ and thus $\partial_k\overline{(f^\varepsilon)}_\ell(X) \to \partial_k\overline{f}_\ell$ as $\varepsilon \to 0$ whenever $\|f^\varepsilon - f\|_p \to 0$. Applying this result with $f = \varrho$, $\boldsymbol{j}$, $\boldsymbol{j}\boldsymbol{u}$, p, E, $(E+p)\boldsymbol{u}$, we get that pointwise in space-time

$$\begin{aligned}
\partial_t\overline{\varrho^\varepsilon}_\ell + \nabla_x\cdot\overline{\mathbf{j}}^\varepsilon_\ell \quad &\longrightarrow \quad \partial_t\bar{\varrho}_\ell + \nabla_x\cdot\overline{\mathbf{j}}_\ell, & (6.124)\\
\partial_t\overline{\mathbf{j}}^\varepsilon_\ell + \nabla_x\cdot\left(\overline{(\boldsymbol{j}^\varepsilon\boldsymbol{u}^\varepsilon)}_\ell + \bar{p}^\varepsilon_\ell\mathbf{I}\right) \quad &\longrightarrow \quad \partial_t\overline{\mathbf{j}}_\ell + \nabla_x\cdot\left(\overline{(\boldsymbol{j}\boldsymbol{u})}_\ell + \bar{p}_\ell\mathbf{I}\right), & (6.125)\\
\partial_t\overline{E}^\varepsilon_\ell + \nabla_x\cdot\left(\overline{((E^\varepsilon + p^\varepsilon)\boldsymbol{u}^\varepsilon)}_\ell\right) \quad &\longrightarrow \quad \partial_t\overline{E}_\ell + \nabla_x\cdot\left(\overline{((E+p)\boldsymbol{u})}_\ell\right), & (6.126)
\end{aligned}$$

as $\varepsilon \to 0$. The coarse-grained Euler equations

$$\begin{aligned}
\partial_t\bar{\varrho}_\ell \quad &+ \quad \nabla_x\cdot\overline{\mathbf{j}}_\ell = 0, & (6.127)\\
\partial_t\overline{\mathbf{j}}_\ell \quad &+ \quad \nabla_x\cdot\left(\overline{(\boldsymbol{j}\boldsymbol{u})}_\ell + \bar{p}_\ell\mathbf{I}\right) = \mathbf{0}, & (6.128)\\
\partial_t\overline{E}_\ell \quad &+ \quad \nabla_x\cdot\left(\overline{((E+p)\boldsymbol{u})}_\ell\right) = 0, & (6.129)
\end{aligned}$$

follow for u, ϱ, $\boldsymbol{u}$ if $\nabla_x\cdot\overline{(\mathbf{T}^\varepsilon)}_\ell$, $\nabla_x\cdot\overline{(\mathbf{T}^\varepsilon\cdot\boldsymbol{u}^\varepsilon)}_\ell$, and $\nabla_x\cdot\overline{(\boldsymbol{q}^\varepsilon)}_\ell$ all vanish as $\varepsilon \to 0$.

We first consider the shear-viscosity contribution to $\nabla \cdot \overline{(\mathbf{T}^\varepsilon)}_\ell$. With the short-hand notation $\eta^\varepsilon(X) := \varepsilon\eta(u^\varepsilon(X), \varrho^\varepsilon(X))$, we can bound this using Cauchy-Schwartz inequality as

$$\begin{aligned}\left|\nabla_x \cdot \overline{(2\eta^\varepsilon \mathbf{S}^\varepsilon)}_\ell(X)\right| &= \frac{2}{\ell}\left|\int \mathrm{d}^{d+1}R\ (\nabla_x \mathcal{G})_\ell(R)\cdot \eta^\varepsilon(X+R)\mathbf{S}^\varepsilon(X+R)\right| \\ &\leq \frac{2}{\ell}\sqrt{\int_{\mathrm{supp}(\mathcal{G}_\ell)} \mathrm{d}^{d+1}R\ \eta^\varepsilon(X+R) \times \int |(\partial\mathcal{G})_\ell(R-X)|^2\ Q^\varepsilon_\eta(dR)},\end{aligned} \tag{6.130}$$

with $Q^\varepsilon_\eta(\mathrm{d}R) = 2\eta^\varepsilon(R)|\mathbf{S}(R)|^2\mathrm{d}^{d+1}R$ denoting the kinetic-energy dissipation measure for $\varepsilon > 0$. Finally, because $Q^\varepsilon_\zeta \geq 0$,

$$\left|\nabla_x \cdot \overline{(2\eta^\varepsilon \mathbf{S}^\varepsilon)}_\ell(X)\right| \leq \frac{2}{\ell}\sqrt{\int_{\mathrm{supp}(\mathcal{G}_\ell)} \mathrm{d}^{d+1}R\ \eta^\varepsilon(X+R) \times \int |(\partial\mathcal{G})_\ell(R-X)|^2\ Q^\varepsilon(dR)} \tag{6.131}$$

with $Q^\varepsilon = Q^\varepsilon_\eta + Q^\varepsilon_\zeta$. Since $\mathcal{G} \in D(\Gamma)$ implies that $S_X|\partial\mathcal{G}|^2 \in D(\Gamma)$ also, then

$$\lim_{\varepsilon\to 0}\int |(\partial\mathcal{G})_\ell(R-X)|^2\, Q^\varepsilon(dR) = \int |(\partial\mathcal{G})_\ell(R-X)|^2\, Q(dR) \tag{6.132}$$

by Assumption 6.1.3. On the other hand, because $\eta(u^\varepsilon, \varrho^\varepsilon) \in L^\infty(\Gamma)$ when η satisfies the smoothness Assumption 6.1.2 with $M = 0$, then the upper bound in (6.130) is proportional to $\varepsilon^{1/2}$. Thus, $\nabla_x \cdot \overline{(2\eta^\varepsilon \mathbf{S}^\varepsilon)}_\ell(X) \to 0$ as $\varepsilon \to 0$. An identical argument using $Q^\varepsilon_\eta \geq 0$ shows that likewise $\nabla_x \overline{(\zeta^\varepsilon \Theta^\varepsilon)}_\ell(X) \to 0$ as $\varepsilon \to 0$, and both results together imply that $\nabla \cdot \overline{(\mathbf{T}^\varepsilon)}_\ell \to 0$ pointwise.

In a similar manner, the shear-viscosity contribution to $\nabla_x \cdot \overline{(\mathbf{T}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)}_\ell$ can be bounded as

$$\begin{aligned}
&\left|\nabla_x \cdot \overline{(2\eta^\varepsilon \mathbf{S}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)}_\ell(X)\right| \\
&\qquad = \frac{2}{\ell}\left|\int \mathrm{d}^{d+1}R\ (\nabla_x \mathcal{G})_\ell(R) \cdot \eta^\varepsilon(X+R)\mathbf{S}^\varepsilon(X+R) \cdot \boldsymbol{u}^\varepsilon(X+R)\right| \\
&\qquad \leq \frac{2}{\ell}\sqrt{\int_{\mathrm{supp}(\mathcal{G}_\ell)} \mathrm{d}^{d+1}R\ \eta^\varepsilon(X+R)|\boldsymbol{u}^\varepsilon(X+R)|^2} \\
&\qquad\qquad\qquad \times\sqrt{\int |(\partial G)_\ell(R-X)|^2\ Q^\varepsilon(dR)},
\end{aligned} \tag{6.133}$$

and an analogous bound holds for $\nabla_x \cdot \overline{(2\zeta^\varepsilon \Theta^\varepsilon \boldsymbol{u}^\varepsilon)}_\ell$. Thus, by Assumption 6.1.3 $\nabla_x \cdot \overline{(\mathbf{T}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)}_\ell \to 0$ pointwise as $\varepsilon \to 0$.

Finally, $\nabla_x \cdot \overline{(\boldsymbol{q}^\varepsilon)}_\ell = -\nabla \cdot \overline{(\kappa^\varepsilon \nabla_x T^\varepsilon)}_\ell$ and the entropy-production measure due to thermal conductivity is defined by $\Sigma^\varepsilon_\kappa(\mathrm{d}R) = \kappa^\varepsilon(R)\left|\frac{\nabla_x T^\varepsilon(R)}{T^\varepsilon(R)}\right|^2 \mathrm{d}^{d+1}R$ for $\varepsilon > 0$. Because $Q^\varepsilon/T^\varepsilon \geq 0$, thus $\Sigma^\varepsilon_\kappa \leq \Sigma^\varepsilon$. Writing $\kappa^\varepsilon \nabla_x T^\varepsilon = \sqrt{\kappa^\varepsilon} T^\varepsilon \cdot \sqrt{\kappa^\varepsilon}\frac{\nabla_x T^\varepsilon}{T^\varepsilon}$ and using a Cauchy-Schwartz estimate similar to (6.133), it follows from the convergence $\Sigma^\varepsilon \xrightarrow{\mathcal{D}} \Sigma$ in Assumption 6.1.3 that $\nabla_x \cdot \overline{(\boldsymbol{q}^\varepsilon)}_\ell \to 0$ pointwise as $\varepsilon \to 0$.

In conclusion, the coarse-grained Euler equations (6.127)–(6.129) hold for all $\ell > 0$. By the results in section 6.2, we have thus proved that $(u, \varrho, \boldsymbol{u})$ form a weak Euler solution. As an aside, we note that it would clearly suffice for this statement to have in Assumption 6.1.3 only the condition on entropy-production $\Sigma^\varepsilon \xrightarrow{\mathcal{D}} \Sigma$ and not the additional assumption $Q^\varepsilon \xrightarrow{\mathcal{D}} Q$. If in Theorem 6.1.5 only the statement (6.24) on entropy balance were made, then this would be more economical in terms of hypotheses. However, to derive the balance equations (6.22) and (6.23) we need the additional convergence statement in Assumption 6.1.3 for Q^ε as we now show.

To derive the balance equations of kinetic energy, internal energy and entropy for the weak Euler solutions, we start with the corresponding eqs.(6.8),(6.9),(6.12) for compressible Navier-Stokes. Then, because the basic fields u^ε, ϱ^ε, $\boldsymbol{u}^\varepsilon$ and their compositions with functions $h^\varepsilon := h(u^\varepsilon, \varrho^\varepsilon)$ satisfying the smoothness assumptions converge strongly in L^p for some $1 \leq p < \infty$ to the corresponding fields u, ϱ, $\boldsymbol{u}$ and $h(u, \varrho)$, it follows directly that

$$
\begin{aligned}
\partial_t \left(\frac{1}{2} \varrho^\varepsilon |\boldsymbol{u}^\varepsilon|^2 \right) + \nabla_x \cdot \left(\left(p^\varepsilon \right.\right. &+ \left.\left. \frac{1}{2} \varrho^\varepsilon |\boldsymbol{u}^\varepsilon|^2 \right) \boldsymbol{u}^\varepsilon \right) \\
&\xrightarrow{\mathcal{D}} \partial_t \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 \right) + \nabla_x \cdot \left(\left(p + \frac{1}{2} \varrho |\boldsymbol{u}|^2 \right) \boldsymbol{u} \right), \\
\partial_t u^\varepsilon + \nabla_x \cdot (u^\varepsilon \boldsymbol{u}^\varepsilon) &\xrightarrow{\mathcal{D}} \partial_t u + \nabla_x \cdot (u \boldsymbol{u}), \\
\partial_t s^\varepsilon + \nabla_x \cdot (s^\varepsilon \boldsymbol{u}^\varepsilon) &\xrightarrow{\mathcal{D}} \partial_t s + \nabla_x \cdot (s \boldsymbol{u}).
\end{aligned}
\tag{6.134}
$$

To see that

$$
\nabla_x \cdot (\mathbf{T}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)\,,\ \nabla_x \cdot \boldsymbol{q}^\varepsilon,\ \nabla_x \cdot \left(\frac{\boldsymbol{q}^\varepsilon}{T^\varepsilon} \right) \xrightarrow{\mathcal{D}} 0,
$$

note that this is equivalent to $\nabla_x \overline{(\mathbf{T}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)}_\ell$, $\nabla_x \overline{\boldsymbol{q}^\varepsilon}_\ell$, $\overline{(\boldsymbol{q}^\varepsilon / T^\varepsilon)}_\ell \to 0$ pointwise. This has already been proved for the first two, and is shown for the third by a very similar Cauchy-Schwartz argument by writing $\boldsymbol{q}^\varepsilon / T^\varepsilon = -\sqrt{\kappa^\varepsilon} \cdot \sqrt{\kappa^\varepsilon} \nabla_x T^\varepsilon / T^\varepsilon$.

Because of the condition $\Sigma^\varepsilon \xrightarrow{\mathcal{D}} \Sigma$ in Assumption 6.1.3, all of the terms in the

Navier-Stokes entropy balance (6.12) converge distributionally and thus one obtains in the limit $\varepsilon \to 0$ the entropy balance (6.24) for the weak Euler solution. Similarly, because of the condition $Q^\varepsilon \xrightarrow{\mathcal{D}} Q$ in Assumption 6.1.3, all of the terms in the Navier-Stokes kinetic energy and internal energy balances (6.8),(6.9) are proved to converge distributionally, except $p^\varepsilon\Theta^\varepsilon$. Thus, this term also converges

$$\begin{aligned} \mathcal{D}\text{-}\lim_{\varepsilon\to 0} p^\varepsilon\Theta^\varepsilon &= \partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2\right) + \nabla_x\cdot\left(\left(p+\frac{1}{2}\varrho|\boldsymbol{u}|^2\right)\boldsymbol{u}\right) + Q \\ &= Q - [\partial_t u + \nabla_x\cdot(u\boldsymbol{u})]. \end{aligned}$$

With the notation $p * \Theta := \mathcal{D}\text{-}\lim_{\varepsilon\to 0} p^\varepsilon\Theta^\varepsilon$ we thus obtain the balances (6.22),(6.23) of kinetic and internal energy for the limiting weak Euler solution. $\square$

6.7 Proof of Theorem 6.1.6

The strategy to prove Theorem 6.1.6 is to use the commutator estimates developed in Section 6.4 to show that Q_{flux} and Σ_{flux} vanish when the Euler solutions possess suitable Besov regularity. Then, we use the "inertial-range" expressions (6.26) to show the dissipation measures Q and Σ also vanish, and that $p * \Theta = p \circ \Theta$.

Energy Flux: We first show that Q_{flux} defined by (6.21) necessarily exists and vanishes for weak Euler solutions satisfying the exponent inequalities (6.29)–(6.31). To show this, simple bounds can be derived for Q_ℓ^{flux} using the expressions (6.106), (6.108) and

Propositions 6.4.2 and 6.4.3. One obtains

$$\|(1/\bar{\varrho}_\ell)\nabla_x \bar{p}_\ell \cdot \bar{\tau}_\ell(\varrho, \boldsymbol{u})\|_{p/3} = O\left(\|1/\varrho\|_\infty \frac{1}{\ell}\|\delta p(\ell)\|_p \|\delta\varrho(\ell)\|_p \|\delta\boldsymbol{u}(\ell)\|_p\right), \quad p \geq 3,$$

$$\|\nabla_x \tilde{\boldsymbol{u}}_\ell\|_p = \frac{1}{\ell}\|\delta\boldsymbol{u}(\ell)\|_p \left[O(1) + O(\|1/\varrho\|_\infty \|\varrho\|_\infty) + O(\|1/\varrho\|_\infty^2 \|\varrho\|_\infty^2)\right], \quad p \geq 1,$$

$$\|\tilde{\tau}_\ell(\boldsymbol{u}, \boldsymbol{u})\|_{p/2} = \|\delta\boldsymbol{u}(\ell)\|_p^2 \left[O(1) + O(\|1/\varrho\|_\infty \|\varrho\|_\infty) + O(\|1/\varrho\|_\infty^2 \|\varrho\|_\infty^2)\right], \ p \geq 2,$$

and thus

$$\|Q_\ell^{\text{flux}}\|_{p/3} = O\left(\frac{1}{\ell}\|\delta p(\ell)\|_p \|\delta\varrho(\ell)\|_p \|\delta\boldsymbol{u}(\ell)\|_p\right) + O\left(\frac{\|\delta\boldsymbol{u}(\ell)\|_p^3}{\ell}\right), \qquad p \geq 3. \tag{6.135}$$

In this latter estimate we absorb the dependence upon the maximum-to-minimum mass ratio $\|1/\varrho\|_\infty \|\varrho\|_\infty$ into the constant factor, since this ratio is ℓ-independent. Assuming the Besov regularity of u, ϱ, $\boldsymbol{u}$ in Theorem 6.1.6 and using Lemma 6.4.5 to get the Besov regularity of p, one thus obtains

$$\|Q_\ell^{\text{flux}}\|_{p/3} = O\left(\ell^{\min\{\sigma_p^u, \sigma_p^\varrho\} + \sigma_p^\varrho + \sigma_p^v - 1}\right) + O\left(\ell^{3\sigma_p^v - 1}\right), \quad p \geq 3.$$

It follows that

$$2\min\{\sigma_p^u, \sigma_p^\varrho\} + \sigma_p^v > 1,\ 3\sigma_p^v > 1,\ \text{for some}\ \ p \geq 3 \Longrightarrow \mathcal{D}\text{-}\lim_{\ell\to 0} Q_\ell^{\text{flux}} = 0.$$

This is enough to infer the first statement of Theorem 6.1.6 that $Q_{\rm flux}$ exists and vanishes for weak Euler solutions, but not enough to conclude that the viscous anomaly vanishes, $Q = 0$. Recall by (6.26) that

$$Q = Q_{\rm flux} + \tau(p, \Theta). \tag{6.136}$$

Therefore, with the exponent inequalities assumed above, we can only conclude

$$Q = \tau(p, \Theta) := p * \Theta - p \circ \Theta. \tag{6.137}$$

In order to show that $Q = 0$, we must make use of the entropy balance, which we consider next.

Entropy Anomaly: We show that $\Sigma_{\rm flux}$ defined by (6.21) necessarily exists and vanishes for weak Euler solutions satisfying the exponent inequalities (6.29)–(6.31). To accomplish this, we next derive bounds on $\Sigma_\ell^{\rm inert*}$ using (6.116)–(6.118) and Propositions 6.4.2, 6.4.3, 6.4.7, and 6.4.8. Expression (6.116) and Propositions 6.4.3, 6.4.7 give:

$$\|I_\ell^{\rm flux}\|_{p/3} = O\left(\frac{1}{\ell}\max\{\|\delta u(\ell)\|_p, \|\delta\varrho(\ell)\|_p\}^2\|\delta\boldsymbol{u}(\ell)\|_p\right).$$

Expression (6.118) and Propositions 6.4.2, 6.4.8 give:

$$\|\Sigma_\ell^{\rm flux}\|_{p/3} \;\; = \;\; O\left(\|\nabla_x\underline{\beta}_\ell\|_p\|\delta u(\ell)\|_p\|\delta\boldsymbol{u}(\ell)\|_p\right) + O\left(\|\nabla_x\underline{\lambda}_\ell\|_p\|\delta\varrho(\ell)\|_p\|\delta\boldsymbol{u}(\ell)\|_p\right)$$

$$= \quad O\left(\frac{1}{\ell}\max\{\|\delta u(\ell)\|_p, \|\delta\varrho(\ell)\|_p\}^2\|\delta\boldsymbol{u}(\ell)\|_p\right), \tag{6.138}$$

while Propositions 6.4.2, 6.4.8 give for the added terms to $\Sigma_\ell^{\rm flux*}$ in (6.117) the estimates

$$\begin{aligned}
\|\partial_t\underline{\beta}_\ell k_\ell\|_{p/3} \quad &= \quad O\left(\|\partial_t\underline{\beta}_\ell\|_p\|\delta\boldsymbol{u}(\ell)\|_p^2\right) = O\left(\frac{1}{\ell}\max\{\|\delta u(\ell)\|_p, \|\delta\varrho(\ell)\|_p\}\|\delta\boldsymbol{u}(\ell)\|_p^2\right), \\
\|\nabla_x\underline{\beta}_\ell\cdot\mathbf{J}_\ell^k\|_{p/3} \quad &= \quad O\left(\|\nabla_x\underline{\beta}_\ell\|_p\|\delta\boldsymbol{u}(\ell)\|_p^2\right) = O\left(\frac{1}{\ell}\max\{\|\delta u(\ell)\|_p, \|\delta\varrho(\ell)\|_p\}\|\delta\boldsymbol{u}(\ell)\|_p^2\right).
\end{aligned}$$

To estimate k_ℓ and $\mathbf{J}_\ell^k$ we here used the expressions (6.106) for $\tilde{\boldsymbol{u}}_\ell$, (6.108) for $\tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u})$ and the similar expression for $\tilde{\tau}_\ell(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u})$ that follows from (6.59). Assuming the Besov regularity of u, ϱ, $\boldsymbol{u}$ in Theorem 6.1.6, one thus obtains from these estimates and the estimate of $\underline{\beta}_\ell Q_\ell^{flux}$ using (6.135) that for any $p\geq 3$

$$\|\Sigma_\ell^{\rm inert*}\|_{p/3} = O\left(\ell^{2\min\{\sigma_p^u,\sigma_p^\varrho\}+\sigma_p^v-1}\right) + O\left(\ell^{\min\{\sigma_p^u,\sigma_p^\varrho\}+2\sigma_p^v-1}\right) + O\left(\ell^{3\sigma_p^v-1}\right).$$

The inequalities (6.29)–(6.31) thus imply that $\Sigma_\ell^{\rm inert*}\to 0$ strong in $L^{p/3}$ as $\ell\to 0$ for the same choice of $p\geq 3$. Because of (6.26), it follows that the non-ideal entropy production also vanishes $\Sigma\equiv 0$.

Viscous Energy Dissipation Anomaly: We now show that $\Sigma=0$ implies that $Q=0$. First note

$$\Sigma^\varepsilon \geq \beta^\varepsilon Q^\varepsilon \geq Q^\varepsilon/\|T^\varepsilon\|_\infty.$$

Because $\|T^\varepsilon\|_\infty$ by Assumption 6.1.1 is bounded by some constant T_0 uniformly in $\varepsilon < \varepsilon_0$, we thus find that

$$\Sigma^\varepsilon \geq Q^\varepsilon / T_0 \geq 0, \quad \varepsilon < \varepsilon_0,$$

and one obtains in the limit $\varepsilon \to 0$ that

$$0 = \Sigma \geq Q/T_0 \geq 0.$$

Thus, the inequalities (6.29)–(6.31) in Theorem 6.1.6 for some $p \geq 3$ imply also $Q \equiv 0$.

Pressure-Dilatation Defect: Lastly, the result $Q = \tau(p, \Theta)$ in (6.137) together with $Q \equiv 0$ implies that $p * \Theta = p \circ \Theta$, as was claimed. □

6.8 Proof of Theorem 6.1.7

We derive Theorem 6.1.7 from a result for more general balance equations (6.35). We consider cases where $\boldsymbol{v} \in L^\infty(\mathbb{T}^d \times [0,T]; \mathbb{R}^m)$, so that $\mathcal{R} = \text{ess.ran.}(\boldsymbol{v})$ is a compact subset of $\mathbb{R}^m$ with $K = \text{conv}(\mathcal{R})$ also compact, and $\mathbf{F} = \mathbf{F}(\boldsymbol{v})$ is a C^1 function on an open set U, $K \subset U \subset \mathbb{R}^m$. Furthermore, the individual components of F_{ia} of $\mathbf{F}$ for $i = 1, \ldots, d$ and $a = 1, \ldots, m$ may not depend upon all of the components u_a, $a = 1, \ldots, m$ of $\boldsymbol{v}$ but only upon a subset. We assume that for each $a = 1, \ldots, m$ the

d-vector $\mathbf{F}_a = (F_{1a}, \ldots, F_{da})$ is a function of the form

$$\mathbf{F}_a(\boldsymbol{v}) = \tilde{\mathbf{F}}_a(u_{b_1^{(a)}}, \ldots, u_{b_{m_a}^{(a)}}), \quad a = 1, \ldots, m \tag{6.139}$$

where the subset $\mathbb{M}_a = \{b_1^{(a)}, \ldots, b_{m_a}^{(a)}\} \subset \{1, \ldots, m\}$ has cardinality $m_a \leq m$, and thus $\mathbf{F}_a$ is constant in the variables u_b for $b \notin \mathbb{M}_a$

We then have the following general result:

Theorem 4* *Suppose that* $\boldsymbol{v} \in L^\infty(\mathbb{T}^d \times [0,T]; \mathbb{R}^m)$ *is a weak solution of (6.35) where* $\mathbf{F} \in C^1(U)$ *with* U *open and* $\mathrm{conv}(\mathrm{ess.ran.}(\boldsymbol{v})) \subset U \subset \mathbb{R}^m$*, and that also* $\mathbf{F}_a$ *satisfies the condition (6.139) for each* $a = 1, \ldots, m$*. If for some* $p \geq 1$

$$u_a \in L^\infty([0,T]; B_p^{\sigma_p^a,\infty}(\mathbb{T}^d)), \quad 0 < \sigma_p^a \leq 1; \qquad a = 1, \ldots, m \tag{6.140}$$

then

$$u_a \in B_p^{\bar{\sigma}_p^a,\infty}(\mathbb{T}^d \times (0,T)), \quad \bar{\sigma}_p^a = \min\{\sigma_p^a, \min_{b \in \mathbb{M}_a} \sigma_p^b\}; \qquad a = 1, \ldots, m. \tag{6.141}$$

Proof We use the notation $\Gamma = \mathbb{T}^d \times [0,T]$ and $R = (\boldsymbol{r}, \tau) \in \Gamma$. Since $L^\infty(\Gamma) \subset L^p(\Gamma)$ for $p \geq 1$, we must only bound the requisite $L^p(\Gamma \cap (\Gamma - R))$ norm in the definition (6.28) of the space-time Besov norm for $|R| < 1$. Because of the invariance of this norm to translations on $\mathbb{T}^d$, Minkowski's inequality gives:

$$\|u_a(\cdot + R) - u_a\|_p \leq \|u_a(\cdot, \cdot + \tau) - u_a\|_p + \|u_a(\cdot + \boldsymbol{r}, \cdot) - u_a\|_p. \tag{6.142}$$

The assumed regularity in (6.140) guarantees that $\|u_a(\cdot + \boldsymbol{r}, \cdot) - u_a\|_p = O(|\boldsymbol{r}|^{\sigma_p^a})$. To estimate the time-increment term, fix an $0 < \ell \leq |\tau|$ and decompose $\boldsymbol{v} = \hat{\boldsymbol{v}}_\ell + \boldsymbol{v}'_\ell$ with $\hat{\boldsymbol{v}}_\ell = \boldsymbol{v} * \check{G}_\ell$ for a spatial mollifier G_ℓ. Applying Minkowski's inequality again,

$$\|u_a(\cdot, \cdot + \tau) - u_a\|_p \;\; \leq \;\; \|\hat{u}_{a,\ell}(\cdot, \cdot + \tau) - \hat{u}_{a,\ell}\|_p + \|u'_{a,\ell}(\cdot, \cdot + \tau) - u'_{a,\ell}\|_p. \tag{6.143}$$

Since $\partial_t u_a + \nabla_x \cdot \mathbf{F}_a = 0$ is satisfied in the sense of distributions or, equivalently, pointwise after space-time mollification (see Proposition 6.2.1), standard approximation arguments show:

$$\begin{aligned} \|\hat{u}_{a,\ell}(\cdot, \cdot + \tau) - \hat{u}_{a,\ell}\|_p \;\; &\leq \;\; |\tau| \|\nabla_x \cdot \hat{\mathbf{F}}_{a,\ell}\|_{L^\infty([0,T];L^p(\mathbb{T}^d))} \\ &= \;\; O(\ell^{\mu_p^a - 1}|\tau|), \quad \mu_p^a = \min_{b \in \mathbb{M}_a} \sigma_p^b. \end{aligned} \tag{6.144}$$

Here we have used the inherited spatial Besov regularity of $\mathbf{F}_a$ with exponent μ_p^a, which follows from a straightforward generalization of Lemma 6.4.5, and the spatial version of Proposition 6.4.3. On the other hand, the term involving the fluctuation fields can be bounded using Proposition 6.4.4 as:

$$\|u'_{a,\ell}(\cdot,\cdot+\tau)-u'_{a,\ell}\|_p \leq 2\|u'_{a,\ell}\|_{L^\infty([0,T];L^p(\mathbb{T}^d))} = O(\ell^{\sigma_p^a}). \tag{6.145}$$

From equations (6.143)–(6.145) we obtain

$$\|u_a(\cdot,\cdot+\tau)-u_a\|_p = O(\ell^{\mu_p^a-1}|\tau|) + O(\ell^{\sigma_p^a}). \tag{6.146}$$

Since $\ell \leq |\tau| < 1$ by assumption, we increase the upper bound in (6.146) by replacing both μ_p^a and σ_p^a with their minimum, $\bar{\sigma}_p^a$, in (6.141). The resulting bound is then optimized by choosing the arbitrary scale $\ell \leq |\tau|$ to be $\ell \propto |\tau|$. Altogether,

$$\|u_a(\cdot,\cdot+\tau)-u_a\|_p = O(|\tau|^{\bar{\sigma}_p^a}), \quad \|u_a(\cdot+\boldsymbol{r},\cdot)-u_a\|_p = O(|\boldsymbol{r}|^{\bar{\sigma}_p^i}). \tag{6.147}$$

It follows from (6.142) and (6.147) that $u_a \in B_p^{\bar{\sigma}_p^a,\infty}(\mathbb{T}^d\times(0,T))$. ☐

Theorem 6.1.7 The result is proved as a corollary of Theorem 4*, specialized to the compressible Euler system with $(u_0,u_1,\ldots,u_d,u_{d+1}) := (\varrho, j_1,\ldots,j_d,E)$ and

$$\begin{aligned} F_{i,0} &:= u_i, \\ F_{i,j} &:= u_0^{-1}u_iu_j + p(u,u_0)\delta_{ij}, \\ F_{i,d+1} &:= (u_{d+1}+p(u,u_0))\,u_0^{-1}u_i. \end{aligned}$$

for $i, j = 1, \ldots, d$ and $u := u_{d+1} - \frac{u_1^2 + \cdots + u_d^2}{2u_0}$. The assumed strict positivity of $\varrho \geq \varrho_0 > 0$, space-time boundedness of $\boldsymbol{v}$, and smoothness of p implies that $\mathbf{F}$ possesses the requisite regularity. It follows that:

$$\varrho \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^j\}, \infty}(\mathbb{T}^d \times (0,T)), \quad \boldsymbol{j}, E \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^j, \sigma_p^E\}, \infty}(\mathbb{T}^d \times (0,T)),$$

Recalling that the fields $\boldsymbol{j}$ and E are algebraically related to u, ϱ, $\boldsymbol{u}$ by $\boldsymbol{j} := \varrho \boldsymbol{u}$ and $E := \frac{1}{2}\varrho|\boldsymbol{u}|^2 + u$, an application of Corollary 6.4.6 shows that we may take $\sigma_p^j = \min\{\sigma_p^\varrho, \sigma_p^v\}$ and $\sigma_p^E = \min\{\sigma_p^u, \sigma_p^\varrho, \sigma_p^v\}$. The inverse relations $\boldsymbol{u} = \varrho^{-1}\boldsymbol{j}$ and $u = E - \varrho^{-1}|\boldsymbol{j}|^2$ and another application of Corollary 6.4.6 yields the space-time regularity (6.33)–(6.34) claimed in Theorem 6.1.7. $\square$

Remark Theorem 4* applies also to solutions of the incompressible Euler equations with velocity $\boldsymbol{u}$ and (kinematic) pressure P satisfying $\boldsymbol{u}, P \in L^\infty(\Gamma)$. Assuming for $q \geq 1$ that $\boldsymbol{u} \in L^\infty([0,T], B_q^{\sigma_q,\infty}(\mathbb{T}^d))$, elliptic regularization of the solutions of the Poisson equation

$$-\triangle P = \partial^2(v_i v_j)/\partial x_i \partial x_j$$

implies that $P \in L^\infty([0,T], B_q^{\sigma_q,\infty}(\mathbb{T}^d))$. Alternatively, this regularity of P follows from boundedness of Calderón-Zygmund operators in Besov-space norms. Theorem 4* yields $\boldsymbol{u} \in B_q^{\sigma_q,\infty}(\mathbb{T}^d \times (0,T))$, so that $\boldsymbol{u}$ is as regular in time as it is in space.

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