The Feynman–Lagerstrom Criterion for Boundary Layers

joint work with

Sameer Iyer and Trinh Nguyen.

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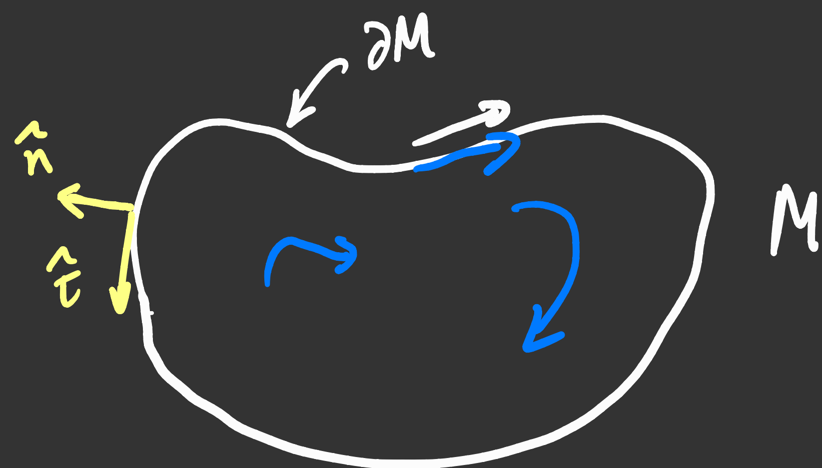
Consider the Navier-Stokes equations on



$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \nu \Delta u^\nu \quad \text{in } M$$

$$\nabla \cdot u^\nu = 0 \quad \text{in } M$$

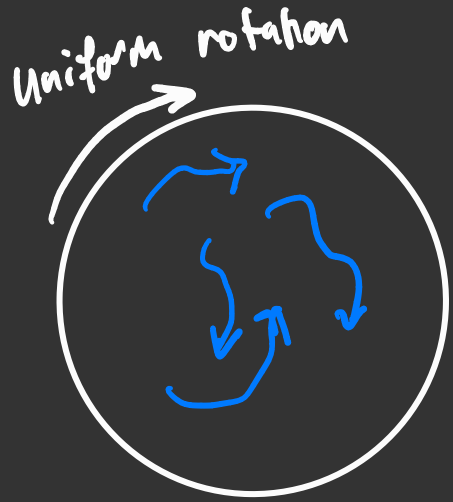
$$u^\nu \cdot \hat{n} = 0 \quad \text{on } \partial M$$

$$u^\nu \cdot \hat{\tau} = f \quad \text{on } \partial M$$

$$u^\nu \big|_{t=0} = u_0 \quad \text{in } M$$

Motion is generated by moving the solid walls, possibly in a <u>non-uniform</u> way.

Consider the simplest situation



$M =$ disk or Radius R

$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \nu \Delta u^\nu \quad \text{in} \quad \{r \le R\}$$

$$\nabla \cdot u^\nu = 0 \quad \text{in} \quad \{r \le R\}$$

$$u^\nu \cdot \hat{n} = 0 \quad \text{on} \quad \{r = R\}$$

$$u^\nu \cdot \hat{\tau} = \tfrac{1}{2}\omega_0 R \quad \text{on} \quad \{r = R\}$$

Theorem: For any initial data u_0, $u^\nu(t) \xrightarrow{t\to\infty} u_{sb} := \frac{1}{2}\omega_0 x^\perp$.

In fact,

$$\|u^\nu(t) - u_{sb}\|_{L^2} \le \|u_0 - u_{sb}\|_{L^2} e^{-\lambda_1 \nu t}$$

↑ independent of ν.

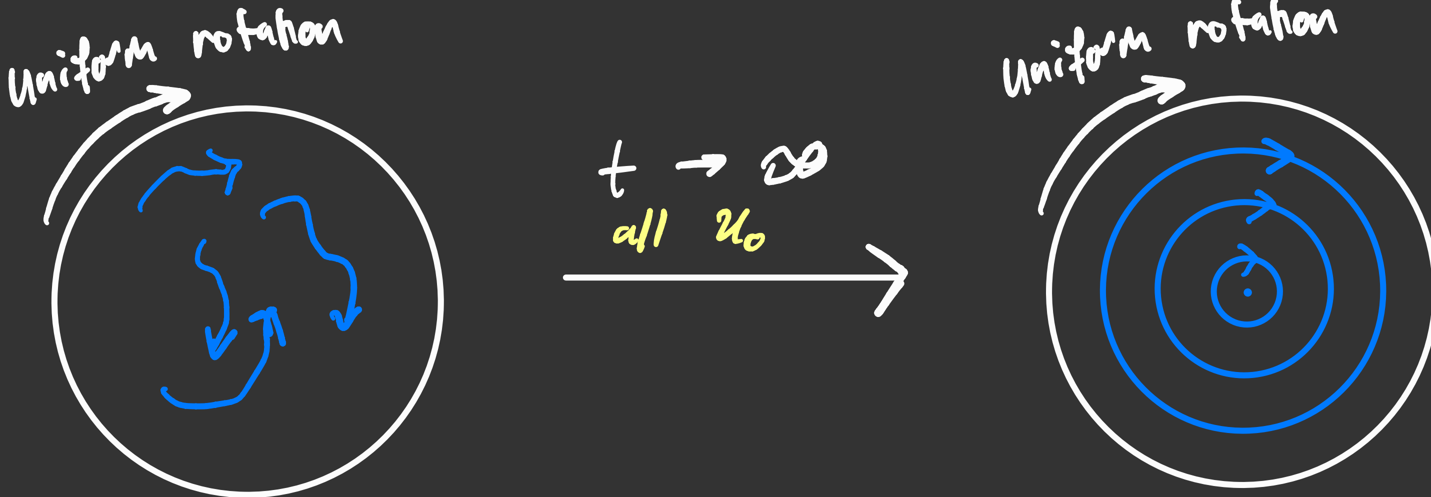
Proof: Let $w = u^\nu - u_{sb}$. Then

$$\partial_t w + u_{sb}\cdot\nabla w + w\cdot\nabla u_{sb} + w\cdot\nabla w = -\nabla q + \nu\Delta w \qquad w|_{\partial M} = 0$$

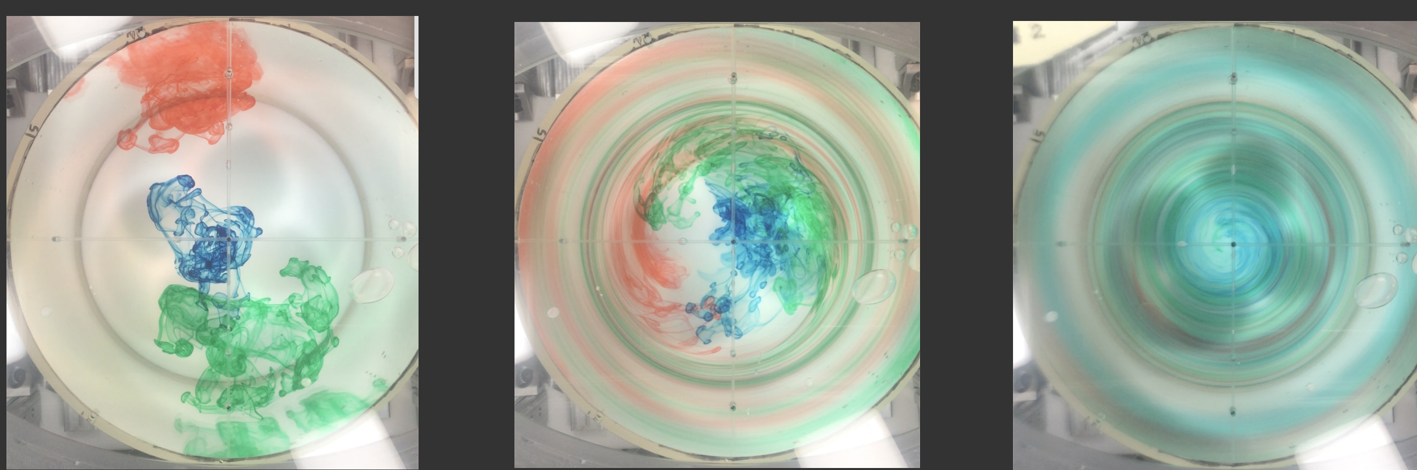
But (!) $\nabla u_{sb} = \omega_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus $w\cdot\nabla u_{sb} = w^\perp$.

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 = -\nu\|\nabla w\|_{L^2}^2 \le -\nu\lambda_1\|w\|_{L^2}^2. \qquad \lambda_1 \sim \frac{1}{R^2}.$$

Thus, we have the **Absence of Turbulence**:



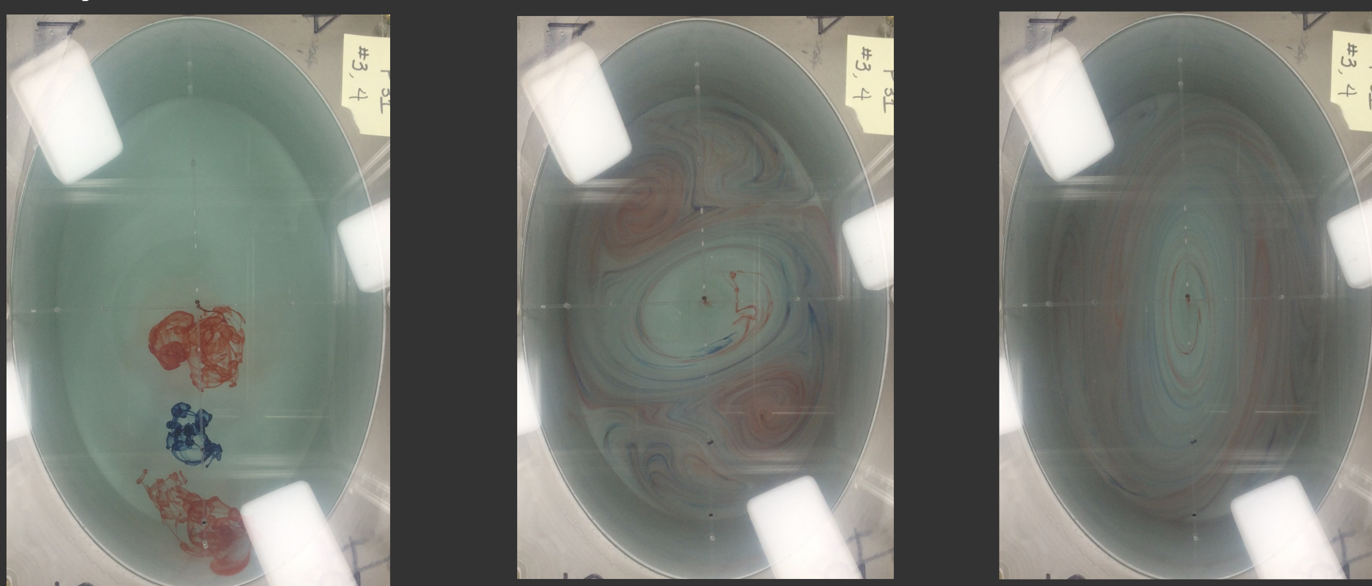
This is analogous to Marchioro's result on $\mathbb{T}^2$ forced by $f = \begin{pmatrix} \sin y \\ 0 \end{pmatrix}$ (e.g. the gravest mode).



Experiments by $\boxed{\text{Diane Henderson}}$, Penn State.

Question: What if the imposed slip is non-constant, or the domain is not perfectly circular?

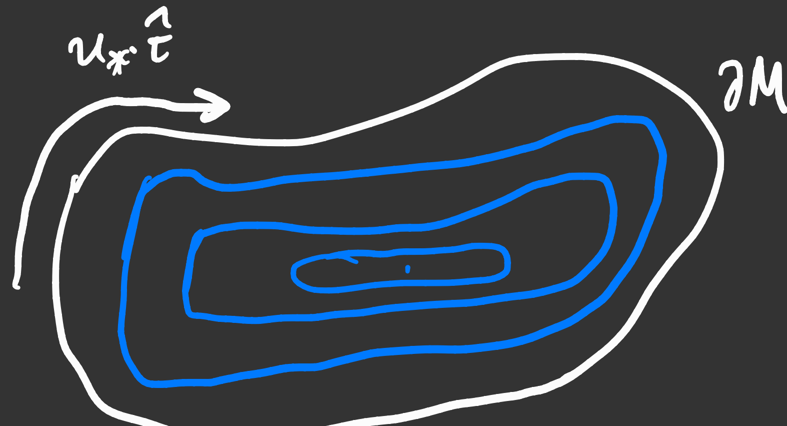
Generally one would expect time dependence to never disappear entirely in these settings. However there always exist stationary states, which will likely shape the large scale features of the flow.



Experiments by $\boxed{\text{Diane Henderson}}$, Penn State.

special case: the state of constant vorticity:

$u_* = K_M[1]$, $\quad K_M = \nabla^\perp \Delta_D^{-1}$



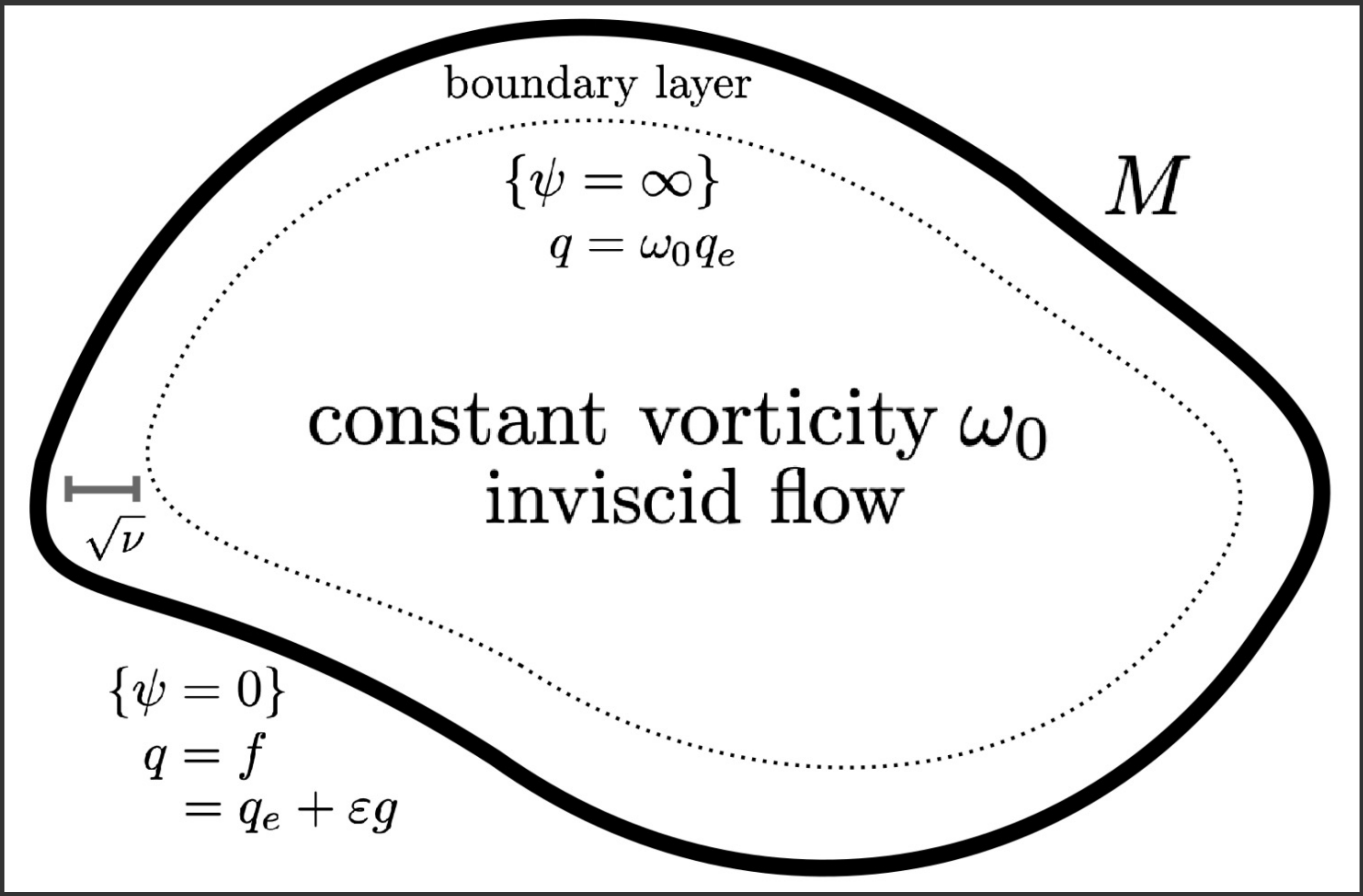
This is a stationary solution of Navier-Stokes for every $\nu \geq 0$. Thus, if taken as initial data and if forced by it's slip velocity, the solution is forever.

Theorem: (Prandtl 1904 & Batchelor 1956)

Let M be simply connected. Suppose u_e is an Euler solution in M with a single stagnation point. Let $\{u^\nu\}$ be a family of steady Navier-Stokes solutions. If $u^\nu \to u_e$, say in C^2, then

$$u_e = \omega_0 u_* \quad \text{for some } \omega_0$$

That is, u_e must have constant vorticity within M.



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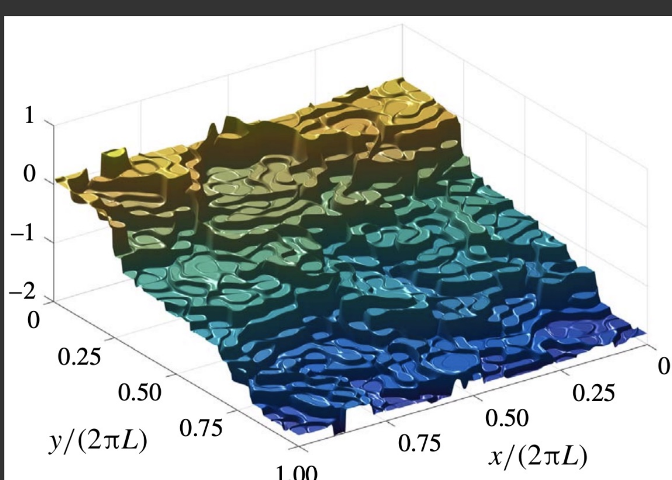
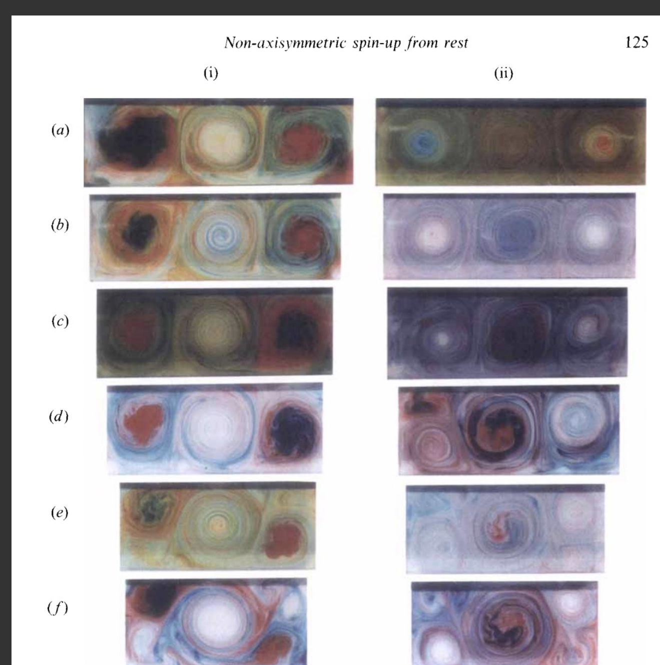
Remark: M can represent closed streamlines of the limiting Euler solution, possible yielding a staircase of vorticity, separated (perhaps) by vortex sheets...



Constantinou and Young, 2017
β-plane turbulence



Henderson, Lopez, Stewart
JFM 1996

Proof of Prandtl-Batchelor Theorem

First note that, under the stated assumptions,

$$\omega_e = F(\psi_e) \quad \text{for a Lip. } F:\mathbb{R}\to\mathbb{R}$$

WLOG suppose $\{\psi_e = 0\}$ is the unique critical point.

Stationary vorticity equation:

$$u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu$$

Integrate over a sublevel set $\{\psi^\nu \le c\}$:

$$0 = \frac{1}{\nu}\int_{\{\psi^\nu \le c\}} u^\nu \cdot \nabla \omega^\nu \, dx = \int_{\{\psi^\nu \le c\}} \Delta \omega^\nu \, dx$$

$$= \int_{\{\psi^\nu = c\}} \partial_n \omega^\nu \, d\sigma$$

$$\xrightarrow{\nu \to 0} \int_{\{\psi_e = c\}} \partial_n \omega_e \, d\sigma = F'(c) \int_{\{\psi_e = c\}} \partial_n \psi \, d\sigma$$

$$= F'(c) \oint_{\{\psi_e = c\}} u_e \cdot dl \iff F'(c) = 0 \quad \forall c$$

This all begs the question

Question : What determines the limiting vorticity ω_0?

This is the content of Feynman's only published work in fluid dynamics (as far as I know):

REMARKS ON HIGH REYNOLDS NUMBER FLOWS IN FINITE DOMAINS[1]

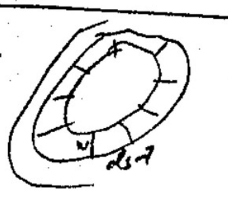
by R. P. FEYNMAN and P. A. LAGERSTROM,
Professor of Aeronautics, California
Institute of Technology, Pasadena, California, U.S.A.

Consider two-dimensional, stationary viscous incompressible flow in a finite domain D on whose boundary B the normal velocity component is zero and the tangential flow component is prescribed to be $u_w(s)$, (s = distance along B). As pointed out by Prandtl in 1905 the limiting flow as Re (Reynolds number) tends to infinity has constant vorticity ω_0 (actually several vortices may occur, each one with different vorticity). Consider the case of only one vortex. The Euler equations for the limiting flow then reduce to Poisson's equation $\nabla^2\psi = -\omega_0$, the solution of which is proportional to ω_0. At B this solution gives a tangential velocity $u_e(s) = \omega_0 f(s)$. Here $f(s)$ depends on the geometry of the domain only and may be considered known. For very large values of Re the vorticity is then essentially constant except in a thin boundary layer near B where the velocity changes rapidly from $u_w(s)$ to $u_e(s)$. One may now determine ω_0 from the requirement that such a boundary layer (periodic in s) be possible. For a circle of radius L simple momentum considerations then lead to the formula $(L\omega_0/2)^2 = \overline{u}_w^{\,2}$. (bar denotes average over B). Explicit formulas are difficult to find for general domains. If $|u_w - u_e| \ll u_w$ an approximate formula for ω_0 is $\overline{\omega_0^2 f^3} = \overline{u_w^2 f}$. The above equations are necessary but not sufficient conditions for existence of a boundary layer as described. Non-existence of such a boundary layer indicates that the limiting flow consists of several vortices.

Incidentally, he seems to have independently derived the Prandtl-Batchelor theorem:

cons. of circulation.

Application to two dimensional flow.



Put $q_x = u = \frac{\partial \psi}{\partial y} = \psi_y$

$q_y = v = -\frac{\partial \psi}{\partial x} = -\psi_x$

$\therefore \omega = \nabla^2 \psi$

$\int q \cdot (\nabla \times \omega)\, d\,vol = 0.$ Apply to thin region between two streamlines which are ~~different~~ adjacent as shown. Let ds = length element. w = separation (is function of s).

$\therefore \oint q \cdot (\nabla \times \omega)\, ds \cdot w = 0$

But q is tangential, and ~~$q w$~~ qw = const by continuity.

$\therefore \oint (\nabla \times \omega)_{tang}\, ds = 0$ But $(\nabla \times \omega)_{tang} = \frac{\partial \omega}{\partial n}$ = Derivative respect to normal.

$\therefore$ **THEO.** $\oint \frac{\partial \omega}{\partial N}\, ds = 0$ The mean normal derivative of vorticity taken around a streamline is zero.

Suppose ν is very small, and there is a region in which ~~no potential~~ the flow is essentially that of Euler ($\nu = 0$). Then ω is constant on a stream-line, ~~it is therefore~~ Therefore it is a function of ψ only,

So $\frac{\partial \omega}{\partial n} = \frac{\partial \omega}{\partial \psi} \cdot \frac{\partial \psi}{\partial n} = \frac{\partial \omega}{\partial \psi} q_{tang}$ But $\frac{\partial \omega}{\partial \psi}$ is constant over the streamline, so

$\int \frac{\partial \omega}{\partial \psi} q_t\, ds = 0 = \frac{\partial \omega}{\partial \psi} \int q_t\, ds$ $\therefore$ Either $\frac{\partial \omega}{\partial \psi} = 0$, or $\int q_t\, ds = 0$. But $\int q_t\, ds$ is the circulation. It may be zero for one streamline, but not for two in succession, unless $\omega = 0$. If $\omega = 0$ then $\frac{\partial \omega}{\partial \psi} = 0$. $\therefore \frac{\partial \omega}{\partial \psi} = 0$. $\therefore \omega$ is the same on all stream lines.

THEO. $\therefore$ (ω cons in pattern) In the free flow region, in the limit $\nu \to 0$, regions whose streamlines can be shrunk to a common point, are regions of constant ω.

And thought about what happens in the case of multiple eddy formation...

The Interior Flow, for ν very small.

Suppose, as we have shown in other places, the flow is $\omega = \text{const}$ in regions except for two kinds of discontinuities:

1. Internal boundries. ω jumps. Here we have (probably) shown that u is continuous, ω is discont.
   p is continuous
   and if we take $\psi = 0$ on all the boundries (all at once, of course) then R is continuous, where $R = p + \frac{1}{2}(u^2+v^2) - \omega\psi$.

$\omega = \psi_x - u_y$
$= \nabla^2\psi$
$u = -\psi_y$
$v = \psi_x$
$u_x = -v_y$

2. Surface boundry layers. u-jumps. Here we have shown that p is continuous
   $q_n \sim 0$ ($q_n = 0$ at surface)

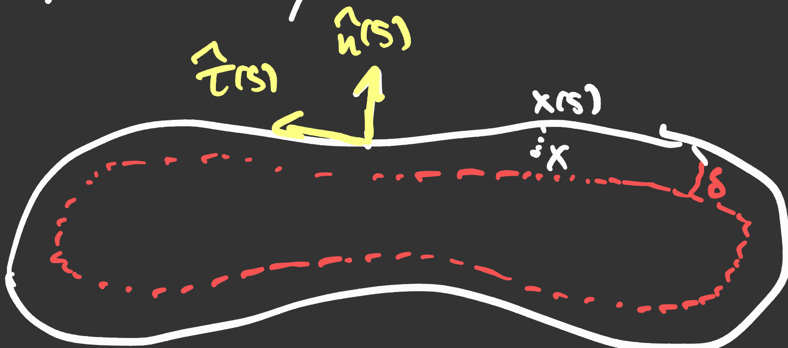
   R jumps, but $\Delta R = +\frac{1}{2}(\Delta u^2)$ (in general $\Delta R = +\frac{1}{2}\Delta q_t^2$)

$q_n = q$ normal
$q_t = q$ tangential to a surface

We call the actual surface velocity f.
We call the ideal surface velocity which eliminates surface boundry layers (but leaves internal boundries unchanged) g. Then ~~ΔR~~ $R_{\text{just inside}} - R_{\text{surface}}$
$= +\frac{1}{2}(g^2 - f^2)$.

Let's revisit Feynman's idea for determining the vorticity value ω_0. First we must derive the Prandtl equations in von-Mises coordinates.

Let $s:[0,L] \to \partial M$ be the arc length parametrization of the boundary. Due to smoothness, $\exists\ \delta$ s.t. if $\text{dist}(x, \partial M) < \delta$, there exists a unique closest boundary point to x, call it $x(s)$.



Let $z = \text{dist}(x, \partial M)$ and s be our local coordinates.

Then

$$x(s,z) = x(s) + z\,\hat{n}(s)$$

and $\gamma(s) = x_1'' x_2' - x_1' x_2''$ is boundary curvature

$J(s,z) = 1 + z\,\gamma(s)$ is Jacobian of change of variables

Let

$$u_\tau(s,z) = u^\nu(x(s,z)) \cdot \hat{\tau}(s)$$

$$u_n(s,z) = u^\nu(x(s,z)) \cdot \hat{n}(s)$$

on disk: $\gamma = -1$, $J(r,\theta) = \frac{1}{r}$, $u_\tau = u_\theta$, $u_n = u_r$

Navier-Stokes near the boundary:

$$\frac{u_\tau}{J}\partial_s u_\tau + u_n \partial_z u_\tau - \frac{\gamma}{J} u_\tau u_n + \frac{1}{J}\partial_s P$$
$$= \nu \left[\frac{1}{J}\partial_z (J \partial_z u_\tau) + \frac{1}{J}\partial_s\left(\frac{1}{J}\partial_s u_\tau\right) - \frac{1}{J}\partial_s\left(\frac{\gamma u_n}{J}\right) - \frac{\gamma}{J}\left(\gamma u_\tau + \partial_s u_n\right) \right]$$

$$\frac{u_\tau}{J}\partial_s u_n + u_n \partial_z u_n - \frac{\gamma}{J} u_\tau^2 + \partial_z P$$
$$= \nu \left[\frac{1}{J}\partial_z (J \partial_z u_n) + \frac{1}{J}\partial_s\left(\frac{1}{J}\partial_s u_n\right) - \frac{1}{J}\partial_s\left(\frac{\gamma u_\tau}{J}\right) - \frac{\gamma}{J}\left(\partial_s u_\tau - \gamma u_n\right) \right]$$

$$\partial_z u_n + \frac{1}{J}\partial_s u_\tau - \frac{\gamma}{J} u_n = 0$$

Now, in a layer of width $\sqrt{\nu}$, we anticipate

$$(u_\tau, u_n) \approx \underbrace{(w_0 q_e(s), 0)}_{\text{Euler}} + \underbrace{(u_\tau^P, u_n^P)}_{\text{Prandtl}}$$

slip profile of unit vorticity Euler

boundary layer

Specifically

$$u_\tau(s,z) \sim w_0 q_e(s) + u_\tau^P\left(s, \frac{z}{\sqrt{\nu}}\right)$$

$$u_n(s,z) = \sqrt{\nu}\, u_n^P\left(s, \frac{z}{\sqrt{\nu}}\right)$$

where $\lim_{z\to\infty} u_\tau^P(s,z) = 0$

Plugging this ansatz into Navier-stokes and using that, with $Z = \frac{z}{\sqrt{\nu}}$:

$$\frac{1}{J} = \frac{1}{1 + z\gamma(s)} \sim 1 - \sqrt{z}\, Z\gamma(s) + \mathcal{O}(\nu)$$

We obtain:

$$\text{(1)} \quad (w_0 q_e(s) + u_\tau^P)\, \partial_s (w_0 q_e(s) + u_\tau^P) + u_n^P\, \partial_Z (w_0 q_e(s) + u_\tau^P) + \partial_s p - \partial_Z^2 u_\tau^P = 0$$

$$\text{(2)} \quad \partial_Z p = 0$$

together with incompressibility

$$\text{(3)} \quad \partial_s (w_0 q_e(s) + u_\tau^P) + \partial_Z u_n^P = 0$$

Taking $Z \to \infty$ in (1), since $p = p(s)$, we obtain

$$\partial_s p = - w_0^2\, q_e(s)\, q_e'(s).$$

Substituting into the above, we arrive at Prandtl's equations:

$$\text{(P1)} \quad (w_0 q_e(s) + u_\tau^P)\, \partial_s (w_0 q_e(s) + u_\tau^P) + u_n^P\, \partial_Z (w_0 q_e(s) + u_\tau^P) - w_0^2 q_e q_e'(s) - \partial_Z^2 u_\tau^P = 0$$

$$\text{(P2)} \quad \partial_s (w_0 q_e(s) + u_\tau^P) + \partial_Z u_n^P = 0$$

(P1) $$(w_0 q_e(s) + u_\tau^P)\, \partial_s (w_0 q_e(s) + u_\tau^P) + u_n^P\, \partial_z (w_0 q_e(s) + u_\tau^P) - w_0^2 q_e q_e'(s) - \partial_z^2 u_\tau^P = 0$$

(P2) $$\partial_s (w_0 q_e(s) + u_\tau^P) + \partial_z u_n^P = 0$$

Now, define von Mises variables (s, ψ):

$$\partial_z \tilde{\Psi} = w_0 q_e(s) + u_\tau^P(s,z) \qquad \psi = \frac{\tilde{\Psi}}{\sqrt{\nu}}$$

$$-\partial_s \tilde{\Psi} = u_n^P(s,z)$$

Let $q = q(\psi, s) = w_0 q_e(s) + u_\tau^P$.

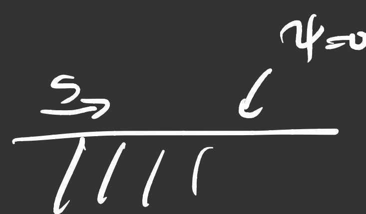
$q(0,s) = f(s)$

NS slip

Prandtl's equations become

pressure term.

$$\partial_s (q^2) - w_0^2 \partial_s (q_e^2) - q\, \partial_\psi^2 (q^2) = 0$$



or, Letting $Q = q^2 - w_0^2 q_e^2$,

$$\boxed{\begin{aligned} &\partial_s Q - q\, \partial_\psi^2 Q = 0 \\ &Q(0,s) = f^2(s) - w_0^2 q_e^2(s) \\ &Q(\infty, s) = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\partial_s Q - q\,\partial_\psi^2 Q = 0 \\ &Q(0,s) = f^2(s) - w_0^2 q_e^2(s) \\ &Q(\infty, s) = 0 \end{aligned}}$$

Feynman-Lagerstrom (1956) condition

w_0 is selected so that ↑ has a periodic solution in s.

In the case of the disk, this is explicit! (Noted also by Batchelor (1956) and Wood (1957))

Since $q_e(s) = \frac{R}{2}$ $\left(u_e = \frac{1}{2}x^\perp\right)$, we have:

$\partial_s Q = 2q\,\partial_s q$. Thus $\partial_s q - \frac{1}{2}\partial_\psi^2 Q = 0$

$$\partial_\psi^2 \int_0^L Q(s,\psi)\,ds = 0 \iff \int_0^L Q(s,\psi)\,ds = 0$$

Evaluating at $\psi = 0$, using $Q(s,0) = f^2(s) - w_0^2 (R/2)^2$,

$$w_0^2 = \frac{1}{(R/2)^2}\int_0^{2\pi R} f^2(\theta)\,d\theta$$

this is only thing remembered from Navier-Stokes.

$$\boxed{\begin{aligned} &\partial_s Q - q\,\partial_\psi^2 Q = 0 \\ &Q(0,s) = f^2(s) - w_0^2 q_e^2(s) \\ &Q(\infty,s) = 0 \end{aligned}}$$

Feynman-Lagerstrom condition

w_0 is selected so that ↑ has a periodic solution in s.

Let us derive a general (nonlinear) criterion:

$$\partial_s Q - w_0 q_e \partial_\psi^2 Q = -(q - w_0 q_e)\,\partial_\psi^2 Q$$

$$= -(q - w_0 q_e)\,\frac{\partial_s Q}{q}$$

$$= \left(1 - \frac{w_0 q_e}{\sqrt{w_0^2 q_e^2 + Q}}\right)\partial_s Q$$

$$\Rightarrow \quad \partial_\psi^2\left[\int_0^L q_e(s)\, Q\, ds\right] = \underbrace{\frac{1}{w_0}\int_0^L \left(1 - \frac{w_0 q_e(s)}{\sqrt{w_0^2 q_e^2 + Q}}\right)\partial_s Q\, ds}_{N^{w_0}(Q)}$$

$$\Rightarrow \quad \int_0^L q_e(s)\, Q(s,\psi)\, ds = \int_\psi^\infty y\, N^{w_0}[Q](y)\, dy \qquad \forall\, \psi \geq 0$$

$$\int_0^L q_e(s)\, Q(s,\psi)\, ds = \int_\psi^\infty y\, N^{\omega_0}[Q](y)\, dy \quad \forall\ \psi \geq 0$$

Evaluating at $\psi = 0$, we find:

$$\boxed{\int_0^L q_e(s)\left(f^2(s) - \omega_0^2\, q_e^2(s)\right) ds = \int_0^\infty y\, N^{\omega_0}[Q](y)\, dy}$$

$f(s) = q_e(s) + \varepsilon g(s)$ in this case $1 - \omega_0^2 = \varepsilon \bar{\omega}_0$

perturbation of unit vorticity

with $\bar{\omega}_0 = O(1)$

We anticipate $Q = \mathcal{O}(\varepsilon)$. Then $N[Q] = \mathcal{O}(Q^2) = \mathcal{O}(\varepsilon^2)$.

Then

$$\omega_0^2 = \frac{\int_0^L q_e(s)\, f^2(s)\, ds}{\int_0^L q_e^3(s)\, ds} + \mathcal{O}\left(|\partial_s \kappa|\, \varepsilon^2 \right)$$

proximate formula for ω_o is $\omega_o^2 f^3 = u_w^2 f$. The above equations are necessary but not sufficient conditions for existence of a boundary layer as described. Non-existence of such a boundary layer indicates that the limiting flow consists of several vortices.

(D., Iyer, Nguyen)

Theorem: Let $q_e(s)$ be any smooth, nonvanishing periodic function. Let $f(s) = q_e(s) + \varepsilon g(s)$.

For all ε sufficiently small, there exist a unique $w_0 \in \mathbb{R}$ and function $q : [0,L) \times \mathbb{R}^+ \to \mathbb{R}$ such that (w_0, q) solves the Prandtl equations

$$\partial_s Q - q\, \partial_\psi^2 Q = 0 \qquad Q = q^2 - w_0^2 q_e^2$$

$$Q(s,0) = f^2(s) - w_0^2 q_e^2(s)$$

$$Q(s,\infty) = 0$$

Moreover, the selected vorticity obeys

$$w_0^2 = \frac{\int_0^L q_e(s) f^2(s)\, ds}{\int_0^L q_e^3(s)\, ds} + w_{err}, \qquad |w_{err}| \le C\varepsilon^2.$$

$$\partial_s Q - q\,\partial_\phi^2 Q = 0 \qquad Q = q^2 - w_0^2 q_e^2$$

$$Q(s,0) = f^2(s) - w_0^2 q_e^2(s)$$

$$Q(s,\infty) = 0$$

rewrite as, with $f(s) = q_e(s) + \varepsilon g(s)$

$$\partial_s Q + w_0 q_e \partial_\phi^2 Q = \left(1 - \frac{w_0 q_e}{\sqrt{w_0^2 q_e^2 + Q}}\right)\partial_s Q$$

$$Q(s,0) = (1-w_0^2)\,q_e^2 + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g(s)$$

$$Q(s,\infty) = 0$$

Recall w_0 is chosen so that Q exist (periodic), Q then implicitly depends on w_0. We study...

the following iteration

$$\partial_s Q_n + w_{0_{n-1}} q_e \partial_\phi^2 Q_n = \overbrace{\left(1 - \frac{w_{0_{n-1}} q_e}{\sqrt{w_{0_{n-1}}^2 q_e^2 + Q_{n-1}}}\right)\partial_s Q_{n-1}}^{F_{n-1}}$$

$$Q_n(s,0) = (1-w_{0_n}^2)\,q_e^2 + 2\varepsilon g(s) q_e(s) + \varepsilon^2 g(s)$$

$$Q(s,\infty) = 0$$

$\overline{w_0}_n = 1 - w_{0_n}^2$ determined by $\left(1 - \frac{w_{0_{n-1}} q_e}{\sqrt{w_{0_{n-1}}^2 q_e^2 + Q_{n-1}}}\right) \partial_s Q_{n-1}$

$$\overline{w_0}_n = \frac{\fint (2 g q_e^2 + \varepsilon g^2 q_e) ds}{\fint q_3^3 ds} + \frac{\fint \int_0^\infty y f(Q_{n-1}, w_{n-1}) ds}{w_{0_{n-1}} \varepsilon \fint q_e^3 ds}$$

to control, we require 1 s-derivative of Q

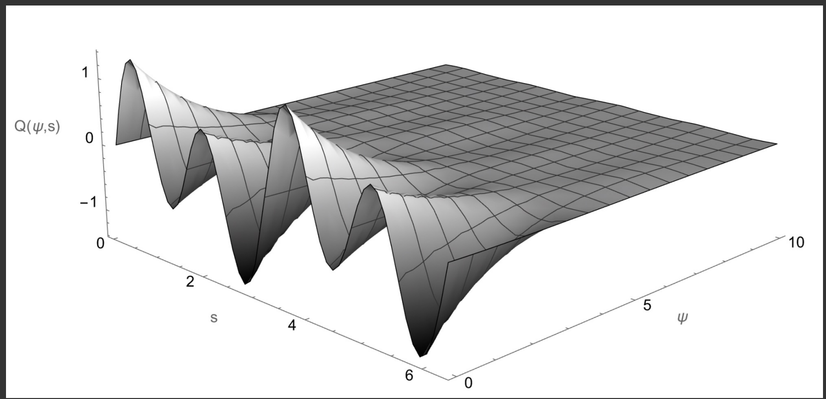
→ On the other hand, if $w_{0_{n-1}}$ is known, we can obtain many derivatives of Q

$$\partial_s Q_n + w_{0_{n-1}} q_e \partial_\psi^2 Q_n = F_{n-1}$$

Since this equation is a "time"-periodic heat.

→ Moreover, to make sense of $\fint \int_0^\infty y f(Q_{n-1}, w_{n-1}) ds$, we require decay of Q in ψ.

→ In fact, the heat evolution gives $Q \sim e^{-\psi}$.



We encode decay and regularity by working in the following space:

For $f(s,\psi): \mathbb{T}_L \times \mathbb{R}_+$,

$$\|f\|^2_{X^{k,m}} = \sum_{k'=0}^{k} \sum_{m'=0}^{m} \Big[\|\langle\psi\rangle^{m'} \partial_s^{k'} f\|^2_{L^2} + \|\langle\psi\rangle^{m'} \partial_s^{k'+1} f\|_{L^2} + \|\langle\psi\rangle^{m'} \partial_\psi \partial_s^{k'} f\|^2_{L^2} + \|\langle\psi\rangle^{m'} \partial_\psi^2 \partial_s^{k'} f\|^2_{L^2} \Big]$$

Note that by Sobolev, $\|f\|_{L^\infty} \lesssim \|f\|_{L^2} + \|\partial_s f\|_{L^2} + \|\partial_\psi f\|_{L^2} \lesssim \|f\|_{X_{1,0}}$.

We construct a solution in $X_{2,50}$.

$$\partial_s Q_n + \omega_0 q_{e_{n-1}} \partial_\psi^2 Q_n = F_{n-1}$$

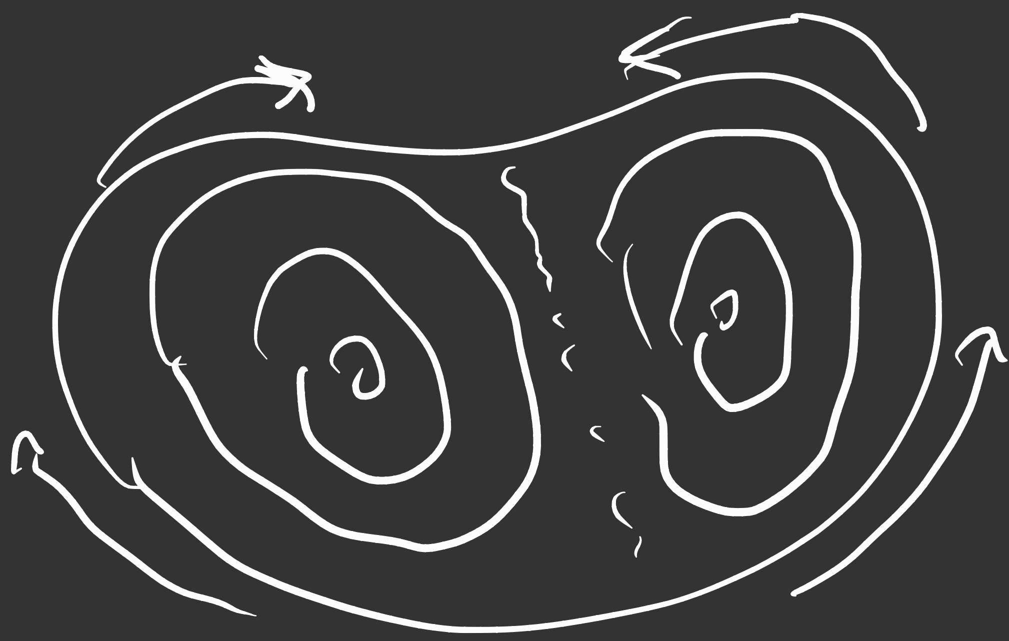
Step 1 multiply by Q, integrate, controls $|\partial_\psi Q|_{L^2}$

Step 2 multiply by $\partial_s Q$, integrate, controls $|\partial_s Q|_{L^2}$

Step 3 given $\partial_s Q$, know $\partial_\psi^2 Q$ (maximal regularity)

Step 4 use $\partial_s Q$ to control $\|Q - \int Q\,ds\|_{L^2}$ by Poincaré

Step 5 Study zero-mode $\int Q\,ds$ seperately. Use Feynman-Lagerstrom, get ODE in ψ. Solve, use to control $\|\int Q\,ds\|_{L^2}$.



Thank-you!