on the support of Anomalous Dissipation

with:

L. De Rosa and M. Inversi
(U Basil)

Theodore D. Drivas
Stony Brook University
Anomalous Dissipation of Energy

The fundamental postulate of Kolmogorov's 1941 theory, the "zeroth law of turbulence" is a non-vanishing dissipation as $Re \to \infty$.

$$
\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \quad \nabla \cdot u = 0
$$

For dimensions $d \geq 3$, the only known a-priori controlled quantities which are controlled is from

$$
\partial_t \left( \frac{1}{2} |u|^2 \right) + \nabla \cdot \left( u \left( \frac{1}{2} |u|^2 + p \right) - \nu \nabla \frac{1}{2} |u|^2 \right) = -\nu |\nabla u|^2 + f \cdot u
$$

provided the solution is smooth. Thus

$$
\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u|^2 \, dx = -\nu \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} f \cdot u \, dx
$$

Gives apriori control of the solution in

$$
u \in L^\infty_t L^2_x \cap L^2_t H^1_x.$$

Leray (1934) used this energy balance for a suitable approximation scheme combined with a compactness argument to prove existence of a global-in-time weak solution of NS. These satisfy $(\star)$ with $\mathcal{E}$. 

What is known is the equality (Duchon-Robert, 2000)

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |u|^2 \right) + \nabla \cdot \left( u \left( \frac{|u|^2}{2} + \rho \right) - \nu \nabla \frac{|u|^2}{2} \right) = -\nu |\nabla u|^2 - D[u'] + u \cdot f
\]

where the \((x,t)\)-distribution \(D[u']\) is defined by a weak form of the Kůržín–Howarth–Monin relation:

\[
D[u'](x,t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \nabla \phi_\varepsilon(r) \cdot \delta u(x,t) |\delta x u(x,t)|^2 dr
\]

where \(\delta u(x,t) = u(x+r,t) - u(x,t)\) and \(\phi_\varepsilon(r) = \frac{1}{2\varepsilon} \phi \left( \frac{r}{\varepsilon} \right)\).

D&R show the distributional limit of \(L^1_{x,t}\) objects is independent of the choice of \(\phi\). Moreover, it is nonnegative. It is a measure!

Assume \(\nabla^3 \to u\). Then, we have a \(\sqrt{\nu} \to 1\) law:

\[
D[u] = \lim_{\nu \to 0} \left( \nu |\nabla u|^2 + D[u'] \right)
\]

Inertial dissipation of Euler solution

anomalous

viscous dissipation
FIG. 1. Normalized dissipation rate for a number of shear flows. Details as found in this work and Refs. 14–16. (a) $C_\varepsilon^+$ [Eq. (3)]; (b) $C_\varepsilon^+$ [Eq. (4)]. □, circular disk, $154 \leq R_\lambda \leq 188$; ▽, pipe, $70 \leq R_\lambda \leq 178$; ◆, normal plate, $79 \leq R_\lambda \leq 335$; △, NORMAN grid, $174 \leq R_\lambda \leq 516$; × NORMAN grid (slight mean shear, $dU/dy \approx U_{\delta+}/(2)$), $607 \leq R_\lambda \leq 1217$; ▽, NORMAN grid (zero mean shear), $425 \leq R_\lambda \leq 1120$; , “active” grid Refs. 14, 15, $100 \leq R_\lambda \leq 731$; , “active” grid, with $L_p \approx 0.1$ m and for Ref. 15 data we estimate $L_p \approx 0.225$ m.
General Anomalous Dissipation Measures

All take the form $\nabla \cdot \mathbf{V} = -D$

- Incompressible Euler from NS $u^0 \rightarrow u$
  \[ \mathbf{V} = \left( \frac{1}{2} |u|^2, \frac{1}{2} |u|^2 + p \right) \]
  \[ D = \lim_{u \rightarrow 0} \frac{1}{|u|^2} \]

- Transported Scalars
  \[ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta \]
  \[ \mathbf{V} = \left( \frac{\theta^2}{2}, \frac{\theta^2}{2} \right) \]
  \[ D = \lim_{\kappa \rightarrow 0} 10\kappa \]

- Compressible Euler
  \[ \partial_t p + \nabla \cdot (p \mathbf{u}) = 0 \]
  \[ \partial_t \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + p \mathbf{I}) = 0 \]
  \[ \partial_t E + \nabla \cdot (\rho E \mathbf{u}) = 0 \]
  \[ \mathbf{V} = (s, \mathbf{u}s) \]
  \[ D = \lim_{s \rightarrow 0} \frac{1}{s} \text{ Dissipation} \]

- Navier-Stokes also!
  "Smooth piece" $\nabla \cdot \mathbf{V} + 10|\mathbf{u}|^2$
  "Rough piece" $D[\mathbf{u}]$.
Fractal Dimensions

Let $N_8(S)$ denote the number of cubes of size 8 which cover $S$.

\[
\text{dim}_B(S) = \lim_{\varepsilon \to 0} \frac{\log N_8(S)}{\log \left( \frac{1}{\varepsilon} \right)}
\]

This is an upper and lower bound (limsup, liminf).

Hausdorff dimension $d_+(S)$ is a more measurement theoretic notion. The basic difference is one can cover with balls of radius $\leq \varepsilon$, not just $= \varepsilon$. Thus:

$\text{dim}_{H}(S) \leq \text{dim}_{B}(S) \leq \overline{\text{dim}}_{B}(S)$

(rational $\varepsilon$ have $d_+ = 0$, $d_B = 1$).

In following statements, you can just think box counting.
The estimate that
\[ \text{dim}_p(\text{dissipation}) \approx 3.87 \]

Occurring on a fractal set? Any restrictions?
Theorem: Bounded weak solutions of any of those equations, on domains \( D \subset \mathbb{R}^d \) which produce entropy anomalously, have \( \dim_H (\text{Spt } D) \geq d \).

Remark: Dissipation measure on bounded domain is defined as a positive measure \( M_\mu (\mathbb{R} \times [0,T]) \) s.t.
\[
\int_0^T \int_\Omega \left[ \frac{1}{2} \frac{d}{dt} \mathbf{\phi} + \left( \frac{1}{2} u_0^2 + p \right) \nabla \cdot \mathbf{\phi} \right] dx \, dt = \int_\Omega \phi \, dM_\mu(T)
\]
for all \( \phi \in C^1_c (\overline{\Omega} \times [0,T]) \). This can be established for limit of strong solutions of NS.

Remark: Theorem holds assuming \( u \in L^p \), then
\[
\dim_H (\text{Spt } D) \geq d + 1 - \frac{p}{p-1}
\]
Also, if \( u \in L^p_t L^q_x \), there are results which can optimize with non-isotropic Hausdorff measure.
Examples

Burgers equation:

\[ u: \mathbb{T} \times \mathbb{R} \to \mathbb{R} \]

\[ \partial_t u + u \partial_x u = \nu \partial_x^2 u \]

Remarks:

- for \( \nu > 0 \), model is globally wellposed
- for \( \nu = 0 \), model shocks in finite time.

Dissipation happening at points moving ctn in time,

\[ \dim_{\text{diss}}(\text{diss}) = 1. \]

Shocks in compressible Euler

Shock surface: Codimension 1 at each time. Smooth evolving

\[ \dim_{\text{H}}(\text{cutting point}) = d. \]
Example:

Constructions: (D. Elgindi - Iyer - Temj
Colombo - Crippa - Sovella)

\[ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta \]

\[ \frac{1}{\epsilon} \int_{\Omega} |\nabla \theta|^2 \, dx \geq c_0 > 0 \]

All the dissipation is happening at one time.

Similar constructions for Navier-Stokes, \((2+\frac{1}{2} \text{ dim})\)
(Brun - DeLellis, Brun-Columbo-Crippa-DeLellis-Sovella)

In all cases, our result shows that at time \(t=1\),

dissipation is full measure in space:

\[ \dim_{H}(\text{diss}) = d \]
Idea of proof.

Theorem: (Furstman) Let \( \mu \) be a pos Borel measure on \( \mathbb{R}^d \times \mathbb{R} \). Let \( \delta_0 > 0 \) and assume \( \forall (x,t) \in \text{Spt} \mu \), and any \( \delta \in (0,\delta_0) \),

\[
\mu(B_\delta(x,t)) \leq \omega(\delta) \delta^S
\]

where \( \omega(\delta) \). Then \( \mu \) is abs-cont w.r.t. \( \mathcal{H}^S \).

If \( \omega(\delta) \) is a modulus, \( \mu(A) = 0 \) for any Borel set \( A \) s.t. \( \mathcal{H}^S(A) < \infty \). E.g., since \( \mu \) nontrivial \( \mu(\text{Spt} \mu) > 0 \),

\[
\operatorname{dim}_H(\text{Spt} \mu) \geq S.
\]

Assume \( \mu = \text{div} V \) is a locally positive measure.

Fix \( x \in \mathbb{R}^d \) and let \( \chi_\delta \) be cutoff localising to \( B_\delta(x) \) s.t. \( \chi_\delta \equiv 1 \) on \( B_\delta(x) \) and \( \| \nabla \chi_\delta \| \leq \delta \).

\[
\mu(B_\delta(x)) \leq \int \chi_\delta(x) \, d\mu = -\int V \cdot \nabla \chi_\delta(x) \, dx
\]

\[
\leq \| V \|_{L^p(B_{2\delta})} \left( \int \chi_\delta(x)^2 \right)^{\frac{1}{2}} \quad q = \frac{d}{p-1}
\]

\[
\leq \omega_V(2\delta) \delta^{\frac{dp-1}{p}} - 1
\]

\[
\Rightarrow \operatorname{dim}_H(\text{Spt} \mu) \geq \frac{d}{p} - 1
\]
Thank you!
Obtaining boundary dissipation measure

0. Let \( S = \mathbb{R} \times (0,T) \)

0. Consider \( C^0_c(S) \) (normed space with sup norm)

0. Consider sequence of functionals on \( C^0_c(S) \) given by integral of dissipation (indeed by \( \mathbf{v} \)), times any test function \( \phi \in C^0_c(S) \).

Call this \( F^\mathbf{v}[\phi] = \int_S \phi \sqrt{1+\mathbf{v}^2} \, dx \, dt \).

0. This functional belongs to dual of \( C^0_c(S) \).

Endow with weak-* topology.

0. Having dissipation bounded in \( L^1 \) space-time gives that \( F^\mathbf{v} \) is a bounded sequence of functionals on the dual.

Banach–Alaoglu implies subsequence converging to a functional \( F \) belonging to some dual.

0. Since along seq \( F^\mathbf{v} \) is positive, \( F \) is positive.

Apply Riesz on \( C^0_c(S) \Rightarrow \exists \) pos measure \( \mu \) on \( S \) s.t. \( F(\phi) = \int \phi \, d\mu \).