On the support of Anomalous Dissipation

with:

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Anomalous Dissipation of Energy

The fundamental postulate of Kolmogorov's 1941 theory, the "zeroth law of turbulence" is a non-vanishing dissipation as $Re \to \infty$.

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \qquad \nabla \cdot u = 0$$

For dimensions $d \geq 3$, the only known a-priori controlled quantities which are controlled is from

$$\partial_t \left(\tfrac{1}{2}|u|^2\right) + \nabla \cdot \left(u \left(\tfrac{1}{2}|u|^2 + p\right) - \nu \nabla \tfrac{1}{2}|u|^2 \right) = -\nu |\nabla u|^2 + f \cdot u$$

provided the solution is smooth. Thus

$$(*) \quad \frac{d}{dt} \int_{\mathbb{T}^d} \tfrac{1}{2}|u|^2 dx = -\nu \int_{\mathbb{T}^d} |\nabla u|^2 dx + \int_{\mathbb{T}^d} f \cdot u \, dx$$

Gives apriori control of the solution in

$$u \in L^\infty_t L^2_x \cap L^2_t H^1_x .$$

Leray (1934) used this energy balance for a suitable approximation scheme combined with a compactness arguement to prove existence of a global-in-time weak solution of NS

- These satisfy $(*)$ with $\leq$.

what is known is the equality (Duchon-Robert, 2000)

$$\partial_t\left(\tfrac{1}{2}|u^\nu|^2\right) + \nabla\cdot\left(u^\nu\left(\frac{|u^\nu|^2}{2} + p^\nu\right) - \nu\nabla\frac{|u^\nu|^2}{2}\right) = -\nu|\nabla u^\nu|^2 - D[u^\nu] + u^\nu\cdot f$$

where the (x,t)-distribution $D[u]$ is defined by a weak form of the Kármán–Howarth–Monin relation:

$$D[u](x,t) = \lim_{\ell\to 0}\frac{1}{4}\int_{\mathbb{T}^d}\nabla\phi_\ell(r)\cdot\delta_r u(x,t)\,|\delta_r u(x,t)|^2\,dr$$

where $\delta_r u(x,t) = u(x+r,t) - u(x,t)$ and $\phi_\ell(r) = \frac{1}{\ell^d}\phi\left(\frac{r}{\ell}\right)$.

D&R show the distributional limit of $L^1_{t,x}$ objects is independent of the choice of ϕ.

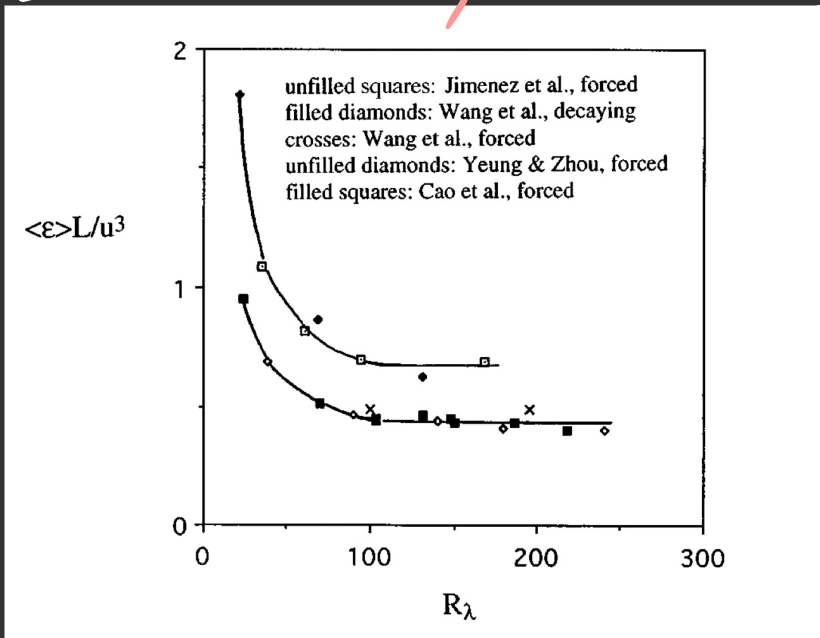
Moreover, it is nonnegative. It is a measure!

Assume $u^\nu \xrightarrow{L^3} u$. Then, we have a 4/5th law:

$$\underbrace{D[u]}_{\text{Inertia dissipation of Euler solution}} = \underbrace{\lim_{\nu\to 0}\left(\nu|\nabla u^\nu|^2 + D[u^\nu]\right)}_{\text{anomalous viscous dissipation}}$$

3

Sreenivasan, 1998



Pearson, Krogstad, de Water, 2001

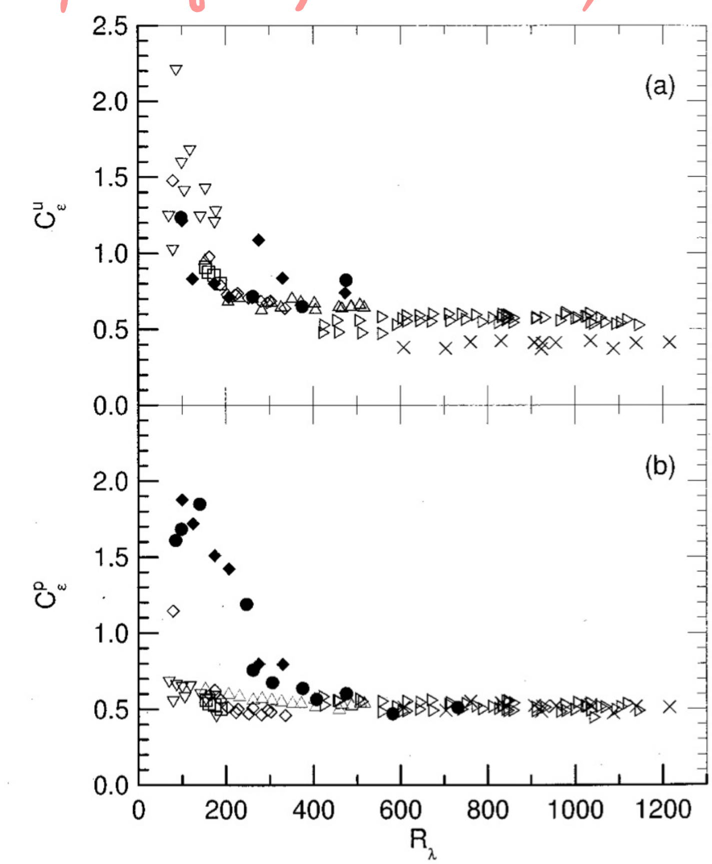
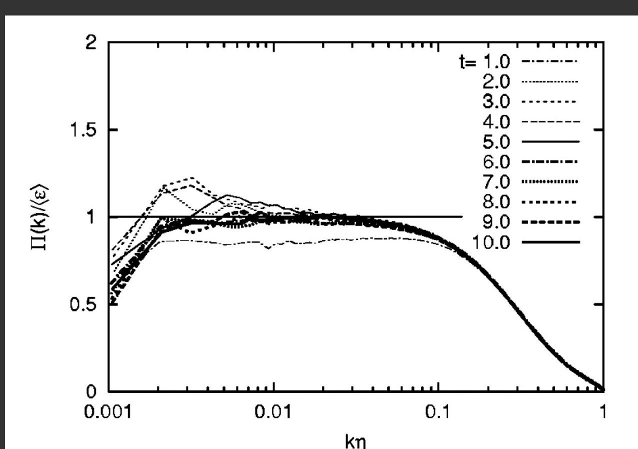
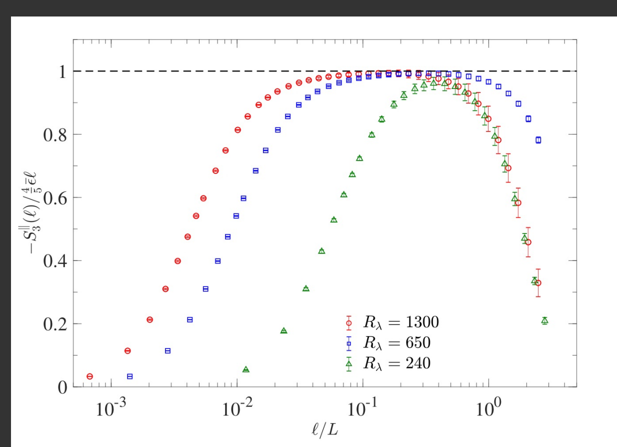


FIG. 1. Normalized dissipation rate for a number of shear flows. Details as found in this work and Refs. 14–16. (a) C_ϵ^u [Eq. (3)]; (b) C_ϵ^p [Eq. (4)]. □, circular disk, $154\lesssim R_\lambda\lesssim 188$; ▽, pipe, $70\lesssim R_\lambda\lesssim 178$; ◇, normal plate, $79\lesssim R_\lambda\lesssim 335$; △, NORMAN grid, $174\lesssim R_\lambda\lesssim 516$; × NORMAN grid (slight mean shear, $dU/dy\approx dU/dy|_{max}/2$), $607\lesssim R_\lambda\lesssim 1217$; ▷, NORMAN grid (zero mean shear), $425\lesssim R_\lambda\lesssim 1120$; ●, "active" grid Refs. 14, 15, $100\lesssim R_\lambda\lesssim 731$; ◆, "active" grid, with L_u estimated by Ref. 16. For Ref. 14 data, we estimate $L_p\approx 0.1$ m and for Ref. 15 data we estimate $L_p\approx 0.225$ m.

4/5th Law:



Kaneda et al 2003



Iyer et al, 2020.

General Anomalous Dissipation Measures

All take the form $\nabla \cdot V = -D$ ← positive measure

- Incompressible Euler from NS $u^\nu \to u$

$$V = \left(\tfrac{1}{2}|u|^2, \left(\frac{|u|^2}{2} + p \right) u \right) \qquad D = \lim_{\nu \to 0} \nu |\nabla u^\nu|^2$$

- Transported Scalars $\qquad |\theta^\kappa|_{L^\infty} \leq C.$

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

$$V = \left(\frac{\theta^2}{2}, u \frac{\theta^2}{2} \right), \qquad D = \lim_{\kappa \to 0} \kappa |\nabla \theta^\kappa|^2$$

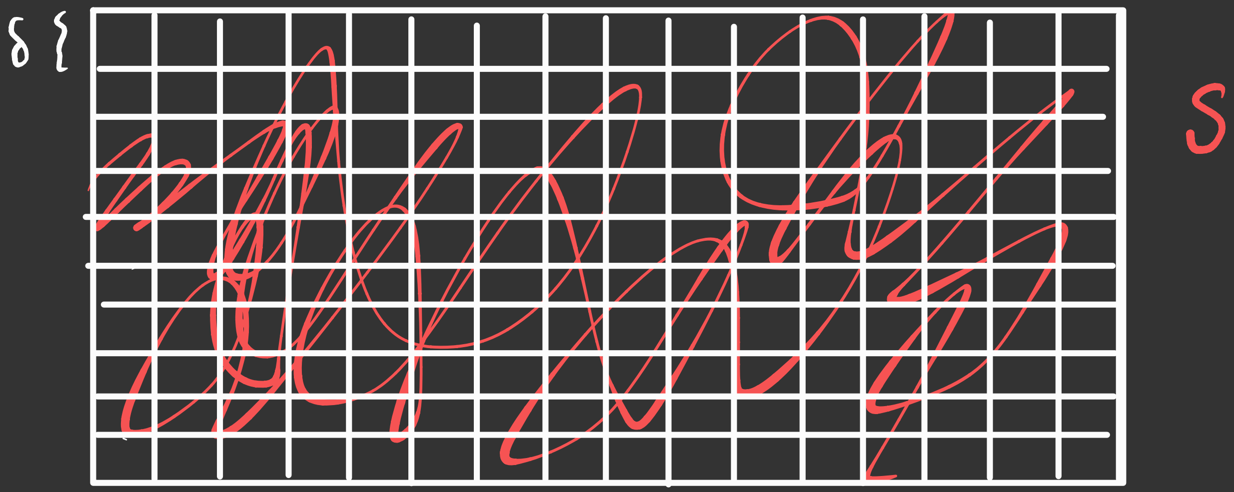
- Compressible Euler

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + pI) = 0$$
$$\partial_t E + \nabla \cdot ((p+E)u) = 0$$

$$V = (s, us) \qquad D = \lim_{\nu, \kappa, 2 \to 0} \text{Dissipation}$$

- Navier-Stokes also! "Smooth piece" $\nu |\nabla u|^2$ + "rough piece" $D[u^\nu]$.

Fractal Dimensions



$N_\delta(S)$ = # of cubes of size δ which cover S

$$\dim_B(S) = \lim_{\delta\to 0} \frac{\log N_\delta(S)}{\log(1/\delta)}$$

e.g. $\dim_B(S) = d$
$N_\delta(S) \sim \delta^{-d}$

(upper and lower with limsup, liminf)

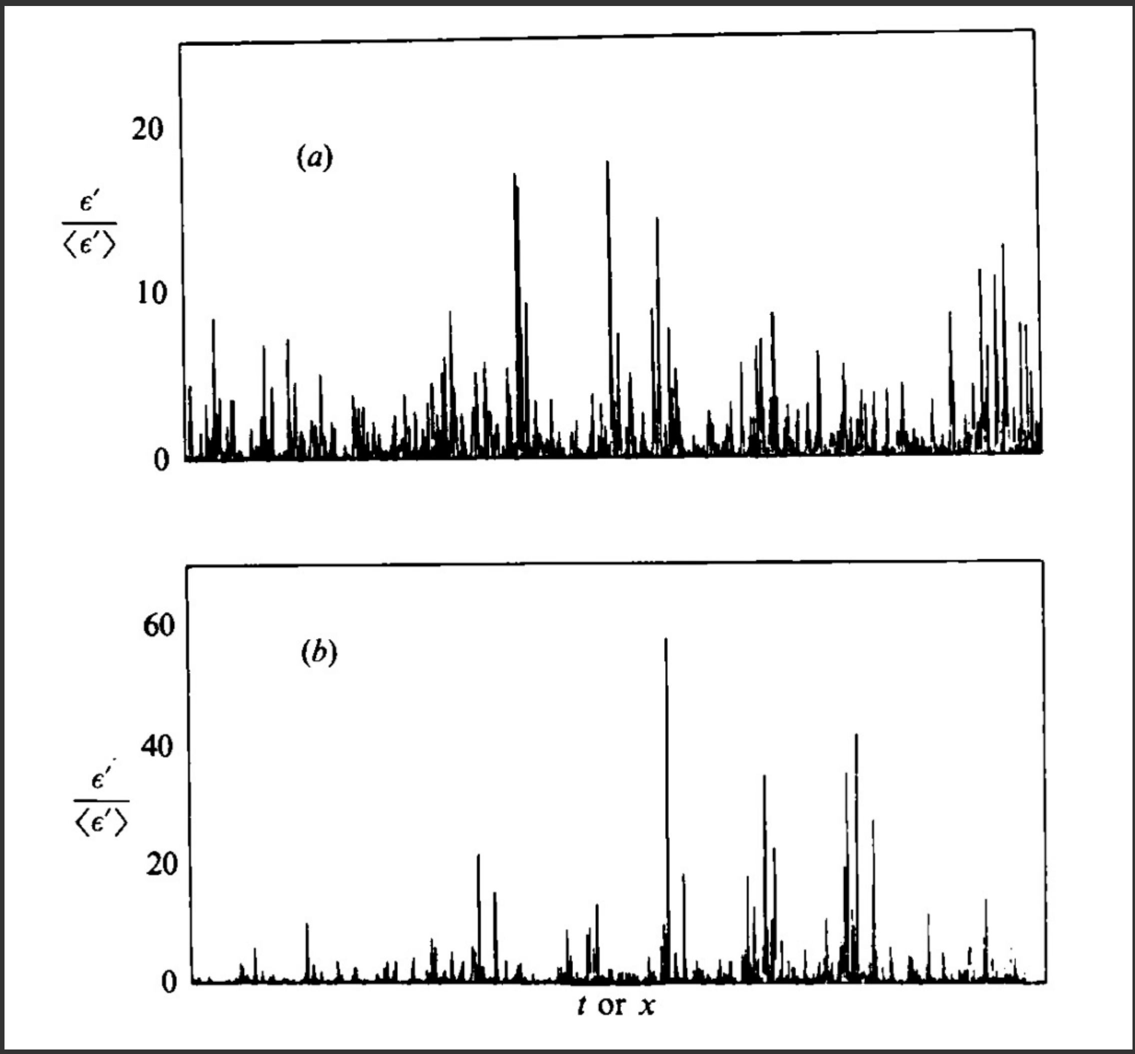
Hausdorff dimension $d_H(S)$ is a more measure theoretic notion. Basic difference is one can cover with balls of radius $\leq \delta$, not just $=\delta$. Thus

$$\dim_H(S) \leq \underline{\dim}_B(S) \leq \overline{\dim}_B(S)$$

(rational #s have $\dim_H = 0$, $\dim_B = 1$)

In following statements, you can just think Box counting.

Meneveau & Sreenivasan 1991



The estimate that

$$\dim_B(\text{dissipation}) \approx 3.87$$

Occurring on a fractal set!

$\boxed{\text{Any restrictions?}}$

Theorem: Bounded weak solutions of any of those equations, on domains $\Omega \subseteq \mathbb{R}^d$ which produce entropy anomalously, have

$$\dim_H(\operatorname{Spt} D) \geq d.$$

↖ space-time

Remark: Dissipation measure on bounded domain is defined as, a positive measure $\mathcal{M}_{loc}(\bar{\Omega} \times [0,T])$ s.t.

$$\int_0^T \int_\Omega \left[\frac{|u|^2}{2} \partial_t \phi + \left(\frac{|u|^2}{2} + p\right) u \cdot \nabla \phi\right] dx\, dt = \int_0^T \int_\Omega \phi \, dD[u]$$

for all $\phi \in C_c^1(\bar{\Omega} \times [0,T])$. This can be established for limits of strong slns of NS.

Remark. Theorem holds assuming $u \in L^p$. then

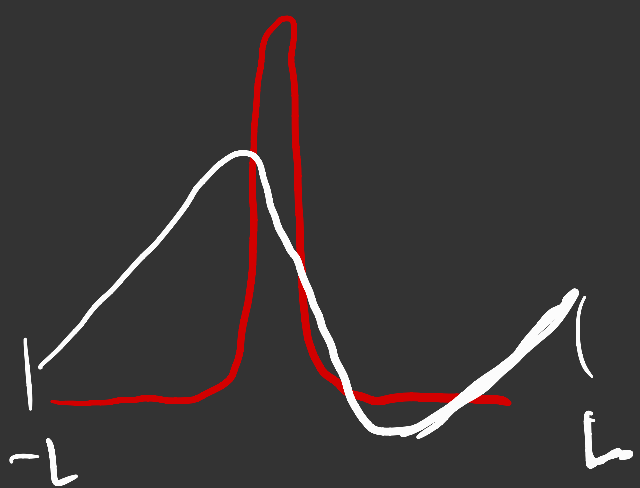
$$\dim_H(\operatorname{Spt} D) \geq d+1 - \frac{p}{p-1}$$

Also, if $u \in L^p_t L^q_x$, there are results which can optimize with non isotropic Hausdorff measure.

Examples Burgers equation: $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u$$

Remarks:
- for $\nu > 0$, model is globally wellposed
- for $\nu = 0$, model shocks in finite time.

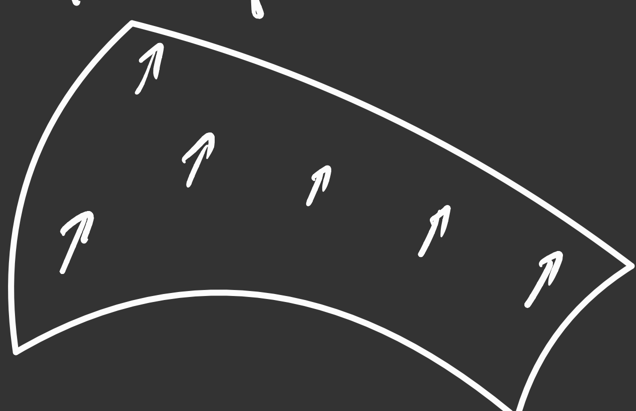


$$\varepsilon^\nu(x,t) = \nu |\partial_x u^\nu|^2 \xrightarrow{\nu \to 0} \frac{(\Delta u)^3}{12} \delta(x)$$

Dissipation happening at points moving cts in time,

$\dim_H(\text{diss}) = 1.$

Shocks in compressible Euler

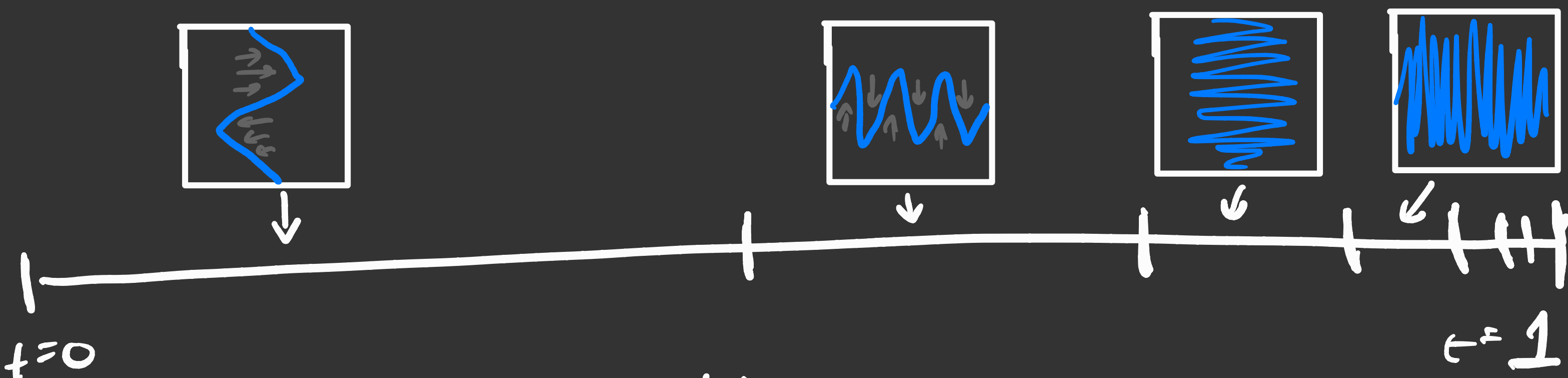


shock surface: codimension 1 at each time. smoothly evolving

$\dim_H(\text{entropy prod}) = d.$

Example:

Constructions: (D. Elgindi - Iyer - Jeong
Colombo - Crippa - Sorella)



$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

$$\int_0^1 \kappa \|\nabla \theta\|_{L^2}^2 \, ds \geq c_0 > 0$$

All the dissipation is happening at one time.

Similar constructions for Navier-Stokes, ($2+\frac{1}{2}$ dim)
(Brue - De Lellis, Brue - Colombo - Crippa - De Lellis - Sorella)

In all cases; our result shows that at time = 1, dissipation is full measure in space!

$$\dim_H(\text{diss}) = d$$

Idea of proof.

Theorem: (Frostman): Let μ be a pos Borel measure on $\mathbb{R}^d \times \mathbb{R}$. Let $\delta_0 > 0$ and assume $\forall (x,t) \in \operatorname{Spt} \mu$, and any $\delta \in (0, \delta_0)$,

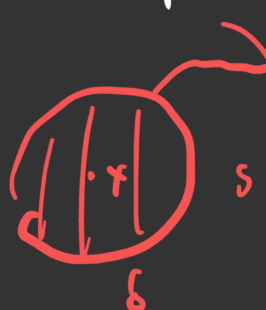
$$\mu(B_\delta(x,t)) \lesssim w(\delta)\, \delta^s$$

where $w(\delta)$. Then μ is abs-cont w.r.t. $\mathcal{H}^s$.
If $w(\delta)$ is a modulus, $\mu(A) = 0$ for any Borel set A st $\mathcal{H}^s(A) < \infty$. E.g. Since μ nontrivial $\mu(\operatorname{Spt}\mu) > 0$.

$$\dim_{\mathcal{H}}(\operatorname{Spt} \mu) \geq s.$$

Assume $\mu = \operatorname{div} V$ is a locally positive measure.
Fix $x \in \mathbb{R}^d$ and let χ_δ be cutoff localizing to $B_\delta(x)$ s.t. $\chi_\delta \equiv 1$ on $B_\delta(x)$ and $|\nabla \chi_\delta| \lesssim \delta^{-1}$.

$$\mu(B_\delta(x)) \underset{\text{pos measure}}{\leq} \int_{\mathbb{R}^d} \chi_\delta(x)\, d\mu = -\int V \cdot \nabla \chi_\delta(x)\, dx$$



$$\leq \|V\|_{L^p(B_{2\delta})} \left(\int_{B_{2\delta}} |\nabla \chi|^q \right)^{1/q} \qquad q = \frac{p}{p-1}$$

$$\lesssim w_V(2\delta)\, \delta^{d\frac{p-1}{p} - 1}$$

$$\Rightarrow \dim_{\mathcal{H}}(\operatorname{Spt} \mu) \geq d\frac{p-1}{p} - 1$$

Thank -you!

Obtaining boundary dissipation measure

- Let $S = \overline{\Omega \times [0,T)}$ ← closed set
- Consider $C_c^0(S)$ (normed space with sup norm)
- Consider sequence of functionals on $C_c^0(S)$ given by integral of dissipation (indexed by ν), times any test function $\phi \in C_c^0(S)$.

  Call this $F_\nu[\phi] = \int_0^T \int_\Omega \phi \, \nu |\nabla u|^2 \, dx \, dt$.

- this functional belongs to dual of $C_c^0(S)$.

  Endow with weak-* topology

- Having dissipation bounded in L^1 space-time gives that F_ν is a bounded sequence of functionals on the dual.

  Banach-Alaoglu implies subsequence converging to a functional F belonging to same dual.

- Since along seq F^ν is positive, F is positive.

  Apply Riezs on $C_c^0(S) \Rightarrow \exists$ pos measure μ on S s.t. $F(\phi) = \int \phi \, d\mu$.