(1) Let $(x,t) \in \mathbb{R}$ and $u(x,t)$ be a smooth function, differentiable in x and t. Suppose that
 (a) u is time-periodic, e.g. there is a $T > 0$ such that $u(x, t+T) = u(x,t)$ for all $t \in \mathbb{R}$
 (b) there are two points $a, b \in \mathbb{R}$ such that $a < b$ and
$$u(a,t) > 0, \qquad u(b,t) < 0 \qquad \forall t \in \mathbb{R}.$$
 Prove that there is a time-periodic solution of the ODE
$$\dot{x}(t) = u(x(t), t).$$

(2) Study the transport equation for the passive scalar $\theta = \theta(x,t)$ with $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$:
$$\partial_t \theta + \vec{u} \cdot \nabla \theta = 0, \qquad \theta(x,0) = \chi_{B_1(0)}(x).$$
for the vector field $\vec{u}$ give by
 (a) $\vec{u}(x_1, x_2) = \beta(x_2, 0)$,
 (b) $\vec{u}(x_1, x_2) = \alpha(x_1, -x_2)$.
Describe the evolution in geometric terms. Which vector field creates small scales faster?

(3) Consider the Cauchy problem for $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$$\partial_t \theta - x \partial_x \theta = 0,$$
$$\theta|_{t=0} = \theta_0,$$
for any given smooth function θ_0. Prove that
 (a) the solution $\theta(t)$ decays as $t \to \infty$ to zero in any $L^p(\mathbb{R})$ for $1 \le p < \infty$,
 (b) $\theta(t)$ does not grow in in $L^\infty(\mathbb{R})$ and $W^{1,1}(\mathbb{R})$ (one derivative in L^1),
 (c) $\theta(t)$ grows indefinitely in $W^{k,p}(\mathbb{R})$ with $k > 1$ and $p \in [1, \infty]$ or with $k = 1, p > 1$.

(4) Let $\vec{u} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a smooth vector field. Let $\Phi_t(a)$ be the corresponding flowmap
$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t(a) = \vec{u}(\Phi_t(a), t), \qquad \Phi_0(a) = a \in \mathbb{R}^d.$$
Prove Liouville's formula
$$\det \nabla \Phi_t(a) = \exp \left(\int_0^t \mathrm{div} \vec{u}(\Phi_s(a), s) \mathrm{d}s \right).$$

(5) Let $\vec{u} : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous vector field satisfying the bound
$$\|\vec{u}(x) - \vec{u}(y)\| \le L \|x - y\|.$$
Show that any time-periodic solution of the ODE
$$\dot{x}(t) = \vec{u}(x(t))$$
which is not a fixed point cannot have period smaller than $2\pi/L$. Show this is sharp.

(6) Let $\Omega = \mathbb{T} \times [0,1]$, v be a nowhere non-constant function, b be a smooth vector field satisfying the bound $|b(x,t)| \le 1$, and fix $\varepsilon > 0$. Consider
$$\partial_t \theta + v(y) \partial_x \theta + \varepsilon b \cdot \nabla \theta = 0.$$
Show that if $b = 0$, then all initial data with $\partial_x \theta_0 \neq 0$ has growing derivative, i.e. $\sup_x |\partial_x \theta(x,t)| \sim t$ as $t \to \infty$. If, on the other hand, given θ_0 if $b \neq 0$ must $\partial_x \theta$ grow?