MAT 307: Advanced Multivariable Calculus

Potential fields : vector fields which are gradients of functions

$$\vec{F}(\vec{r}) = \nabla f(\vec{r})$$

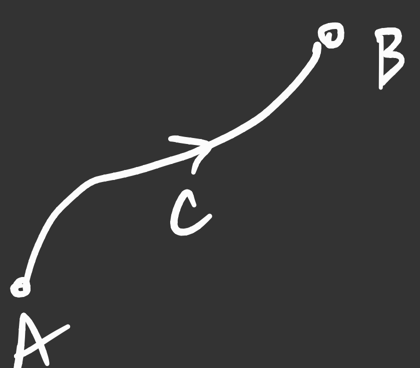
$$\vec{F}(x,y) = (f_x(x,y), f_y(x,y))$$

The function $f(x,y)$ is called the potential.

Ex: $\vec{F}(x,y) = (x,y)$ $\qquad f(x,y) = \frac{1}{2}(x^2+y^2)$

$\nabla f(x,y) = (x,y) = \vec{F}$

Observation:



$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$

To see why this is, let's introduce a parametrization of

$$C: \quad \vec{r} = \vec{r}(t) \qquad a \leq t \leq b$$
$$\vec{r}(a) = A \qquad \vec{r}(b) = B$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = (x(t), y(t))$

$$= \int_a^b \Big(x'(t) f_x(x(t), y(t)) + y'(t) f_y(x(t), y(t)) \Big) dt$$

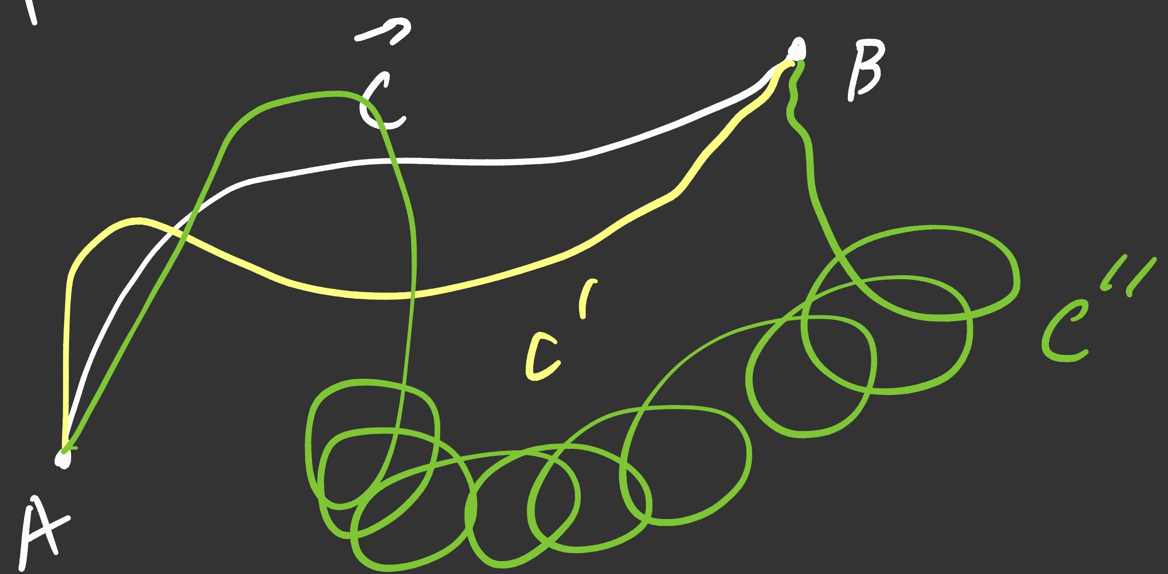
by chain rule

$$= \int_a^b \frac{d}{dt} f(x(t), y(t))\, dt$$

fundamental theorem of calculus : differentiation is inverse to integration

$$= f(B) - f(A)$$

Thus, as soon as we know the potential for a potential vector field, we immediatly know the work done by it along a curve by evaluating the potential at two end points.



$$\oint_C \vec{F}\cdot d\vec{r} = f(B) - f(A) \quad , \quad \vec{F} = \nabla f$$

$$\shortparallel$$

$$\int_{C'} \vec{F}\cdot d\vec{r}$$

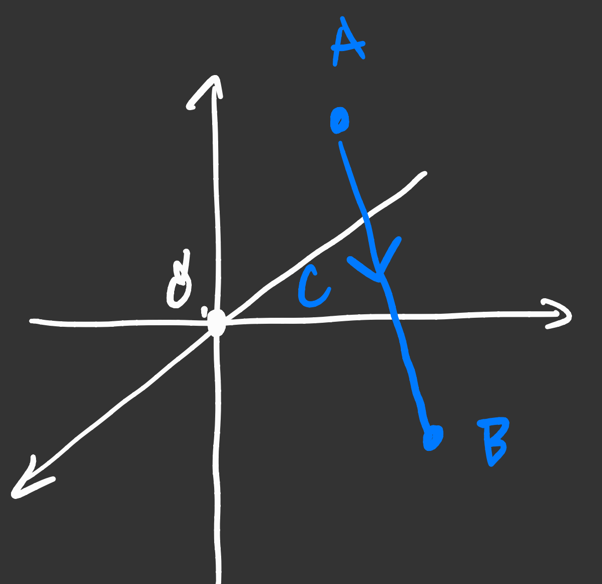
Thus $\oint_C \vec{F}\cdot d\vec{r}$ does not depend on C, provided C connects A and B.

In 3d, the same holds.

$$\vec{F}(\vec{r}) = \nabla f(\vec{r}) = (f_x, f_y, f_z)$$

$\oint_C \vec{F} \cdot d\vec{r}$ depends only on A and B.

<u>Example</u>: $\vec{F}(\vec{r}) = -\dfrac{\vec{r}}{\|\vec{r}\|^3} = \left(\dfrac{-x}{\sqrt{x^2+y^2+z^2}}, \dfrac{-y}{\sqrt{x^2+y^2+z^2}}, \dfrac{-z}{\sqrt{x^2+y^2+z^2}} \right)$



$\oint_C \vec{F} \cdot d\vec{r} = ?$

↑ looks hard...

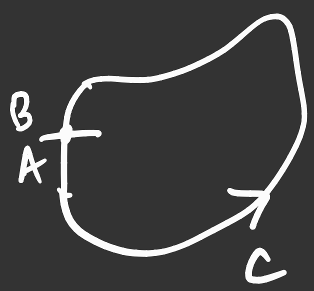
But: consider $f(r) = \dfrac{1}{\|\vec{r}\|} = (x^2+y^2+z^2)^{-1/2}$.

$$\nabla f = (f_x, f_y, f_z) = \left(\frac{-x}{\sqrt{x^2+y^2+z^2}}, \frac{-y}{\sqrt{x^2+y^2+z^2}}, \frac{-z}{\sqrt{x^2+y^2+z^2}} \right) = \vec{F} \; !$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = \frac{1}{|OB|} - \frac{1}{|OA|}$$

Integral along a closed curve



$$\vec{F} = \nabla f$$

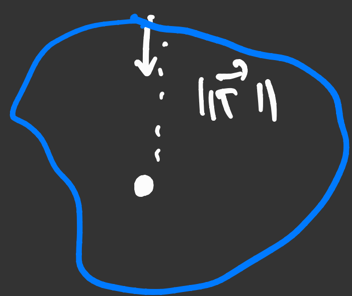
$$\oint_C \vec{F} \cdot dr = f(B) - f(A)$$

$$= 0 \quad \text{as } A = B.$$

Hence $\oint_C \vec{F} \cdot dr = 0$,

integral of a potential field over a closed curve is zero.

Note: $\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$



Work done moving a body along a closed curve in a gravitational field is zero.

Most "perpetual motion" machines are based on the idea of moving a body in a field on a closed curve and hope to get some work done. For example

heavier part should slide down due to gravity and cause the whole chain to turn and to produce some work (by gravitation).

But, according to our theorem, the gravitational work done here is zero!

Nothing will be done. Oh well...

Suppose the vector field $\vec{F}(\vec{r})$ is such that

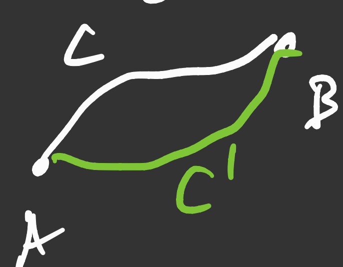
$$\oint_C \vec{F}\cdot d\vec{r} = 0 \quad \text{for any closed curve } C.$$

Then $\vec{F}$ is potential.

This is the reverse theorem to what we now know, that $\oint_C \nabla f \cdot d\vec{r} = 0$.

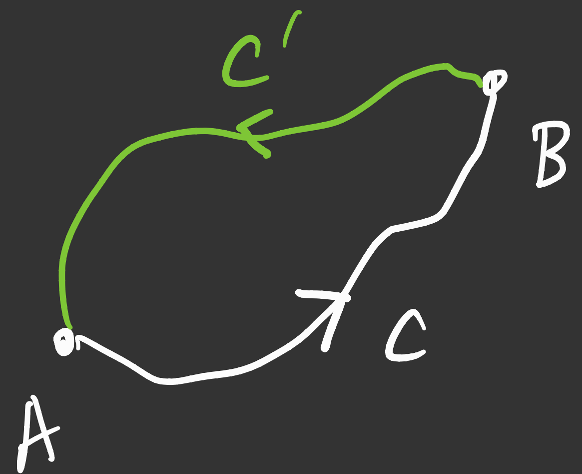
Proof: 1 The above field $\vec{F}$ is such that $\int_C \vec{F}\cdot dr$ is path independent

e.g.



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$$

Indeed, to see this consider the following curve



$D =$ C and C' together

Call that closed curve from $A \to A$, D.
Now, since D is a closed path, so

$$\oint_D \vec{F} \cdot d\vec{r} = 0$$

changes sign because orientation reversed

but

$$\oint_D \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot dr - \int_{C'} \vec{F} \cdot d\vec{r}$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot dr.$$

C and C' are arbitrary, thus integral does not depend.

(2) We want to prove that $\vec{F}$ is potential. To do this, we must produce some potential function, f.

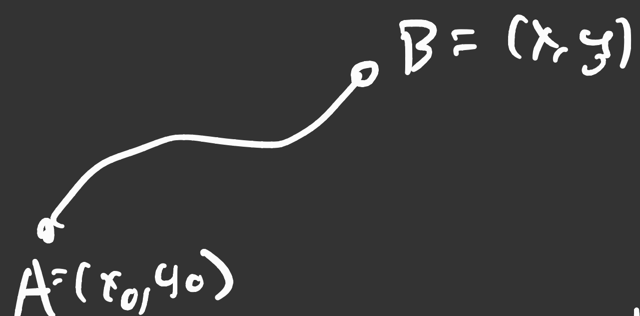
Define the function $f(\vec{r})$ as follows.

- fix $A = (x_0, y_0)$ and vary $B = (x, y)$

define

$$f(x,y) = f(B) = \int_C \vec{F} \cdot d\vec{r}$$

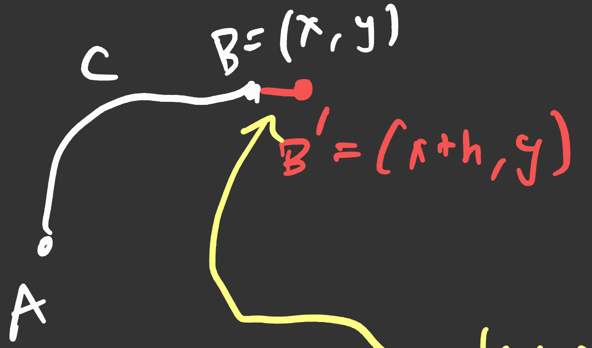
where C is any curve from A to B.



By (1), this integral does not depend on the curve, only on the point B. Thus it is well defined.

Suppose $\vec{F}(x,y) = (P(x,y),\ Q(x,y))$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t) \right] dt$$



$$f(B') = \int_a^b \vec{F}\cdot d\vec{r} + \int_b^{b+h} \vec{F}\cdot d\vec{r}$$

$$\int_b^{b+h} P(x(t),y(t))\,dt$$

where we move along the segment with unit speed.
$x'(t)=1$

Thus

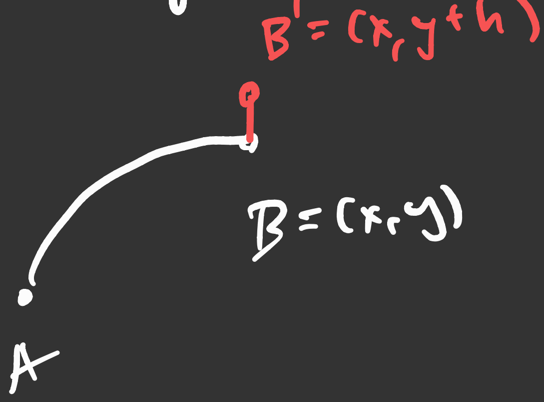
$$f(B') = f(B) + \int_b^{b+h} P(x(t),y(t))\,dt$$

$$\approx f(B) + P(B)\,h$$

But

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(B')-f(B)}{h} = P(B) = P(x,y)$$

Similarly



we find $\frac{\partial f}{\partial y} = Q(x,y)$

Thus $\nabla f = \left(P(x,y),\ Q(x,y) \right)$

$= \vec{F}$

and we found our potential field by computing the integral of $\vec{F}$ along any path connecting A and $B = (x, y)$.

Two questions

- How do you know if a field is potential.
- How to find the potential function.

Conditions of potentiality of a vector field

$\vec{F} = (P, Q, R)$

How to know if it is potential?

Two problems:

1. Which properties are necessary for potentiality? Namely, what are the properties common to all potential fields?

2. Which conditions are sufficient? Namely, if a field has this property, then it is potential.

In particular, properties that are pler than "integral along every closed curve is zero"

Suppose $F(x,y) = (P(x,y), Q(x,y))$ is potential.
This means that there is an f so that

$$P(x,y) = \frac{\partial f}{\partial x}(x,y) \qquad Q(x,y) = \frac{\partial f}{\partial y}(x,y)$$

Consider $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

But partial derivatives commute (Clairot thm)
so that $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus, given a field in the plane,
you can look at $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$.
If equal, the field is potential.
If not, then the field is not potential.

Ex: $P = e^x \cos y \qquad Q = e^x \sin y$

$\vec{F} = (P, Q)$

$$\frac{\partial P}{\partial y} = -e^x \sin y \qquad \frac{\partial Q}{\partial x} = e^x \sin x$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not potential.

But

$$P = e^x \sin y \qquad Q = e^x \cos y$$

$$\frac{\partial P}{\partial y} = e^x \cos y \qquad \frac{\partial Q}{\partial x} = e^x \cos y$$

They are equal, so $\vec{F} = (P, Q)$ is potential. The potential function is

$P = f_x$ and $Q = f_y$

$f = e^x \sin y$ works!

What happens in 3D? What is condition for potentiality?

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}, \quad R = \frac{\partial f}{\partial z}$$

From $\frac{\partial P}{\partial y} = f_{yx} \quad \frac{\partial Q}{\partial x} = f_{xy}$

$$\Rightarrow \boxed{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0}$$

$$\frac{\partial P}{\partial z} = f_{zx} \quad \frac{\partial R}{\partial x} = f_{xz}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0}$$

$$\frac{\partial Q}{\partial z} = f_{zy} \quad \frac{\partial R}{\partial y} = f_{yz}$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0}$$

These identities are familiar, they are the three components of curl $\vec{F}$!

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

<u>Conclusion:</u> If $\vec{F}$ is potential, then curl $\vec{F} = 0$.

Note:

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

in 2D defines a vector field in 3D

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + 0\,\vec{k}.$$

What is curl?

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y).$$

Thus 3D condition for potentiality reduces to the 2D condition:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Def: A vector field $\vec{F}(\vec{r})$ is called irrotational if $\operatorname{curl} \vec{F} = 0$.

Theorem: Potential fields are irrotational.

Is the opposite true?

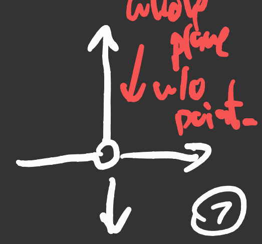
To some extent, it is true (in general, not).

It depends on which domains we consider, e.g. if the vector field is defined on all space, then yes. But in more complicated domains, it can be false.

Def: Domain D in the plane is called simply connected if it contains no holes.

e.g. D = whole plane, half plane, interior of circle.

NOT:



Simple but deep example

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Vector field is defined everywhere but at the origin. Thus, it's domain is not simply connected.

Is $\vec{F}$ irrotational? $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$

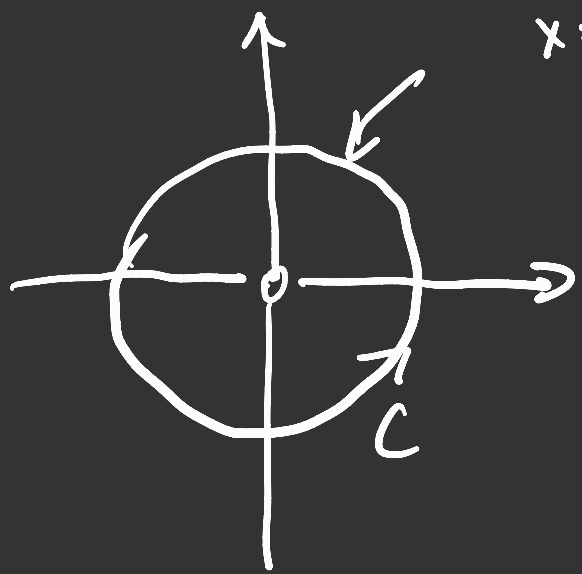
$$\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2} + \frac{y(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow$ irrotational.

Lets integrate



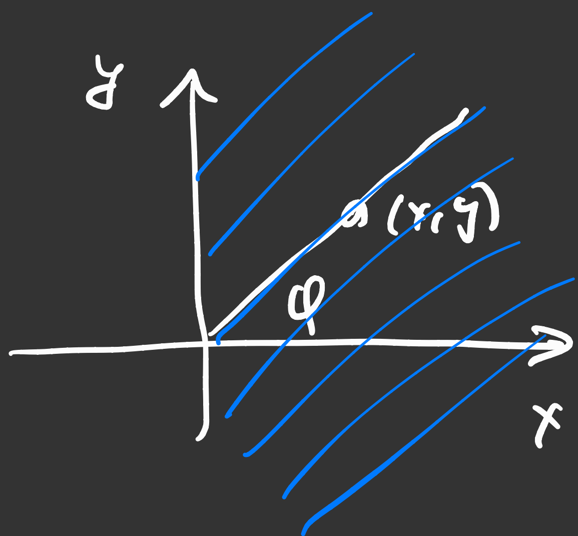
$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi.$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \frac{(-\sin t)}{\sin^2 t + \cos^2 t} (-\sin t)\, dt$$

$$+ \int_0^{2\pi} \frac{(\cos t)\ (\cos t)}{\sin^2 t + \cos^2 t}\, dt = 2\pi \neq 0.$$

Something is wrong! If it were potential, this should be zero!

Consider a function that looks like it's potential:



$$\varphi(x,y) = \arctan\left(\frac{y}{x}\right) \qquad x > 0.$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = P$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = Q$$

Thus $\vec{F}(x,y) = \nabla\phi$ if $x > 0$.

If $y>0$

Now $\quad \phi = \operatorname{arccot}\left(\frac{x}{y}\right) \qquad y>0.$

One can check that

$$\vec{F}(x,y) = \nabla\phi \qquad \text{for } y>0.$$

Thus, in right half plane, the potential is defined by

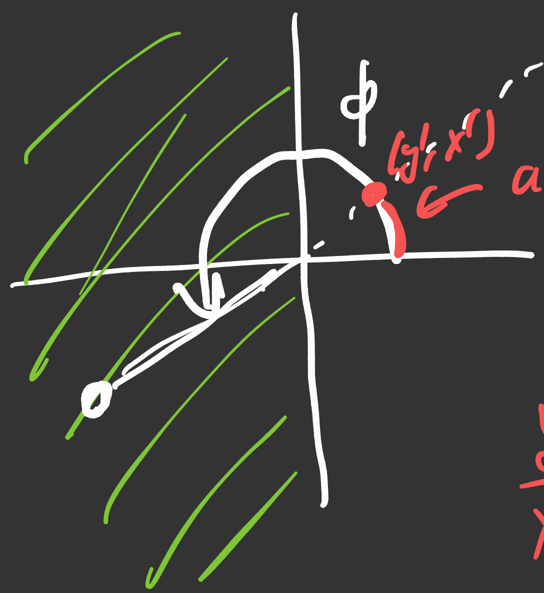
$$\phi = \arctan\left(\frac{y}{x}\right)$$

in upper half plane

$$\phi = \operatorname{arccot}\left(\frac{x}{y}\right)$$

In the overlapping region, both functions give same result.

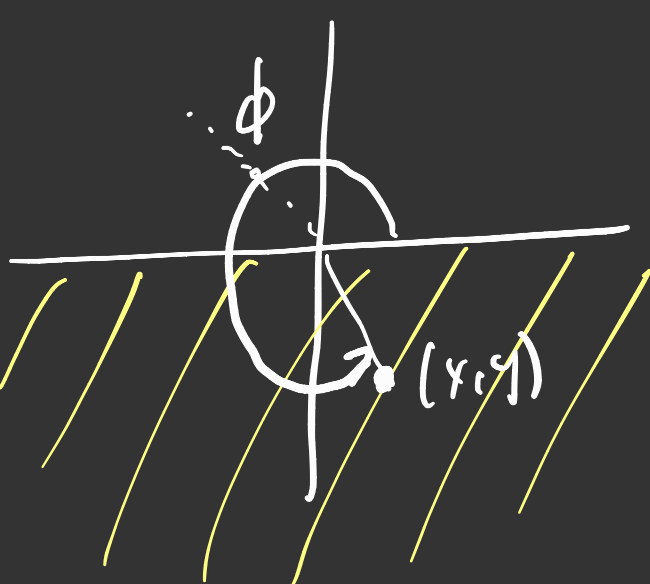
In the left half plane $(x<0)$



$$\frac{y'}{x'} = \frac{y}{x}$$

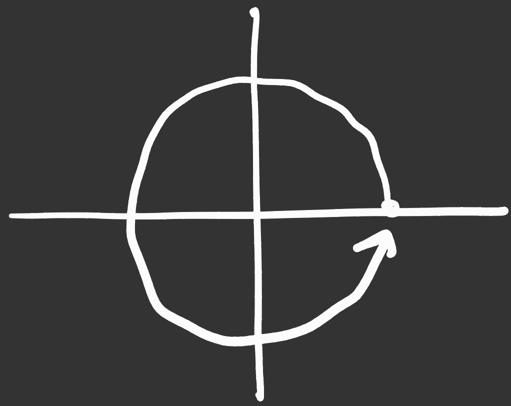
$$\phi = \arctan\left(\frac{y}{x}\right) + \pi. \qquad x<0$$

In lower half plane $(y<0)$



$$\phi = \operatorname{arccot}\left(\frac{x}{y}\right) + \pi$$

Thus when we make a full turn



the angle becomes 2π (not 0).

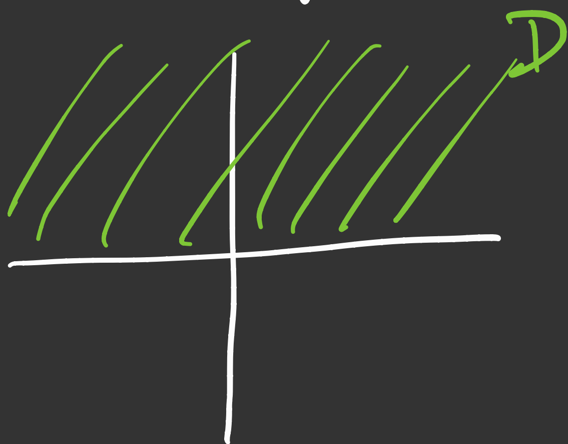
Thus, the "function" angle is not a single function. It grows from 0 to 2π in one rotation, and if you continue it keeps growing (if you want to keep it continuous).

Thus it is not a function, rather

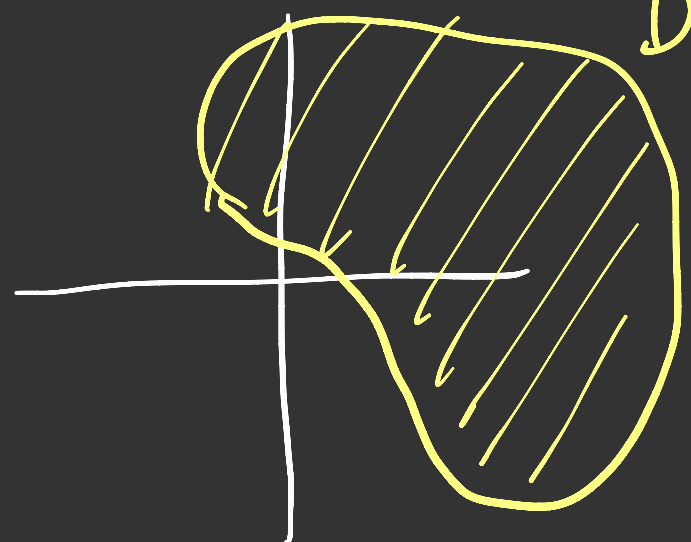
$$\phi(x,y), \quad \phi(x,y) + 2n\pi \qquad n \in \mathbb{Z}$$

are all "branches" of the angle.

For it to be a well defined function, must restrict the domain so that φ is single valued, for instance



or



Domain should have no holes, so that φ is defined everywhere and would be continuously differentiable.

Theorem: If a field is irrotational in a simply connected domain, it is potential in this domain.

<u>Problem</u> Suppose $\vec{F}$ is irrotational field.
How to find its potential $f(\vec{r})$?

<u>Example</u> $\vec{F}(x,y) = (4x^3 + 6x^2y,\ 2x^3 + 6y^2)$

$$P = 4x^3 + 6x^2y \qquad Q = 2x^3 + 6y^2$$

$$\partial_y P = 6x^2 \quad , \quad \partial_x Q = 6x^2$$

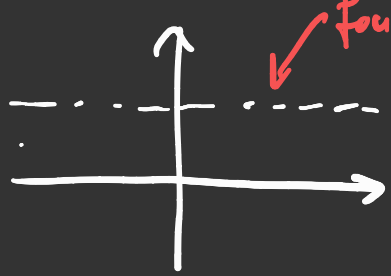
$\Rightarrow$ x is irrotational defined on all the plane $\Rightarrow$ potential.

There exists f such that $P = \partial_x f$, $Q = \partial_y f$.

$$\frac{\partial f}{\partial x} = 4x^3 + 6x^2y$$

$$\Rightarrow \quad f(x,y) = \int (4x^3 + 6x^2y)\,dx + C$$

found f here



$$= x^4 + 2x^3y + C$$

So $C = g(y)$

$$f(x,y) = x^4 + 2x^3y + g(y)$$

Must find $g(y)$. Use...

$$\frac{\partial f}{\partial y} = Q$$

$$\frac{\partial f}{\partial y} = 2x^3 + g'(y) \quad \text{But} \quad Q = 2x^3 + 6y^2$$

Thus

$$2x^3 + g'(y) = 2x^3 + 6y^2$$

here is where irrotationality appears. If done for an arbitrary vector field, will get

$g'(y) = h(x,y)$

inconsistent

$$\Rightarrow \quad g'(y) = 6y^2$$

$$\Rightarrow \quad g(y) = \int 6y^2 + C = 2y^3 + C$$

Thus

$$f(x,y) = x^4 + 2x^3y + 2y^3 + C$$

arbitrary, since P, Q don't depend.

3D examples:

$$\vec{F}(x,y,z) = (3x^2y,\ x^3+6y^2z,\ 2y^3-3z^2)$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 3x^2y & x^3+6y^2z & 2y^3-3z^2 \end{vmatrix} = (6y^2-6y^2,\ 0,\ 3x^2-3x^2) = (0,0,0)$$

Thus $\vec{F}$ is irrotational. There exists f such that

$$f_x = 3x^2y \Rightarrow f = x^3y + g(y,z) \quad ①$$

$$\Downarrow$$

$$f_y = x^3+6y^2z \qquad f_y = x^3 + g_y$$

$$\Downarrow$$

$$f_z = 2y^3-3z^2 \qquad g_y = 6y^2z$$

$$\Downarrow$$

$$g(y,z) = 2y^3z + h(z)$$

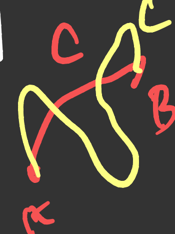
$$f = x^3y + 2y^3z + h(z)$$

$$f_z = 2y^3 + h'(z) \Rightarrow h'(z) = -3z^2$$

$$h = z^3 + C$$

$$\boxed{f(x,y,z) = x^3y + 2y^3z + z^3 + C}$$

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$



This comes up in ODE: exact equations.