The Feynman-Lagerstrom Criterion for Boundary Layers

joint work with
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Consider the Navier-Stokes equations on $M$

$$
\partial_t u^v + u^v \cdot \nabla u^v = -\nabla p^v + \nu \Delta u^v \quad \text{in} \ M
$$

$$
\nabla \cdot u^v = 0 \quad \text{in} \ M
$$

$$
u^v \cdot \hat{n} = 0 \quad \text{on} \ \partial M
$$

$$
u^v \cdot \hat{t} = f \quad \text{on} \ \partial M
$$

$$
u^v \big|_{t=0} = u_0 \quad \text{in} \ M
$$

Motion is generated by moving the solid walls, possibly in a non-uniform way.
Consider the simplest situation

**Uniform rotation**

\[ M = \text{disk or radius } R \]

\[ \begin{align*}
\partial_t u^v + u^v \cdot \nabla u^v &= -\nabla p^v + v \Delta u^v & \text{in } & \{ r \leq R \} \\
\nabla \cdot u^v &= 0 & \text{in } & \{ r \leq R \} \\
u^v \cdot n &= 0 & \text{on } & \{ r = R \} \\
u^v \cdot \hat{e} &= \frac{1}{2} \omega_0 R & \text{on } & \{ r = R \}
\end{align*} \]

**Theorem:** For any initial data \( u_0 \), \( \tilde{u}(+) \xrightarrow{t \to \infty} u_{\text{sb}} := \frac{1}{2} \omega_0 x^\perp \).

In fact,

\( \| u(+) - u_{\text{sb}} \|_2 \leq \| u_0 - u_{\text{sb}} \|_2 e^{-\lambda_1 v t} \) independent of \( v \).

**Proofs:** Let \( W = u^v - u_{\text{sb}} \). Then

\[ \partial_t w + u_{\text{sb}} \cdot \nabla w + w \cdot \nabla u_{\text{sb}} + w \cdot \nabla w = -\nabla \omega + v \Delta w \quad w|_{\partial M} = 0 \]

But (1) \( \nabla u_{\text{sb}} = \omega_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Thus \( w \cdot \nabla u_{\text{sb}} = w^\perp \).

\[ \Rightarrow \quad \frac{1}{2} \frac{d}{dt} \| w \|_2^2 = -v \| w \|_2^2 \leq -\lambda_1 \| w \|_2^2. \quad \lambda_1 \sim \frac{1}{v^2}. \]
Thus, we have the **Absence of Turbulence**:

Uniform rotation

\[ t \to \infty \quad \forall u_0 \]

Uniform rotation

This is analogous to Marchioro's result on \( \mathbb{T}^2 \)

forced by \( f = (\sin \phi)(\epsilon) \) (e.g. the gravest mode).

Experiments by Diane Henderson, Penn State.

**Question**: What if the imposed slip is non-constant, or the domain is not perfectly circular?
Generally, one would expect time dependence to never disappear entirely in these settings. However, there always exist stationary states, which will likely shape the large scale features of the flow.

Experiments by Diane Henderson, Penn State.

Special case: the state of constant vorticity:

\[ u_x = K[1], \quad k_m = \sigma^{1/2} \delta_0 \]

This is a stationary solution of Navier-Stokes for every \( v \). Thus, if taken as initial data and if forced by its slip velocity, the solution is forever...
Theorem: (Prandtl 1904 & Batchelor 1956)

Let $M$ be simply connected. Suppose $u_e$ is an Euler solution in $M$ with a single stagnation point. Let $\mathcal{U}$ be a family of steady Navier-Stokes solutions. If $u^n \to u_e$ in $C^2$, then

$$u_e = \omega_0 u_f$$

for some $\omega_0$.

That is, $u_e$ must have constant vorticity within $M$.

\[ \psi = \infty \]
\[ q = \omega_0 q_e \]
\[ \sqrt{\nu} \]
\[ \psi = 0 \]
\[ q = f \]
\[ = q_e + \varepsilon g \]

constant vorticity $\omega_0$

inviscid flow
Theorem: (Prandtl 1904 & Batchelor 1956)

Let $M$ be simply connected. Suppose $\mathbf{u}_e$ is an Euler solution in $M$ with a single stagnation point. Let $\mathbf{U}^{\infty}_3$ be a family of steady Navier-Stokes solutions. If $\mathbf{u}^{\infty} \rightarrow \mathbf{u}_e$, say in $C_2$, then

$$\mathbf{u}_e = \omega_0 \mathbf{u}_x$$

for some $\omega_0$.

That is, $\mathbf{u}_e$ must have constant vorticity within $M$.

Remark: $M$ can represent closed streamlines of the limiting Euler solution, possibly yielding a staircase of vorticity, separated (perhaps) by vortex sheets...

Constantinou and Young, 2017
8-place turbulence

Henderson, Lopez, Stewart
JFM 1996
Proof of Prandtl-Batchelor Theorem

First note that, under the stated assumptions,

\[ \omega_c = F(\Psi_c) \text{ for a } \text{Lip. } F: \mathbb{R} \to \mathbb{R} \]

WLOG suppose \( \Psi_c = 0 \) is the unique critical point.

Stationary vorticity equation:

\[ u^v \cdot \nabla \omega^v = \nu \Delta \omega^v \]

Integrate over a sublevel set \( \Psi^v \leq c^3 \):

\[ 0 = \frac{1}{\nu} \int u^v \cdot \nabla \omega^v \, dx = \int \Delta \omega^v \, dx \]

\[ \forall \Psi^v \leq c^3 \]

\[ = \int \partial_n \omega^v \, d\sigma \]

\[ \forall \Psi^v = c^3 \]

\[ \nu > 0 \]

\[ \rightarrow \int \partial_n \omega_c \, d\sigma = F' \left( c \right) \int \partial_n \psi \, d\sigma \]

\[ \forall \Psi^v \leq c^3 \]

\[ = F'(c) \int \partial_n \psi \, dl \]

\[ \forall \Psi^v = c^3 \]

\[ \Leftrightarrow F'(c) = c \]
This all begs the question

**Question**: What determines the limiting vorticity $\omega_0$?

This is the content of Feynman’s only published work in fluid dynamics (as far as I know):

REMARKS ON HIGH REYNOLDS NUMBER FLOWS IN FINITE DOMAINS

by R.P. FEYNMAN and P.A. LAGERSTROM,
Professor of Aeronautics, California Institute of Technology, Pasadena, California, U.S.A.

Consider two-dimensional, stationary viscous incompressible flow in a finite domain $D$ on whose boundary $B$ the normal velocity component is zero and the tangential flow component is prescribed to be $u_w(s)$, ($s =$ distance along $B$). As pointed out by Prandtl in 1905 the limiting flow as $Re$ (Reynolds number) tends to infinity has constant vorticity $\omega_0$ (actually several vortices may occur, each one with different vorticity). Consider the case of only one vortex. The Euler equations for the limiting flow then reduce to Poisson’s equation $\nabla^2 \psi = -\omega_0^2 \psi$, the solution of which is proportional to $\omega_0^2$. At $B$ this solution gives a tangential velocity $u_\infty(s) = \omega_0 f(s)$. Here $f(s)$ depends on the geometry of the domain only and may be considered known. For very large values of $Re$ the vorticity is then essentially constant except in a thin boundary layer near $B$ where the velocity changes rapidly from $u_w(s)$ to $u_\infty(s)$. One may now determine $\omega_0$ from the requirement that such a boundary layer (periodic in $s$) be possible. For a circle of radius $L$ simple momentum considerations then lead to the formula $(L\omega_0/2)^2 = \bar{u}_w^2$. (bar denotes average over $B$). Explicit formulas are difficult to find for general domains. If $|u_w - u_\infty| \ll u_w$ an approximate formula for $\omega_0$ is $\omega_0^2 f^3 = u_w^2 f$. The above equations are necessary but not sufficient conditions for existence of a boundary layer as described. Non-existence of such a boundary layer indicates that the limiting flow consists of several vortices.
Incidentally, he seems to have independently derived the Prandtl-Batchelor theorem:

Application to two-dimensional flow.

\[ \text{Put } \frac{\partial u}{\partial z} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial x} \]

\[ \Phi = \frac{\partial w}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial z} = 0 \]

\[ \int \frac{\partial w}{\partial x} \, ds = 0 \quad \text{Apply to their region between two streamlines which are deformed adjacent as shown. Let } ds = \text{length element} \]

\[ \text{W} = \text{separation (in functions of) } \]

\[ \text{But } \phi \text{ is tangential, and } \frac{\partial w}{\partial x} = \text{con} \text{. by continuity.} \]

\[ \int \frac{\partial w}{\partial y} \, ds = 0 \quad \text{But } \frac{\partial w}{\partial y} \text{ tangential, perpendicular to normal.} \]

\[ \text{Thm. } \int \frac{\partial w}{\partial N} \, ds = 0 \quad \text{The mean normal derivative of potential} \]

\[ \text{taken around a streamline is zero.} \]

Streamlines are everywhere and there is a region in which non-potential

the stream is essentially that of Euler (w = 0). Then w is constant

on a streamline. It is therefore therefore it is a function of only

\[ \frac{\partial w}{\partial m} = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} \text{, say.} \]

\[ \int \frac{\partial w}{\partial y} \, ds = 0 = \int \frac{\partial w}{\partial x} \, ds \text{ either.} \]

\[ \text{But } \frac{\partial w}{\partial x} \text{ is constant over the streamline, so} \]

\[ \text{circulation. It may be zero for one streamline, but not for two in} \]

\[ \text{succession, unless } w = 0. \text{ If } w = 0, \text{ then } \frac{\partial w}{\partial x} = 0. \]

\[ \text{If } w = 0 \text{ in the} \]

\[ \text{stream on all stream lines.} \]

\[ \text{Thm. (In the free flow region, in the limit } w \rightarrow 0, \text{ regions whose streamlines} \]

\[ \text{can be adjoined to a common point are regions of constant } w. \]
And thought about what happens in the case of multiple eddy formation...

The Internal Flow, for very small...

Suppose as we have shown in other places, the flow is \( \omega = \text{const} \) in regions except for two kinds of discontinuities:

1. Internal boundaries. Here we have (probably) shown that \( \omega \) is continuous and discontinuous and if we take \( \nu = 0 \) on all the boundaries (all at once, of course) then \( R \) is continuous when \( R = \rho + \frac{1}{2}(u^2 + v^2) - \omega^2 \).

2. Surface boundary layers. U-jumps. Here we have shown that \( \rho \) is continuous \( \frac{\partial}{\partial z} \omega = 0 \) (\( \omega = 0 \) at surface)

\[
R \text{ jumps, but } \Delta R = \pm \Delta \theta \text{ (in general } \Delta R = \pm \Delta \theta \text{)}
\]

We call the actual surface velocity \( f \).
We call the surface velocity which eliminates surface boundary layers (but leaves internal boundaries unchanged) \( g \). Then \( R^2 \text{ just inside } \partial \text{-Surface } = \pm \frac{1}{2} g^2 - f^2 \).
Let's revisit Feynman's idea for determining the vorticity value \( \omega \). First we must derive the Prandtl equations in von-Mises coordinates.

Let \( s: [0, L] \rightarrow \Omega \) be the arclength parametrization of the boundary. Due to smoothness, \( \exists \, s \) s.t. if \( \text{dist}(x, \Omega) < \delta \), there exists a unique closest boundary point to \( x \), call it \( x(s) \).

Let \( z = \text{dist}(x, \Omega) \) and \( s \) be our local coordinates. Then

\[
X(s, z) = x(s) + z \hat{n}(s)
\]
and

\[
\Gamma(s) = X_i' X_j' - X_i X_j' 
\]
is boundary curvature

\[
J(s, z) = 1 + z \Gamma(s) 
\]
is Jacobian of change of variables.

Let

\[
\begin{align*}
\hat{u}_c(s, z) &= \hat{u}(X(s, z)) \cdot \hat{n}(s) \\
\hat{u}_n(s, z) &= \hat{u}(X(s, z)) \cdot \hat{t}(s)
\end{align*}
\]
on disk: \( \gamma = -1, \quad J(\gamma \theta) = \frac{1}{\gamma}, \quad \hat{u}_c = u \theta, \quad \hat{u}_n = u r \)
Navier-Stokes near the boundary:

\[
\begin{align*}
\frac{u_T}{J} \partial_z u_T + u_n \partial_z u_T - \frac{x}{J} u_T u_n + \frac{1}{J} \partial_z p \\
= & \; v \left( \frac{\partial}{\partial z} \left( \Gamma \partial_z u_T \right) + \frac{1}{J} \partial_z \left( \frac{1}{J} \partial_z u_T \right) - \frac{1}{J} \partial_z \left( \frac{\partial u_T}{\partial z} \right) - \frac{1}{J} \left( \frac{\partial u_T}{\partial z} \right) \right) \\
= & \; v \left[ \frac{1}{J} \partial_z \left( \Gamma \partial_z u_T \right) + \frac{1}{J} \partial_z \left( \frac{1}{J} \partial_z u_T \right) - \frac{1}{J} \partial_z \left( \frac{\partial u_T}{\partial z} \right) - \frac{1}{J} \left( \frac{\partial u_T}{\partial z} \right) \right] \\
\partial_z u_n + \frac{1}{J} \partial_z u_T - \frac{x}{J} u_T = 0
\end{align*}
\]

Now, in a layer of width \( \sqrt{v} \), we anticipate

\[
(U_T, u_n) \approx (w_0 q e(s), 0) + (u_T^P, u_n^P)
\]

Specifically

\[
U_T(s, z) \sim w_0 q e(s) + u_T^P(s, \frac{z}{\sqrt{v}}) \\
U_n(s, z) = \sqrt{v} U_n^P(s, \frac{z}{\sqrt{v}})
\]

where \( \lim_{z \to \infty} u_T^P(s, z) = 0 \)
Plugging this ansatz into Navier-Stokes and using that, with $Z = \frac{3}{\sqrt{n}}$:
\[
\frac{1}{I} = \frac{1}{1 + 2g(\xi)} \sim 1 - \sqrt{2} g(s) + o(r)
\]

We obtain:
\[
\begin{align*}
(\omega_0 q_0(s) + u_t^P) \partial_s (\omega_0 q_0(s) + u_t^P) \\
+ u_n^P \partial_z (\omega_0 q_0(s) + u_t^P) + \partial_3 p - \partial_z u_n^P &= 0 \\
\partial_z p &= 0
\end{align*}
\]

together with incompressibility
\[
\begin{align*}
\partial_s (\omega_0 q_0(s) + u_t^P) + \partial_z u_n^P &= 0
\end{align*}
\]

Taking $\xi \to \infty$ in (1), since $p = p(s)$, we obtain
\[
\partial_s p = -\omega_0^2 q_0(s) q_0'(s).
\]

Substituting into the above, we arrive at Prandtl’s equations:
\[
\begin{align*}
(\omega_0 q_0(s) + u_t^P) \partial_s (\omega_0 q_0(s) + u_t^P) \\
+ u_n^P \partial_z (\omega_0 q_0(s) + u_t^P) - \omega_0^2 q_0 q_0'(s) - \partial_z u_n^P &= 0 \\
\partial_s (\omega_0 q_0(s) + u_t^P) + \partial_z u_n^P &= 0
\end{align*}
\]
\[
\begin{align*}
& (w_0 q_e(s) + u_t^p) \partial_s (w_0 q_e(s) + u_t^p) \\
& + u_n^p \partial^2_z (w_0 q_e(s) + u_t^p) - w_0^2 q_e q_e(s) - \partial^2_z u_e = 0 \\
& \partial_s (w_0 q_e(s) + u_t^p) + \partial_z u_n^p = 0
\end{align*}
\]

Now, define von Mises variables \((s, \psi)\):
\[
\begin{align*}
\partial_z \psi &= w_0 q_e(s) + u_t^p(s, z) \\
-\partial_s \psi &= u_n^p(s, z)
\end{align*}
\]

Let \( q = q(t, s) = w_0 q_e(s) + u_t^p \).  

Prandtl's equations become
\[
\begin{align*}
\partial_s (q^2) - & w_0^2 \partial_s (q_e^2) - q \partial_\psi p(q^2) = 0 \\
\text{or, Letting } & Q = q^2 - w_0^2 q_e^2,
\end{align*}
\]

\[
\begin{align*}
\partial_s Q - & q \partial_\psi Q = 0 \\
Q(0, s) &= f_2(s) - w_0^2 q_e^2(s) \\
Q(\infty, s) &= 0
\end{align*}
\]
\[ \frac{\partial}{\partial s} Q - \frac{1}{2} \frac{\partial^2}{\partial \varphi^2} Q = 0 \]

\[ Q(0,s) = \frac{f^2(s)}{2} - \omega_0^2 q_e^2(s) \]

\[ Q(\infty,s) = 0 \]

**Feynman-Lagerstrom (1956) condition**

\( \omega_0 \) is selected so that \( Q \) has a periodic solution in \( s \).

In the case of the disk, this is explicit! (Noted also by Batchelor, 1956) and Wood (1957)

Since \( q_e(s) = \frac{R}{2} \) \( (u_e = \frac{1}{2} x^1) \), we have:

\[ \frac{\partial}{\partial s} Q = 2q \frac{\partial}{\partial s} q. \]  

Thus \[ \frac{\partial^2}{\partial \varphi^2} Q = 0 \]

\[ \frac{\partial^2}{\partial \varphi^2} \int_0^L Q(s,\varphi) ds = 0 \iff \int_0^L Q(s,\varphi) ds = 0 \]

Evaluating at \( \varphi = 0 \), using \( Q(s,0) = f^2(s) - \omega_0^2 q_e^2(s/2)^2 \)

\[ \omega_0^2 = \frac{1}{(R/2)^2} \int_0^{2\pi} \frac{f^2(\theta)}{2} d\theta \]  

This is only thing remembered from Navier-Stokes.
\[ \begin{align*} 
\frac{d^2}{ds^2} Q - q^2 \frac{d}{dy} Q &= 0 \\
Q(0, s) &= f^2(s) - \omega_0 q_e^2(s) \\
Q(\infty, s) &= 0
\end{align*} \]

Feynman-Lagerstrom condition

\( \omega_0 \) is selected so that it has a periodic solution in \( s \).

Let us derive a general (nonlinear) criterion:

\[ \frac{d^2}{ds^2} Q - \omega_0 q_e \frac{d^2}{dy} Q = -\left( q - \omega_0 q_e \right) \frac{d^2}{dy} Q \]

\[ = -(q - \omega_0 q_e) \frac{d^2}{dy} \left( \frac{1}{\sqrt{\omega_0^2 q_e^2 + Q}} \right) \frac{d}{ds} Q \]

\[ \Rightarrow \quad \frac{d^2}{dy} \left[ \int_0^1 q_e(s) Q ds \right] = \frac{1}{\omega_0} \int_0^1 \left( 1 - \frac{\omega_0 q_e(s)}{\sqrt{\omega_0^2 q_e^2 + Q}} \right) \frac{d}{dy} Q ds \]

\[ \Rightarrow \quad \int_0^1 q_e(s) Q(0, s) ds = \int_0^\infty N_{\omega_0}^{\omega_0}(Q) dy \quad \text{for} \quad \psi > 0 \]
Evaluating at $\psi = 0$, we find:

\[
\int_0^L q_e(s) \left( f^2(s) - \omega_0^2 q_e^2(s) \right) ds = \int_0^\infty y N^{\omega_0}[\psi](y) dy
\]

\[f(s) = q_e(s) + \varepsilon g(s)\]

in this case

\[1 - \omega_0^2 = \varepsilon \bar{\omega}_0\]

with \(\bar{\omega}_0 = O(1)\)

We anticipate \(Q = O(\varepsilon)\). Then

\[N[\psi] = O(\varepsilon^2) = O(\varepsilon^2)\]

Then

\[\omega_0^2 = \frac{\int_0^L q_e(s) f^2(s) ds}{\int_0^L q_e^3(s) ds} + O^1(12, k|\varepsilon^2)\]
proximate formula for $\omega_0$ is $\omega_0^2 f^3 = u_w^2 f$. The above equations are necessary but not sufficient conditions for existence of a boundary layer as described. Non-existence of such a boundary layer indicates that the limiting flow consists of several vortices.

(D. Iyer, Nguyen) 

**Theorem:** Let $g_\varepsilon(s)$ be any smooth, nonvanishing periodic function. Let $f(s) = g_\varepsilon(s) + \varepsilon y(s)$. For all $\varepsilon$ sufficiently small, there exist a unique $w_0 \in \mathbb{R}$ and function $g: [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ such that $(w_0, g)$ solves the Prandtl equations

$$
\begin{align*}
\frac{3}{2} Q - q \frac{\partial^2}{\partial x^2} Q &= 0 \\
Q(s, 0) &= f^2(s) - w_0^2 g_\varepsilon^2(s) \\
Q(s, \infty) &= 0
\end{align*}
$$

Moreover, the selected vorticity obeys

$$
\begin{align*}
\omega_0^2 &= \frac{\int_0^L g_\varepsilon(s) f^2(s) ds}{\int_0^L g_\varepsilon^2(s) ds} + \omega_{err}, \\
|\omega_{err}| &\leq C \varepsilon^2.
\end{align*}
$$
\[ \frac{\partial Q}{\partial t} - q \frac{\partial^2 Q}{\partial y^2} = 0 \quad Q = \gamma^2 - \omega_0^2 q_e^2 \]

\[ Q(s, 0) = \frac{f}{q_e} - \omega_0^2 q_e^2(s) \]
\[ Q(s, \infty) = 0 \]

Rewrite as \( s \), with \( f(s) = \gamma(s) + 3 \gamma(s) \)

\[ \frac{\partial Q}{\partial t} + \omega_0 q_e \frac{\partial^2 Q}{\partial y^2} = \left( 1 - \frac{\omega_0 q_e}{\sqrt{\omega_0^2 q_e^2 + Q}} \right) \frac{\partial Q}{\partial s} \]

\[ Q(s, 0) = (1 - \omega_0^2) q_e^2 + 2 \gamma q(s) q_e(s) + s^2 J(s) \]
\[ Q(s, \infty) = 0 \]

Recall \( \omega_0 \) is chosen so that \( Q \) exist (periodic), \( Q \) then implicitly depends on \( \omega_0 \). We study...

the following iteration

\[ \frac{\partial \xi^n}{\partial t} + \omega_0 q_e \frac{\partial^2 \xi^n}{\partial y^2} = \left( 1 - \frac{\omega_0 q_e}{\sqrt{\omega_0^2 q_e^2 + \xi^n_{n-1}}} \right) \frac{\partial \xi^n}{\partial s} \]

\[ \xi^n(s, 0) = (1 - \omega_0^2) q_e^2 + 2 \gamma q(s) q_e(s) + s^2 J(s) \]
\[ \xi^n(s, \infty) = 0 \]
\( w_{n+1} = 1 - w_n^2 \) determined by

\[
\overline{w}_{on} = \frac{\int (2gq^2 + \varepsilon g^2 qe) ds}{\int q^3 ds} + \frac{\int_{\theta}^{\infty} y \cdot f(a_{n-1}, w_{n-1}) ds}{w_{n-1}^2 \int q^3 ds}
\]

→ To control, we require 1 derivative of \( Q \)

→ On the other hand, if \( w_{n-1} \) is known, we can obtain many derivatives of \( Q \)

\[
\partial_s Q_n + w_0 q e \partial_s^2 Q_n = F_{n-1}^n
\]

Since this equation is a "time"-periodic heat.

→ Moreover, to make sense of \( \int_0^\infty y \cdot f(a_{n-1}, w_{n-1}) ds \), we require decay of \( Q \) in \( \psi \).

→ In fact, the heat evolution gives \( Q \psi^2 e^\psi \).
We encode decay and regularity by working in the following space:

For \( \mathcal{F}(s, \gamma) : T_L \times \mathbb{R}_+ \)

\[
\| \hat{f} \|_{X^{k,m}} = \sum_{k=0}^{K} \sum_{m=0}^{M} \left[ \| \langle \psi \rangle^{m} \partial_x^k \partial_y^m \|_{L_2}^2 + \| \langle \psi \rangle^{m} \partial_x^k \partial_y^m \|_{L_2}^2 \right]
\]

Note that by Sobolev, \( \| \hat{f} \|_{X^{k,m}} \leq \| \hat{f} \|_{L_2} + \| \partial_x \hat{f} \|_{L_2} + \| \partial_y \hat{f} \|_{L_2} \leq 1 \),

We construct a solution in \( X_{2,50} \):

\[
\partial_y^2 \phi_n + \omega_0 \partial_y^2 \partial_y^2 \phi_n = F_n
\]

**Step 1**

Multiply by \( \phi \), integrate, controls \( \| \partial_y \phi \|_{L_2} \).

**Step 2**

Multiply by \( \partial_x \phi \), integrate, controls \( \| \partial_x \phi \|_{L_2} \).

**Step 3**

Given \( \partial_x \phi \), know \( \partial_y^2 \phi \) (maximal regularity).

**Step 4**

Use \( \partial_x \phi \) to control \( \| \phi - \text{fads} \|_{L_2} \) by Poincare.

**Step 5**

Study zero-mode \( \text{fads} \) separately. Use Feynman-Lagrange, get ode in \( \psi \). Solve, use to control \( \| \text{fads} \|_{L_2} \).
Thank-you!