Emergence of order in fluid motion

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"When a continuous medium is deprived of its physical properties (elasticity, thermal and electrical conductivity, and so on) its property of occupying a definite position in space remains, as do elementary interactions through the mutual pressure of its parts, due to Aristotle's principle that it is impossible for two bodies to occupy the same space. It is amazing that it is these elementary interactions that cause the most complicated effects, including turbulence."

– V. I. Yudovich [33]

Ideal fluids are perhaps the most basic of physical media; their defining property is that they "occupy space" and are otherwise free. Although the equations of motion were derived by Euler in 1757 [17], fluids have been on the minds of thinkers for millennia [1] and exhibit some of the most complex behavior in the observable world. Turbulence, for example, was considered by R. Feynman to be the last great problem of classical physics [18].

Fluids in the world around us typically occupy either three-dimensional volumes or are (effectively) confined to two-dimensional surfaces. While most everyday experiences with fluid phenomena are three dimensional, many geophysical and astrophysical systems such as oceanic currents, large-scale weather patterns, and planetary atmospheres are, to good approximation, two-dimensional. Fluid behavior is highly sensitive to the dimension! In two dimensions there is a trend towards order, with vortices emerging and merging into super structures (see Figure 1), shedding light on the formation and persistence of phenomena such as the jet stream and hurricanes. Three-dimensional fluids are, by contrast, marked by chaotic, turbulent behaviors with few coherent features, if any [20]. In this article, we aim to convey some mathematical results on the behavior of fluids in two dimensions.

Definition of fluid motion

Like nearly all natural systems, the Euler equations arise from a *principle of least action*. Namely, an ideal fluid move from place to place in order to minimize expended energy, while maintaining occupied volume. This perspective was first enunciated by Lagrange in 1755, and has a beautiful underlying geometric nature. In his seminal work, Arnold [2] interpreted ideal fluid motion as geodesic flow on the group of volume-preserving diffeomorphisms. In a sense, fluids move along the "great arcs" of the manifold of all possible rearrangements of space. We begin with a quick derivation of this result for ideal (incompressible and inviscid) fluids by analogy with finite dimensional systems.

The viewpoint is as follows: consider first a collection of N particles of unit mass moving in d-dimensional Euclidean space $\mathbb{R}^d$. This system can be thought of as the motion of a *single* particle moving in the higher dimensional space $\mathbb{R}^{N\times d} = \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, the Cartesian product configuration space. If no external forces are present, according to Newton's law of motion the curve traced out by this "particle" is a straight line. Newton's law in turn follows from Hamilton's principle, a primal axiom of mechanics, which states that the system moves in a way to extremize its 'action' (kinetic energy less potential energy). In the case above, where the particles don't interact and are devoid of all physical properties, the action is solely the kinetic energy. This makes clear why the motion is, in fact, a geodesic on $\mathbb{R}^{N\times d}$ endowed with the Euclidean (kinetic energy) metric.

Now, suppose that this particle is not free to move everywhere in space, but is rather forced to remain on some surface or submanifold of the space $S \subset \mathbb{R}^{N\times d}$. For example, the bob of a pendulum is forced to live on a circle on the plane, and the double pendulum to a two-dimensional torus in four-dimensional configuration space. How then does the particle move? According to Hamilton's principle, subject to the aforementioned constraints, one arrives at *d'Alembert's principle* for the position of the particle $X_t : \mathbb{R} \to \mathbb{R}^{N\times d}$ at time $t \in \mathbb{R}$:

$$\ddot{X}_t \perp T_{X_t} S, \tag{1}$$

$$X_t \in S. \tag{2}$$

Here $T_{X_t}S$ is the tangent space to S at the point X_t. That is, the acceleration is orthogonal to the constraint. Such constrained particles follow nothing but geodesic curves on the embedded surface S, endowed with the induced Euclidean metric.

Now for the fluid. It is thought of as infinitely many particles ($N = \infty$) experiencing a force between them produced by the constraint that volumes are neither expanded nor contracted. The state of the system is described then as a mapping of some fluid domain $M \subset \mathbb{R}^d$ to itself, namely $X_t(a) : \mathbb{R}\times M \to M$, where $a \in M$ serves as a continuum label for the particle. The configuration space S, thought of as "surface" in $L^2(M;\mathbb{R}^d)$, namely L^2 mappings of $M \to \mathbb{R}^d$, is the set of diffeomorphisms preserving volume:

$$S = \big\{X_t : M \to \mathbb{R}^d \ : \ X_t(a) \in M, \quad \det \nabla X_t(a) = 1 \text{ for all } a \in M\big\}.$$

That is, $S = \mathsf{SDiff}(M)$. The tangent space to S at a "point" $X \in \mathsf{SDiff}(M)$ is formally

$$T_X S = \{u \circ X \ : \ u : \mathbb{R}^d \to \mathbb{R}^d, \quad \text{div } u = 0, \ u \text{ tangent to the boundary of } M\}.$$

According to the Helmholtz-Hodge decomposition, the normal space (orthogonal complement of $T_X S$ with respect to the L^2, kinetic energy, metric), is

$$(T_X S)^\perp = \{\nabla p \circ X \ : \ p : \mathbb{R}^d \to \mathbb{R}\}.$$

Thus, from d'Alembert's principle (1)–(2) we arrive at (Lagrangian form of) the *Euler equations*:

$$\ddot{X}_t = -\nabla p_t \circ X_t, \tag{3}$$

$$\det \nabla X_t = 1. \tag{4}$$

That is, ideal fluid flow moves along geodesics on the "submanifold" of volume preserving diffeomorphisms. This perspective is that of V. I. Arnold [2, 3], and gives a clear illustration of the principle underlying V. I. Yudovich's quote. In the above, we have ignored the important issues associated to regularity of the mappings, which is required to give precise geometric meaning [15]. It is, in any case, probably better to say we have *defined* ideal fluid motion in the above discussion rather than derived it.

In the above discussion, we have described the state of the fluid through its configurations, with $\dot{X}_t(a)$ representing the velocity at time t of a particle that started at position $a \in M$ at time 0. It is very useful to change perspective slightly and consider the velocity of whatever particle now occupies a fixed point $x \in M$ at time t. Introducing this velocity field $u : \mathbb{R}\times M \to \mathbb{R}^d$ via $u(t,x) := \dot{X}_t(X_t^{-1}(x))$, we arrive at the *Eulerian form* of the Euler equations:

$$\partial_t u + u\cdot\nabla u = -\nabla p, \tag{5}$$

$$\nabla\cdot u = 0, \tag{6}$$

$$u\cdot n|_{\partial M} = 0, \tag{7}$$

supplied with initial data u_0. The unknown function p, named the pressure, is a Lagrange multiplier which enforces the condition $X_t \in \mathsf{SDiff}(M)$ at every instant $t \in \mathbb{R}$ or, in terms of the velocity vector field, that u remain divergence-free (6). This Eulerian formulation of the equations of motion can be interpreted in regimes that are too irregular for our definition to apply. This is the case for *turbulence*, which is a singular regime of rough and tumble flow.

We will refer to the situation where the velocity field is differentiable, so that all terms in the equations (5)–(7) make naïve sense, as a *non-turbulent regime*. It is a regime in which the Euler equations give a definite prediction for the motion of the fluid. Being variational in nature (arising due to Hamilton's principle), this regime is subject to the conservation laws implied by continuous symmetries via Noether's theorem [26, 3]. Specifically, *time translation invariance* implies energy conservation:

$$\mathsf{E}[u(t)] = \mathsf{E}[u_0] \ \text{ where } \ \mathsf{E}[u] := \frac{1}{2}\int_M |u(x)|^2 dx$$

while *particle relabelling symmetry* implies conservation of circulation (Kelvin's theorem):

$$\mathsf{K}_{X_t(\Gamma)}[u(t)] = \mathsf{K}_\Gamma[u_0] \ \text{ where } \ \mathsf{K}_\Gamma[u] := \oint_\Gamma u\cdot d\ell$$

for any rectifiable loop $\Gamma \subset M$. By Stokes' theorem, this line integral is equivalent to the flux of the curl of the velocity vector field u, the *vorticity* ω, through any co-moving bounding surface. Letting the loop become infinitesimal about any given point $x \in M$, with arbitrary orientation, we deduce that the vorticity itself is transported. In two dimensions one can identify the vorticity with a scalar field $\omega = \nabla^\perp\cdot u$

where $\nabla^\perp = (-\partial_2, \partial_1)$ and in three dimensions with a vector field $\omega = \nabla \times u$. These are transported by u:

(8) $$d = 2: \qquad \partial_t \omega + u \cdot \nabla \omega = 0,$$

(9) $$d = 3: \qquad \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u.$$

When M has trivial homology, these equations form a closed description of two and three-dimensional fluid motion upon specifying the initial vorticity $\omega|_{t=0} = \omega_0$ and the non-local *Biot-Savart* law for recovering the velocity field $u = \mathrm{curl}^{-1}\omega$. It is a classical result of Lichtenstein and Gunther dating from near the turn of the century that if the vorticity is initially regular enough, say $C^\alpha(M)$ for any non-integer $\alpha > 0$, then there exists a unique solution remaining in that class for some (possibly short) period of time.

This local existence is dimension–independent. However, the vectorial nature of the vorticity causing the presence of the right-hand-side in (9) is the source of great differences between two and three dimensional fluid motion. The most basic manifestation of this discrepancy occurs for the question of *long time existence*. Indeed, the celebrated result of Beale-Kato-Majda [5] says a classical solution can persist if and only if the maximum vorticity is under control in a time-integrated sense (namely be in $L^1(0, T; L^\infty(M))$). How does this result differentiate between two and three-dimensions? The Euler equation in two-dimensions (8) says that the vorticity is transported along particle (Lagrangian) trajectories, namely

$$\omega(t) = \omega_0 \circ X_t^{-1}, \qquad \dot{X}_t = u(X_t, t), \qquad X_0 = \mathrm{id}.$$

Since u is divergence-free, the configuration X_t is an area-preserving diffeomorphism. It therefore follows that all vorticity values, as well as any integral moment of it (called Casimirs), are conserved:

$$\mathrm{I}_f[\omega(t)] = \mathrm{I}_f[\omega_0] \ \text{ where } \ \mathrm{I}_f[\omega] := \int_M f(\omega(x))dx$$

for any continuous $f : \mathbb{R} \to \mathbb{R}$. Uniform-in-time boundedness of the vorticity follows and consequently, in two-dimensions global existence holds:

For any non-integer $\alpha > 0$ and all $\omega_0 \in C^\alpha(M)$ there is a unique solution $\omega \in C^\alpha(\mathbb{R} \times M)$ with $\omega|_{t=0} = \omega_0$.

On the other hand, in three dimensions we now know by the work of Elgindi [16] that a finite time singularity occurs. Namely,

There exists an $\alpha > 0$, an $\omega_0 \in C^\alpha(\mathbb{R}^3)$ and a $T_* > 0$ such that $\omega(T_*)$ ceases to be in $L^1(0, T_*; L^\infty(\mathbb{R}^3))$.

In summary, with global existence in two-dimensions known, all the dynamical systems questions about long term behavior become admissible. In three dimensions, an entirely different approach is evidently called for.

Emergence of order

Understanding the long term dynamics of fluids in two-dimensions is fundamental to weather prediction, climate science and astrophysics. A rather mysterious feature is observed at long times: the fluid forms *coherent structures* (hurricanes, jet streams, etc). See Figure 1 for an example on the two-torus of *vortex merging* leading to the formation of a vortex dipole pair.

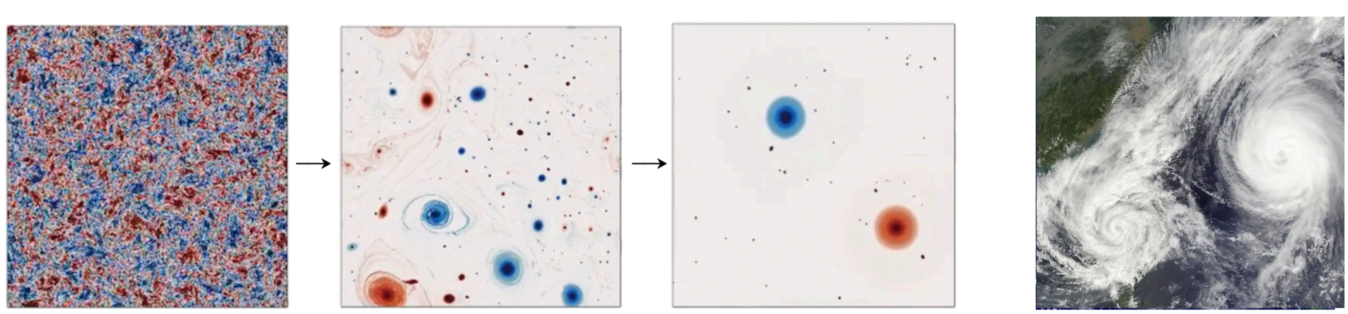

Figure 1. Vorticity in two-dimensions; emergence of a dipole over time from simulation [9, 6] and a pair of hurricanes.

There are many natural questions that arise. By which mechanism do coherent structures form? What is their shape? How many are there? Mathematically, it is surprising that these structures emerge since there is formally no friction mechanism in the Euler equations that could allow relaxation towards special states. In fact, the equations take the form of a time reversible, infinite dimensional Hamiltonian system. In finite dimensions, such reduction of complexity is forbidden by Poincaré recurrence.

The answer to this puzzle lies in the infinite dimensional nature of the Euler system. This feature allows a hidden *inviscid damping* facilitated by mixing of vorticity at long time. Singularities, the (infinitely) long term effect of mixing of vorticity, enable information to "exit" the system – ever finer filamentation generates a mille feuille of singularities that, in turn, average out inconsequential detail.

Yudovich [31] opened the door to studying long time dynamics by proving that Euler equations define a compact dynamical system on the phase space

$$X := \{\omega \in L^\infty(M) \ : \ \|\omega\|_{L^\infty(M)} \leq 1\}$$

endowed with the weak-$*$ topology, recalling that the definition of weak-$*$ convergence for a sequence is convergence upon integrating against arbitrary $L^1(M)$ functions. Indeed, the Euler equations (8) have a unique solution in X for all time $t \in \mathbb{R}$ and the solution map $S_t : X \to X$ is weak-$*$ continuous. As $t \to \infty$, since $\omega(t) = S_t(\omega_0)$ satisfies $\|\omega(t)\|_{L^\infty(M)} = \|\omega_0\|_{L^\infty(M)}$, we have weak-$*$ convergence

$$\omega(t_i) \overset{*}{\rightharpoonup} \overline{\omega} \quad \text{along subsequences} \quad t_i \to \infty.$$

Weak-$*$ limits $\overline{\omega}$ can forget oscillations, leaving behind only a coarse-grained representative of the vorticity. Denoting the weak-$*$ closure in $L^\infty(M)$ by $\overline{(\cdot)}^*$, for any $\omega_0 \in X$ we introduce the Omega limit set

$$\Omega_+(\omega_0) := \bigcap_{s\geq 0} \overline{\{S_t(\omega_0), t \geq s\}}^*.$$

This set collects all the weak-$*$ limits (e.g. all possible 'coarsened' persistent motions) as $t \to \infty$ along the solution $\omega(t) = S_t(\omega_0)$ passing through $\omega_0 \in X$. For a more global view, a (weak-$*$) attractor for the 2D Euler equations is naturally defined [9]

$$\Omega_+(X) := \bigcup_{\omega_0 \in X} \Omega_+(\omega_0).$$

The ultimate goal is to understand $\Omega_+(X)$: what kind of motions can survive indefinitely? This is a toy model for understanding the sorts of spots and stripes that might appear in the atmosphere of Jupiter or Saturn, and what type of weather patterns in Earth's atmosphere are long lived.

We take a moment to discuss what is observed from numerical simulations and physical observations about $\Omega_+(X)$. From generic initial data, large scale coherent structures emerge through vortex mergers over time. The formation and persistence of these features is a manifestation of the *inverse energy cascade* first postulated by Kraichnan [21]. These observations show that a great deal of diversity is lost over time – there is an apparent 'contraction' in phase space as a *decrease in entropy* causing the trend towards order. This phenomenon demands explanation from first principles. By what mechanism can information be lost? While energy and all Casimirs are conserved at finite time, weak-$*$ limits $\overline{\omega} \in \Omega_+(\omega_0)$ have the same energy as the initial data but may forget their initial Casimirs. Indeed, by lower-semicontinuity of $\mathrm{I}_f(\cdot)$ for convex f, the only information remembered is the inequality

$$\mathrm{I}_f(\overline{\omega}) \leq \liminf_{i\to\infty} \mathrm{I}_f(\omega(t_i)) = \mathrm{I}_f(\omega_0) \quad \text{for any convex } f.$$

We define *mixing* of vorticity by strict loss:

Definition We say that an Euler solution $\omega(t)$ **mixes** if there exists a weak-$*$ limit $\bar{\omega} \in \Omega_+(\omega_0)$ such that $\mathrm{I}_f(\overline{\omega}) < \mathrm{I}_f(\omega_0)$ for some strictly convex function f.

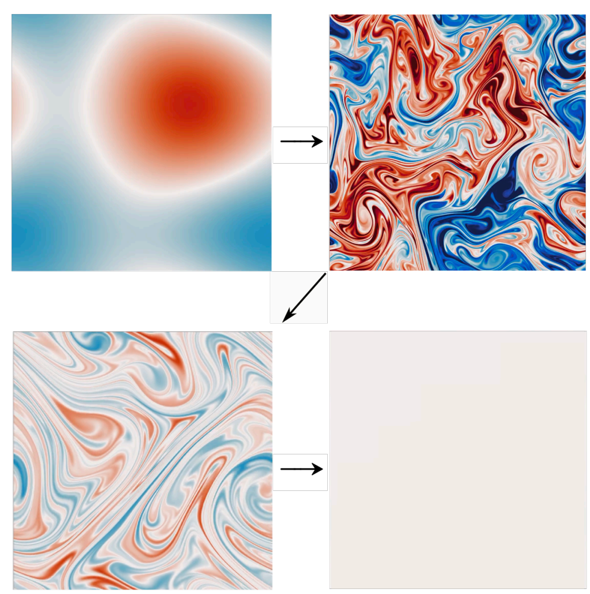

Figure 2. Perfect mixing (long time convergence to its space average) of a dye by a random velocity field [6].

Mixing is the mechanism by which "information" can be lost at long time. See Figure 2 for an extreme manifestation of this effect for randomly stirred dye. Notice the sharpening of the interfaces over time. Such *perfect* mixing is not possible for the vorticity of an ideal fluid since long time limits cannot lose energy – this is why some coherent features must remain. The following conjecture of Šverák [30] and Shnirelman [29] formalizes the intimate relation between mixing and long time behavior in Euler:

Conjecture *(Irreversibility and Entropy Decrease)*

(a) From generic data $\omega_0 \in L^\infty(M)$, the vorticity field **mixes** at infinite time.

(b) For any initial data $\omega_0 \in L^\infty(M)$, the Omega limit set $\Omega_+(\omega_0)$ consists of vorticities fields that generate Euler solutions which **do not mix** at infinite time, i.e.

$$\Omega_+(X) = \{\text{vorticities that ``do not mix'' as } t \to \infty\}.$$

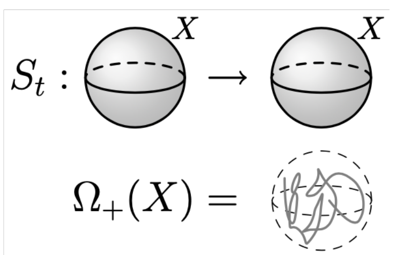

Figure 3. Due to finite-time reversibility, Euler preserves X. However the limit set $\Omega_+(X)$ is conjectured to be a sparse subset of X due to long-time irreversibility. [9]

Some known examples of non-mixing Euler solutions in the attractor $\Omega_+(X)$ are stationary, time periodic, quasiperiodic and chaotic states. According to the Conjecture, these represent a 'sparse' set in the entire X phase space (see Figure 3). As such, the Conjecture gives a notion of the perceived entropy decrease at long time – a "second law of thermodynamics" for ideal fluids.

We remark that, even if the conjecture were established, it tells very little about the shape of long time states. There are, however, some informed guesses. In his celebrated 1949 paper, Lars Onsager proposed a statistical approach based on large but finite dimensional approximations to predict specific most likely configurations of the flow. These ended up being a class of stable steady configurations (isolated vortices or jets) of fluid, see Figure 4. Note that stable steady states conform to the symmetries of space on which the fluid flows [7, 14]. This partly explains why hurricanes and cyclones are near circular, and Jovian jets are nearly zonal. Similarly, Shnirelman [27] (see also [8]) proved that if energy preservation were the *only* limiting factor to mixing of vorticity then stable steady states would emerge at long times. Both results suggest that large scale features should emerge from general data, but both are a step removed from the true dynamics.

Figure 4. Steady, stable fluid flows in a channel [3].

For Euler itself, very little is understood about the set $\Omega_+(\omega_0)$ for nearly any ω_0. Currently it is even unknown if $\Omega_+(X) \neq X$! The only setting where more can be understood is nearby stable *steady states*. In fact, nearby certain stable steady states on the channel, Bedrossian and Masmoudi successfully established some form of the conjecture, at least as it applies to data which are close in (essentially) the analytic topology [4]. They were able to prove *inviscid damping* (a fluid analogue of Landau damping) of perturbations towards (another) steady state via mixing. However such perturbations form a meager set in the natural phase space X.

Towards the full conjecture, it makes sense to establish some weaker features implied by the conjectured mixing. For example, mixing requires creation of small scales in the vorticity, which results from the stretching and folding of vortex lines by coherent velocity structures. It can be measured, for example, by norm growth in stronger topologies (say C^α). Yudovich enunciated a conjecture in the direction of genericity of this phenomenon [34, 23]:

Conjecture *"There is a 'substantial set' of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow."*

Norm growth, although weaker than mixing, already represents a qualitative expression of irreversibility for the Euler equations [28]. By now there are many examples of growth of voriticity gradients for specific Euler solutions (See [24, 19, 9]), but they are typically not stable under general perturbations and are thus may be non-generic. Recent work with T. Elgindi and I-J. Jeong establishes Yudovich's conjecture nearby stable steady states on annuli (Fig 4):

Theorem [10] *Let M be an annular surface and ω_* be a stable steady state. Then, for any $\alpha > 0$ there exists $\varepsilon > 0$ such that the set of initial data*

$$\left\{\omega_0 \in C^\alpha(M) \ : \ \sup_{t\geq 1} \frac{\|\omega(t)\|_{C^\alpha}}{|t|^{\alpha-}} = \infty\right\}$$

is dense in $\{\omega \in C^\alpha(M) \ : \ \|\omega - \omega_*\|_{C^\alpha(M)} \leq \varepsilon\}$.

This theorem follows from a strong understanding of the stability of qualitative properties such as twisting of the corresponding flowmaps. Of course, one would like to promote this norm growth to mixing.

Returning to the question of how the flow actually looks at long time, through careful numerics, Modin and Viviani [22] claim that the "typical" element of $\Omega_+(X)$ is a localized vortex blob moving like a point-vortex (a singular vortex carrying the same net circulation). Moreover, the number of such blobs *typically* coincides with the largest number of point vortices for which the motion is integrable. This claim represents a major refinement of the previous Conjecture saying that not only is $\Omega_+(X)$ meagre, but also the elements take specific shape (possibly removing some exceptional cases). Potentially being the "ghost" behind long term limits, these observations motivate the study of point and singular vortex dynamics as a proxy for coherent structures [11]. If M is the torus or the disk, these are dipole pairs. With D. Glukhovskiy and B. Khesin, we show that dipoles effectively follow geodesics away from solid boundaries [12] and behave at the boundary like a modified billiard system in which the "billiard ball" travels along the boundary for some distance depending on the incidence angle before reflecting, while preserving the billiard rule of equality of the angles of incidence and reflection [13]. See Figure 5.

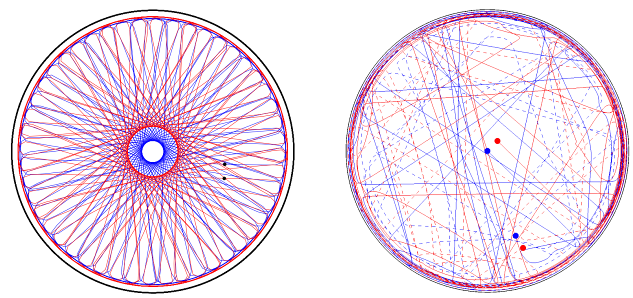

Figure 5. Billiard-like motion of dipole pairs on disk. [13]

In the end one thing seems certain: the world of ideal fluids, simple as they are to define, appears to be unimaginably rich!

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References

[1] Archimedes and S. T. L. Heath, The works of Archimedes, CUP Archive (1897).

[2] Arnold, V. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. In Annales de l'institut Fourier (Vol. 16, No. 1, pp. 319-361), 1966

[3] Arnold, V. I., and Khesin, B. A.: Topological methods in hydrodynamics. Vol. 125. Springer Science & Business Media, 1999.

[4] Bedrossian, J. and Masmoudi, N. Inviscid damping and the asymptotic stability of planar shear flows in the 2d Euler equations, Publications mathématiques de l'IH ES 122 (2015), no. 1, 195–300.

[5] Beale, J. T., Kato, T. and Majda, A. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Communications in Mathematical Physics 94.1 (1984): 61-66.

[6] Constantinou, Navid, Gregory Wagner, Lia Siegelman, Brodie Pearson, and André Palóczy. GeophysicalFlows. jl: Solvers for geophysical fluid dynamics problems in periodic domains on CPUs GPUs. Journal of Open Source Software 6, no. 60 (2021).

[7] Constantin, P., Drivas, T. D., and Ginsberg, D. Flexibility and rigidity in steady fluid motion, Communications in Mathematical Physics 385 (2021), no. 1, 521–563.

[8] Dolce, M., and Drivas, T. D. (2022). On maximally mixed equilibria of two-dimensional perfect fluids. Archive for Rational Mechanics and Analysis, 246(2), 735-770.

[9] Drivas, T. D., and Elgindi, T. M. (2023). Singularity formation in the incompressible Euler equation in finite and infinite time. EMS Surveys in Mathematical Sciences, 10(1), 1-100.

[10] Drivas, T. D., Elgindi, T. M., and Jeong, I. J. (2024). Twisting in Hamiltonian flows and perfect fluids. Inventiones mathematicae, 238(1), 331-370.

[11] Drivas, T. D., Elgindi, T. M., and La, J. (2023). Propagation of singularities by Osgood vector fields and for 2D inviscid incompressible fluids. Mathematische Annalen, 387(3), 1691-1718.

[12] Drivas, T. D., Glukhovskiy, D. and Khesin. Singular vortex pairs follow magnetic geodesics. International Mathematics Research Notices 2024.14 (2024): 10880-10894.

[13] Drivas, T. D., Glukhovskiy, D. and Khesin. B. Pensive Billiards, Point Vortices, And The Silver Ratio. Forum of Mathematics, Sigma. Vol. 13. Cambridge University Press, 2025.

[14] Drivas, T. D. and Nualart, M. (2024). A geometric characterization of steady laminar flow. arXiv preprint arXiv:2410.18946.

[15] Ebin, D. G., and Marsden, J.. Groups of diffeomorphisms and the motion of an incompressible fluid. Annals of Mathematics 92.1 (1970): 102-163.

[16] Elgindi, T. M. Finite-time Singularity Formation for $C^{1,\alpha}$ Solutions to the Incompressible Euler Equations on $\mathbb{R}^3$. Annals of Mathematics, 194(3), 647-727, (2021).

[17] Euler, L. Principes généraux du mouvement des fluides. Académie Royale des Sciences et des Belles, 11, 274–315 (1757)

[18] Feynman, R P., Leighton, R.B. and Sands, M. The Feynman lectures on physics, Vol. II: Basic books, 2011.

[19] Kiselev, A. and Šverák, V. (2014). Small scale creation for solutions of the incompressible two-dimensional Euler equation. Annals of mathematics, 180(3), 1205-1220.

[20] Kolmogorov, A. N. Dissipation of energy in the locally isotropic turbulence. In Dokl. Akad. Nauk. SSSR, vol. 32, pp. 19-21. 1941.

[21] Kraichnan, R. H. Inertial ranges in two-dimensional turbulence, The Physics of Fluids 10 (1967), no. 7, 1417–1423.

[22] Modin, K., and Viviani, M. (2022). Canonical scale separation in two-dimensional incompressible hydrodynamics. Journal of Fluid Mechanics, 943, A36.

[23] Morgulis, A., Shnirelman, A. I. and Yudovich, V. I. Loss of smoothness and inherent instability of 2d inviscid fluid flows, Communications in Partial Differential Equations 33 (2008), no. 6, 943–968.

[24] Nadirashvili, N. S. (1991). Wandering solutions of Euler's D-2 equation. Functional Analysis and Its Applications, 25(3), 220-221.

[25] Onsager, L. Statistical hydrodynamics, Il Nuovo Cimento (1943-1954) 6 (1949), no. 2, 279–287.

[26] Salmon, R. Hamiltonian fluid mechanics. Annual review of fluid mechanics 20.1 (1988): 225-256.

[27] Shnirelman, A. I. Lattice theory and flows of ideal incompressible fluid, Russ. J. Math. Phys 1 (1993), no. 1, 105–113.

[28] Shnirelman, A. I. Evolution of singularities, generalized Liapunov function and generalized integral for an ideal incompressible fluid. American Journal of Mathematics 119.3 (1997): 579-608.

[29] Shnirelman, A. I. On the long time behavior of fluid flows, Procedia IUTAM 7 (2013), 151–160.

[30] Šverák, V. (2012). Selected Topics in Fluid Mechanics, lectures notes of an introductory graduate course taught in 2011.

[31] Yudovich, V. I. Non-stationary flows of an ideal incompressible fluid, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 3 (1963), no. 6, 1032–1066.

[32] Yudovich, V. I. The loss of smoothness of the solutions of euler equations with time, Dinamika Splosn. Sredy Vyp 16 (1974), 71–78.

[33] Yudovich, V. I. Global solvability versus collapse in the dynamics of an incompressible fluid. Mathematical events of the twentieth century. Springer, Berlin, Heidelberg, 2006. 501-528.

[34] Yudovich, V. I. On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid, Chaos: An Interdisciplinary Journal of Nonlinear Science 10 (2000), no. 3, 705–719.