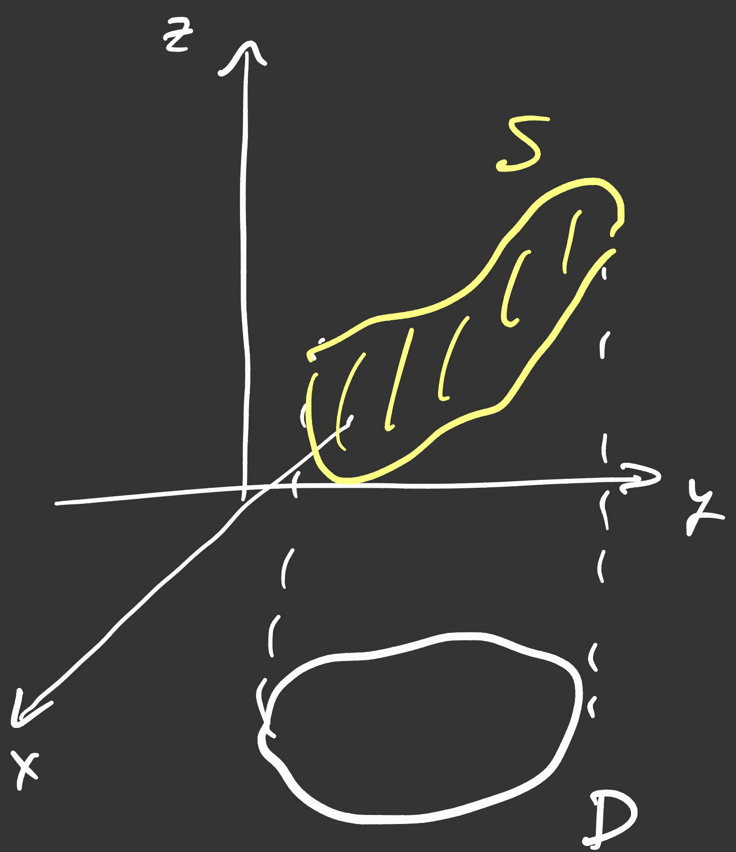
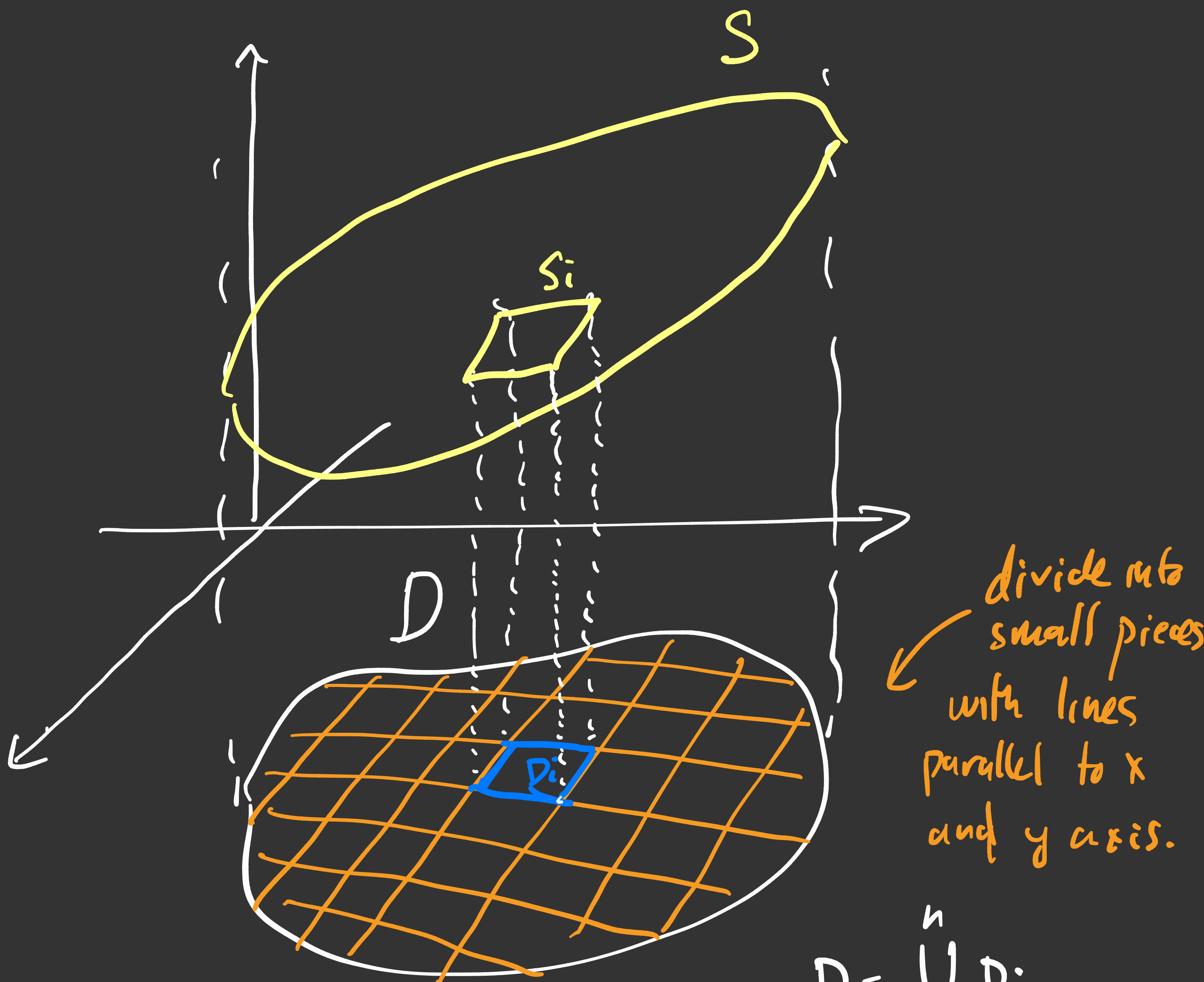
Surface Area

$S: \quad z = f(x,y) \qquad (x,y) \in D \subset \mathbb{R}^2$

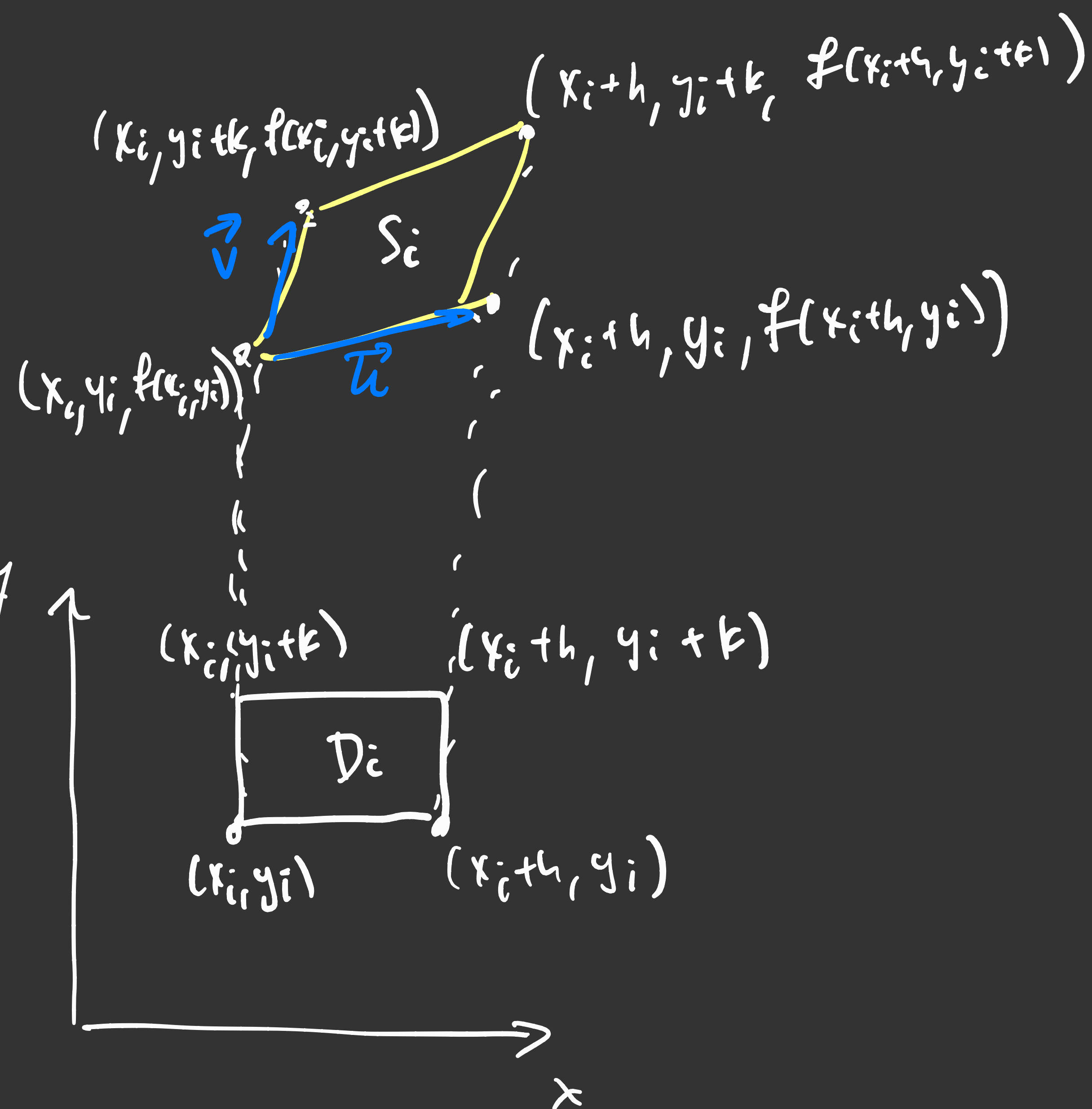


How to find the area of this piece of surface?



$$D = \bigcup_{i=1}^{n} D_i$$

What is the area of the piece S_i?



Let's find area of the surface S_i:

$$A(S_i) = A(S_i')$$

where S_i' is parallelogram spanned by $\vec{u}$ and $\vec{v}$

when $\vec{u} = (h, 0, f(x_i+h, y_i) - f(x_i, y_i))$

$\vec{v} = (0, k, f(x_i, y_i+k) - f(x_i, y_i))$

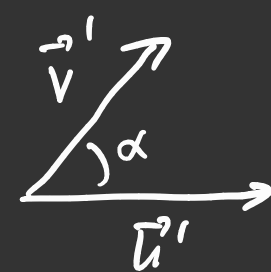
$$\vec{u} = (h, 0, f(x_i + h, y_i) - f(x_i, y_i))$$
$$\approx (h, 0, f_x(x_i, y_i)\, h) = \vec{u}\,'$$
$$\vec{v} = (0, k, f(x_i, y_i + k) - f(x_i, y_i))$$
$$\approx (0, k, f_y(x_i, y_i)\, k) = \vec{v}\,'$$

Thus, we further approximate:

$$A\left(\begin{matrix}\text{parallelogram} \\ \text{spanned by}\end{matrix}\ \vec{u}, \vec{v}\right) \approx A\left(\begin{matrix}\text{parallelogram} \\ \text{spanned by}\end{matrix}\ \vec{u}\,', \vec{v}\,'\right)$$

Now,

$$A(\vec{u}\,', \vec{v}\,') = \|\vec{u}\,'\|\,\|\vec{v}\,'\|\,|\sin\alpha|$$
$$= \|\vec{u}\,' \times \vec{v}\,'\|$$



$$\vec{u}\,' \times \vec{v}\,' = \begin{vmatrix} i & j & k \\ h & 0 & f_x h \\ 0 & k & f_y k \end{vmatrix} = (-f_x hk, -f_y hk, hk)$$
$$= (-f_x, -f_y, 1)\, hk$$

$$A(\vec{u}\,', \vec{v}\,') = \sqrt{h^2 k^2 (f_x^2 + f_y^2 + 1)} = hk\sqrt{f_x^2 + f_y^2 + 1} = hk\sqrt{|\nabla f|^2 + 1}$$

$$\approx A(S_i)$$

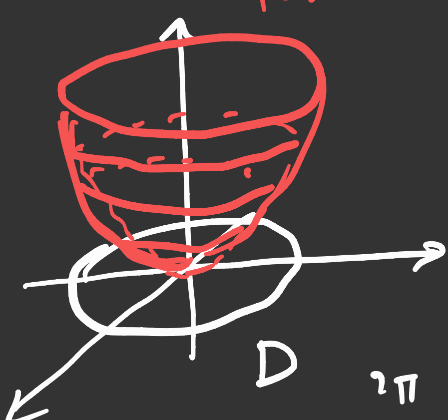
$$A(S) \approx \sum_{i=1}^{n} A(S_i) \approx \sum_{i=1}^{n} hk\sqrt{\|\nabla f(x_i, y_i)\|^2 + 1}$$

$$\approx \iint_D \sqrt{f_x^2 + f_y^2 + 1}\; dA$$

Example: $D = \{x^2 + y^2 \le 1\}$

$f(x,y) = x^2 + y^2$

Paraboloid



$$A(S) = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\; dA$$

$$= \iint_D \sqrt{4(x^2+y^2) + 1}\; dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \sqrt{4r^2+1}\; r\,dr$$

$s = r^2$

$ds = 2r\,dr$

$$= \pi \int_0^1 \sqrt{4s+1}\; ds = \pi \, \frac{2}{3}\cdot\frac{1}{4}(4s+1)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5^{3/2} - 1)$$

Ex: $\quad D: \; x^2+y^2 \le 1$

$\quad\quad S: \; z = \sqrt{1-x^2-y^2}$

$$A(S) = \iint_D \sqrt{\left(\frac{-2x}{2\sqrt{1-x^2-y^2}}\right)^2 + \left(\frac{-2y}{2\sqrt{1-x^2-y^2}}\right)^2 + 1}\; dA$$

$$= \iint_D \sqrt{\frac{x^2+y^2}{1-x^2-y^2} + 1}\; dA$$

$$= \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r\,dr\,d\theta = 2\pi \int_0^1 \frac{r\,dr}{\sqrt{1-r^2}}$$

$s = r^2$

$ds = 2r\,dr$

$$= \pi \int_0^1 \frac{ds}{\sqrt{1-s}} = \pi \left(-2\sqrt{1-s}\right)\Big|_0^1 = 2\pi.$$

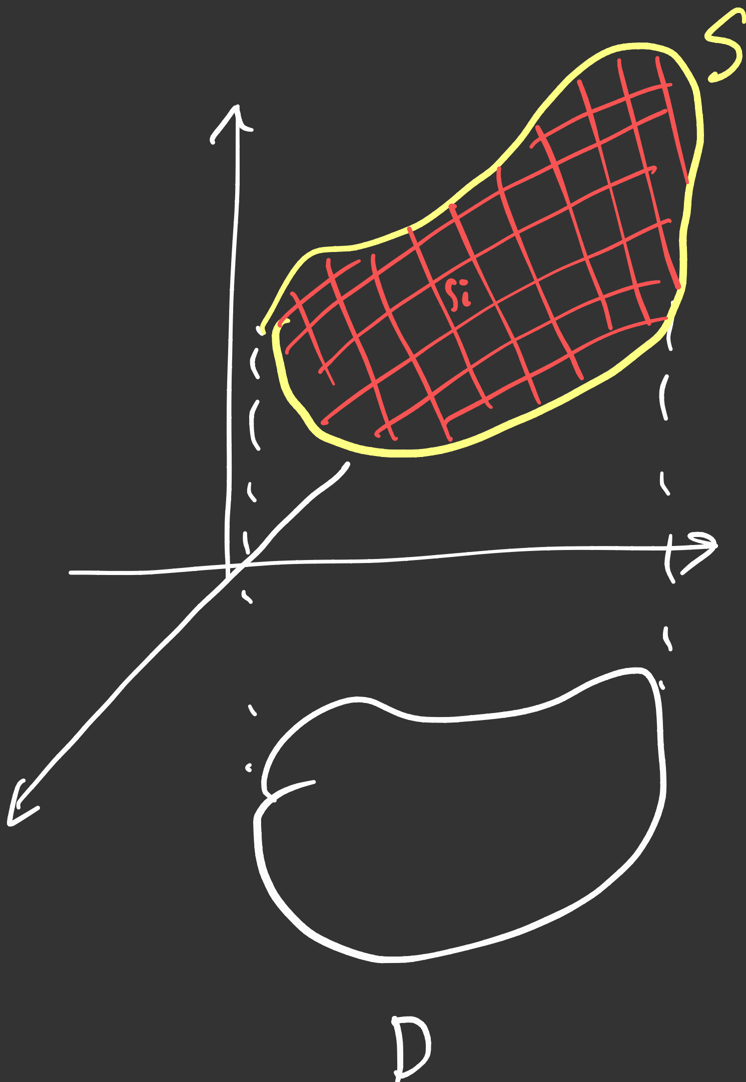
Area of upper hemisphere $= 2\pi$

$\Rightarrow$ Area of whole sphere $= 4\pi$

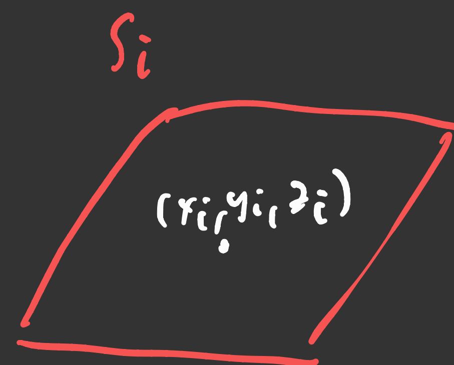
This is a result of Archimedes!

Surface integral

$S = z = f(x,y)$ $\qquad (x,y) \in D \subset \mathbb{R}^2$



$G(x,y,z) \leftarrow$ function on the surface S.



$$\sum_{i=1}^{N} \text{Area}(S_i)\, G(x_i, y_i, z_i) \xrightarrow[\text{max size } S_i \to 0]{N \to \infty} \iint_S G(x,y,z)\, dS$$

area element of surface

If $G = 1$, we have the usual area.

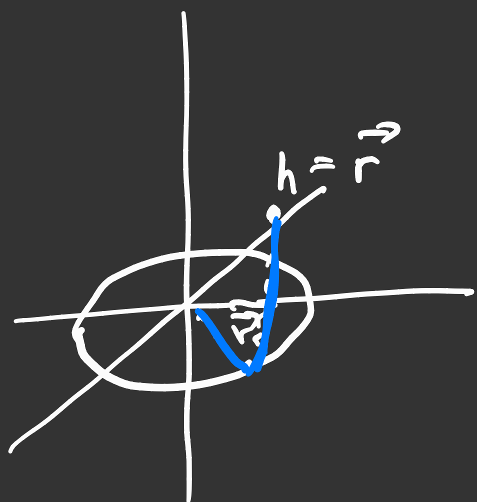
If S is a warped sheet of metal, G can represent the thickness of the sheet. Then $\iint_S G\, dS$ is the total volume of material making up the sheet.

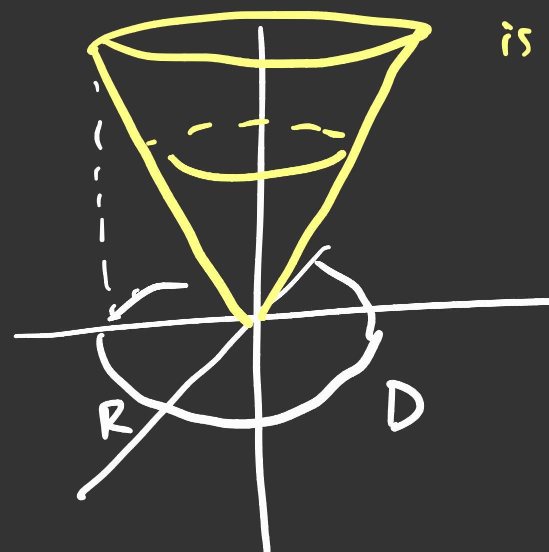
$$\iint_S G(x,y,z)\, dS = \iint_D \underbrace{\sqrt{1+f_x^2+f_y^2}}_{\text{area}}\ \overbrace{G(x,y,f(x,y))}^{\text{thickness}}\ \underbrace{dA}_{\text{area}}$$

" $dS = \sqrt{1+f_x^2+f_y^2}\ dA$ "

Example $\quad S: \; z = \sqrt{x^2+y^2}$

$D: \; x^2+y^2 \le R^2$





$z = \sqrt{x^2+y^2}$ is a cone

$$G(x,y) = \rho = \text{constant}$$

$$f(x,y) = \sqrt{x^2+y^2} \qquad f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{Mass} = \rho \; \text{Area}(S) = \rho \iint_D \sqrt{f_x^2+f_y^2+1}\, dA$$

$$= \rho \iint_D \sqrt{2}\, dA = \sqrt{2}\rho \; \text{Area}(D)$$

$$= \sqrt{2}\pi R^2 \rho .$$

Thus $\quad dS = \sqrt{2}\, dA.$

Now we find center of mass.

Moments relative to x-y plane:

$$M_{xy} = \iint_S z\rho \, dS$$

$$M_{yz} = \iint x\rho \, dS$$

$$M_{xz} = \iint_S y\rho \, dS$$

Center of Mass

$$\bar{x} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M}.$$

$$M_{xy} = \iint_S z\rho \, dS = \int_D \sqrt{x^2+y^2}\,\rho\sqrt{2}\, dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^R r\rho\, r\,dr\,d\theta = 2\pi\rho\sqrt{2}\int_0^R r^2 dr$$

$$= 2\pi\rho\sqrt{2}\,\frac{R^3}{3}.$$

$$M_{yz} = \iint_S x\rho\, dS = \int_0^{2\pi}\int_0^R r\cos\theta\, \rho\sqrt{2}\, r\, dr\, d\theta$$

$$= \rho\sqrt{2}\int_0^{2\pi}\cos\theta\, d\theta \int_0^R r^2\, dr$$

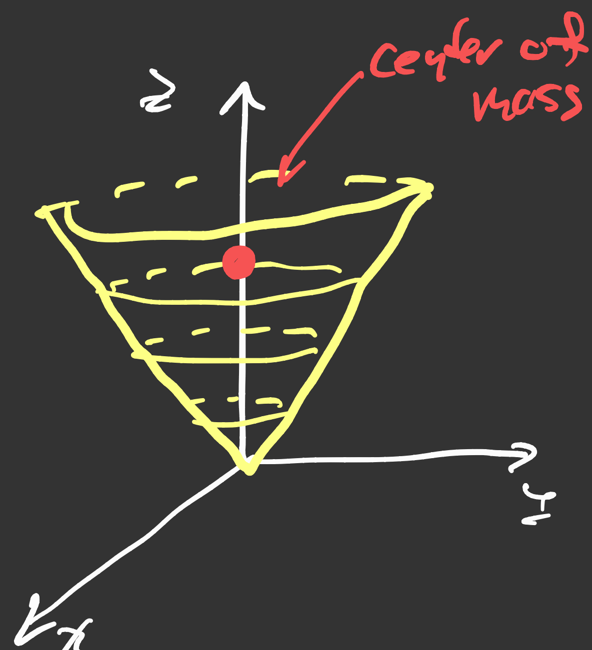
$$= \frac{\rho\sqrt{2}}{2}\,\frac{R^3}{3}\underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{0} = 0$$

$M_{xy} = 0$ also.

$$\bar{x} = \frac{M_{yz}}{M} = 0$$

$$\bar{y} = \frac{M_{xz}}{M} = 0$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{2}{3}R.$$



It is natural that $\bar{x}, \bar{y} = 0$ as the cone is rotationally symmetric. Thus, COM if were somewhere off z axis, it would move if cone is turned. But a turned cone is same as cone!