Gradient

$\vec{r} = (x,y)$ is position vector

$$f(x,y) = f(\vec{r})$$

Def: $\nabla f(\vec{r}) = \nabla f(x,y)$

$$= \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$

$$= \frac{\partial f}{\partial x}(x,y)\,\vec{i} + \frac{\partial f}{\partial y}(x,y)\,\vec{j}$$

Ex: $f(x,y) = x^2 - 2y^2$

$$\nabla f(x,y) = (2x, -4y)$$

The symbol ∇ is called "nabla".
It is an ancient phoenician letter.
It was supposed to represent the head of a bull.

Also symbolizes a musical instrument.
But for us, it is the vector of partials.

Linear approximation: $\vec{r} = (x, y)$ $\vec{r}_0 = (x_0, y_0)$

$$f(\vec{r}) - f(\vec{r}_0) \approx f_x(\vec{r}_0)(x - x_0) + f_y(\vec{r}_0)(y - y_0)$$

$$= \nabla f(\vec{r}_0) \cdot (x - x_0, y - y_0)$$

since

$$\nabla f(\vec{r}_0) = (f_x(\vec{r}_0), f_y(\vec{r}_0))$$

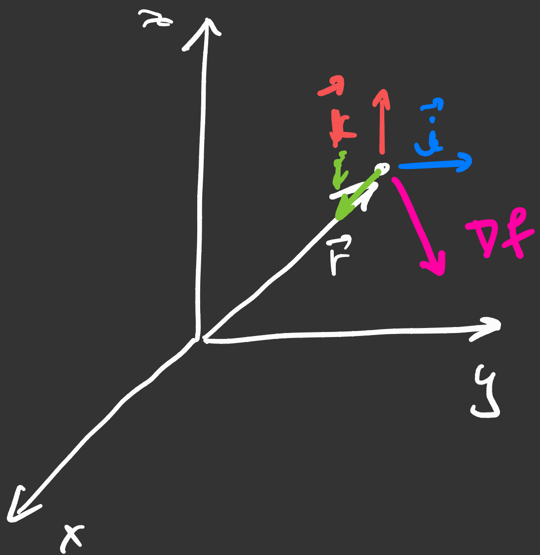
Thus

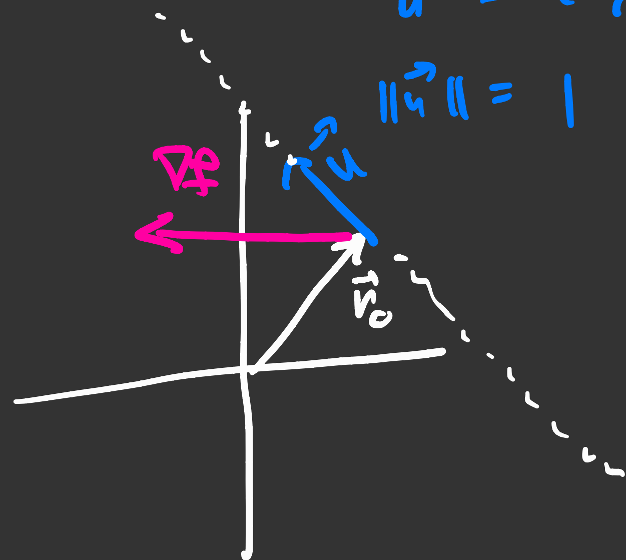
$$\boxed{f(\vec{r}) - f(\vec{r}_0) \approx \nabla f(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0)}$$

In particular

$$f_x = \nabla f \cdot \vec{i} \qquad f_y = \nabla f \cdot \vec{j}$$

Directional Derivative





$$l = \vec{r}_0 + t\vec{u}$$

$$f(\vec{r}_0 + t\vec{u}) = f(x_0 + ta, y_0 + tb)$$

$$\frac{d}{dt} f(\vec{r}_0 + t\vec{u}) = f_x(x_0 + ta, y_0 + tb)\, a + f_y(x_0 + ta, y_0 + tb)\, b$$

$$\frac{d}{dt} f(\vec{r}_0 + t\vec{u}) = f_x a + f_y b$$

$$= \nabla f(x_0, y_0) \cdot \vec{u}$$

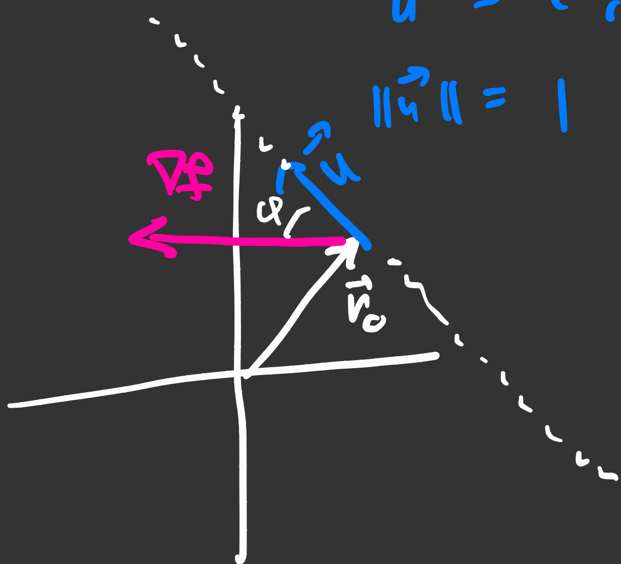
component of ∇f on $\vec{u}$

Derivative in direction of a unit vector

$$D_u f(x_0, y_0) = \vec{u} \cdot \nabla f(x_0, y_0)$$

$\vec{u} = (a, b)$

$\|\vec{u}\| = 1$



$D_u f$ is rate of change of f as we move in direction $\vec{u}$.

$$\nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\varphi$$

$$-1 \leq \cos\varphi \leq 1$$

$\uparrow$ if $\vec{u}$ and ∇f are opposite directions

$\uparrow$ when $\vec{u}$ and ∇f are same direction

$\therefore$ $D_u f(\vec{r}_0)$ is maximal if $\vec{u} \parallel \nabla f(\vec{r}_0)$

$D_u f(\vec{r}_0)$ is minimal if $-\vec{u} \parallel \nabla f(\vec{r}_0)$

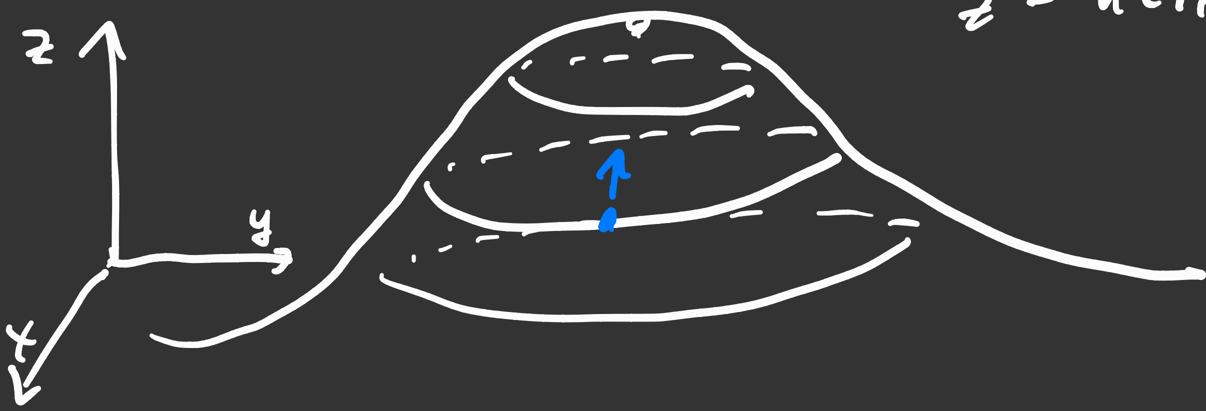
i.e. $\vec{u}$ and $\nabla f(\vec{r}_0)$ antiparallel.

Maximal growth if you move along gradient.

Maximal decay if you move opposite.

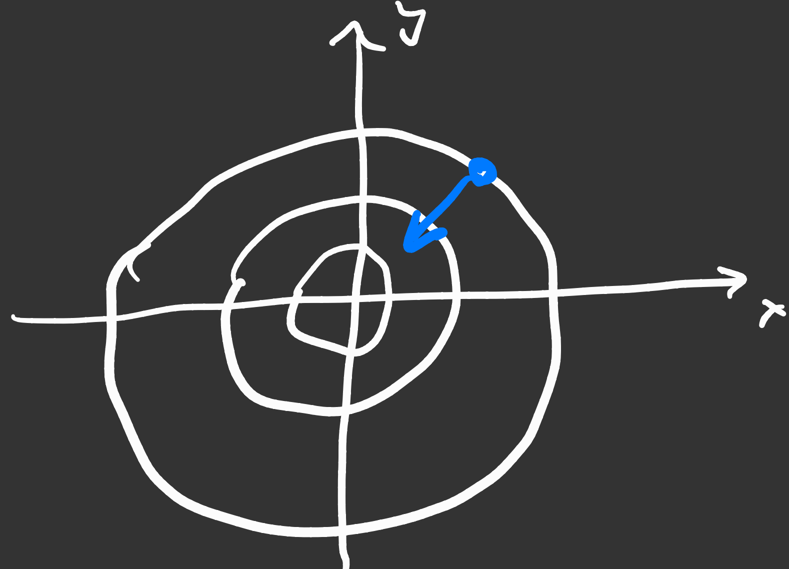
Example

$z = h(x,y)$



To climb the fastest, you should go in the direction of $\nabla f(x_0, y_0)$.

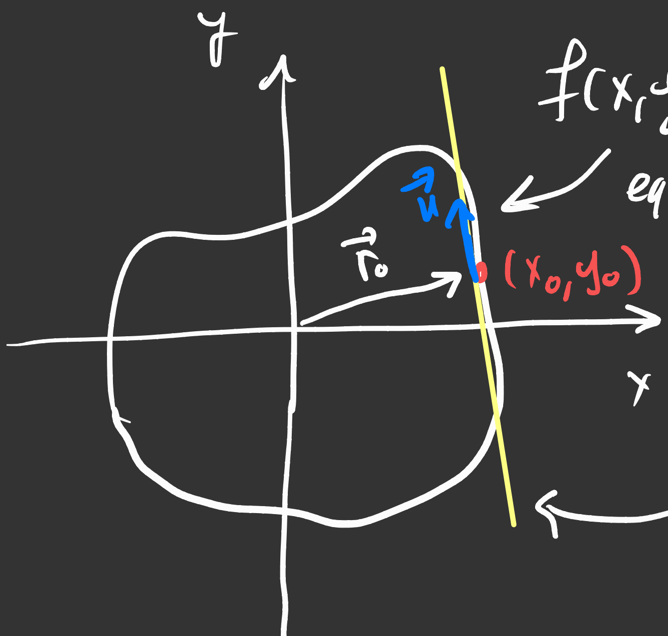
To decend the fastest, go in direction of $-\nabla f(x_0, y_0)$.



How to find a point where a function is maximal?
choose initial point, and move up the gradient!
Basis of $\boxed{\text{Gradient method}}$ in optimization.

MAT 307: Advanced Multivariable Calculus — Lecture 10

Tangent lines and planes



$f(x,y) = 0$ equation for curve.

tangent line to curve at (x_0, y_0)

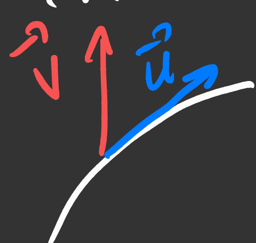
Equation for tangent is

$$\vec{r} = \vec{r}_0 + t\vec{u}$$

Recall then that

$$\frac{d}{dt} f(r_0 + tu)\Big|_{t=0} = D_u f(\vec{r}_0) = \nabla f(\vec{r}_0) \cdot \vec{u}$$

What can we say about $\vec{u}$ if it is in direction of tangent.



$D_{\vec{v}} f \neq 0$ since $f(x,y) \sim$ dist from point to curve.

if $\vec{v}$ is not tangent, the distance grows with some rate.

For $\vec{u}$ tangent to the curve $\{f(x,y)=0\}$

$$D_u f(\vec{r}_0) = 0 = \nabla f(\vec{r}_0) \cdot \vec{u} = (f_x, f_y) \cdot \vec{u}$$

This means, the tangent line equation is

$$\partial_x f\,(x-x_0) + \partial_y f\,(y-y_0) = 0$$

is the equation for the line which is orthogonal to the gradient.

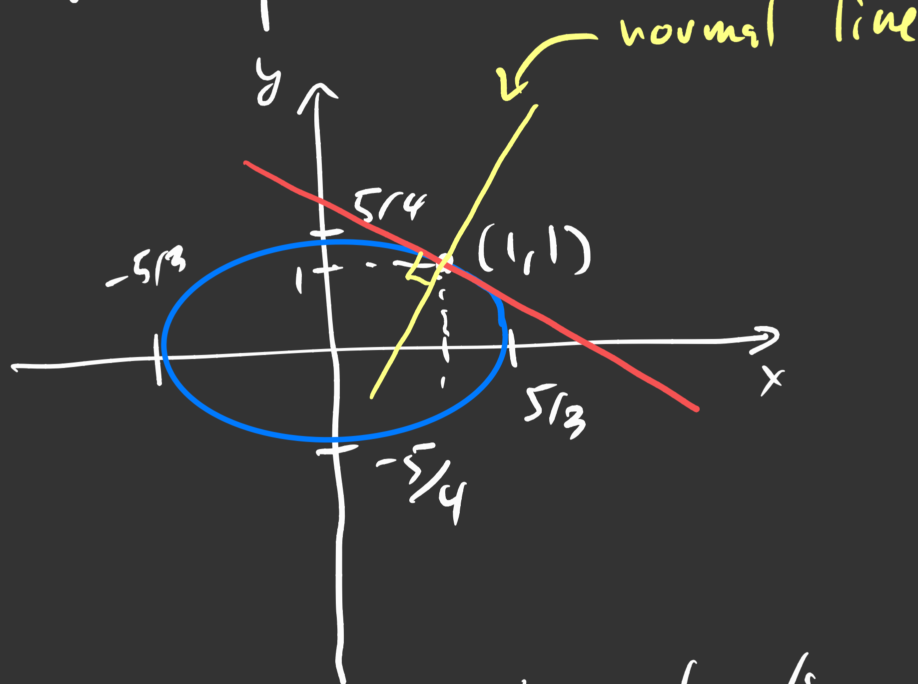
The equation for the normal line is

$$\vec{r} = \vec{r}_0 + \nabla f(\vec{r}_0)\, t$$

Example: $9x^2 + 16y^2 = 25$

$$\Rightarrow \quad \left(\frac{3x}{5}\right)^2 + \left(\frac{4y}{5}\right)^2 = 1$$

This is an ellipse whose axes are



Let's find equation for tangent line at (1,1).

$$f(x,y) = 9x^2 + 16y^2 - 25$$

$$\partial_x f = 18x, \quad \partial_y f = 32y \qquad x_0 = 1, \ y_0 = 1$$

$$18(x-1) + 32(y-1) = 0$$

$$\Rightarrow \quad 9(x-1) + 16(y-1) = 0 \quad \leftarrow \text{tangent line}$$

normal line

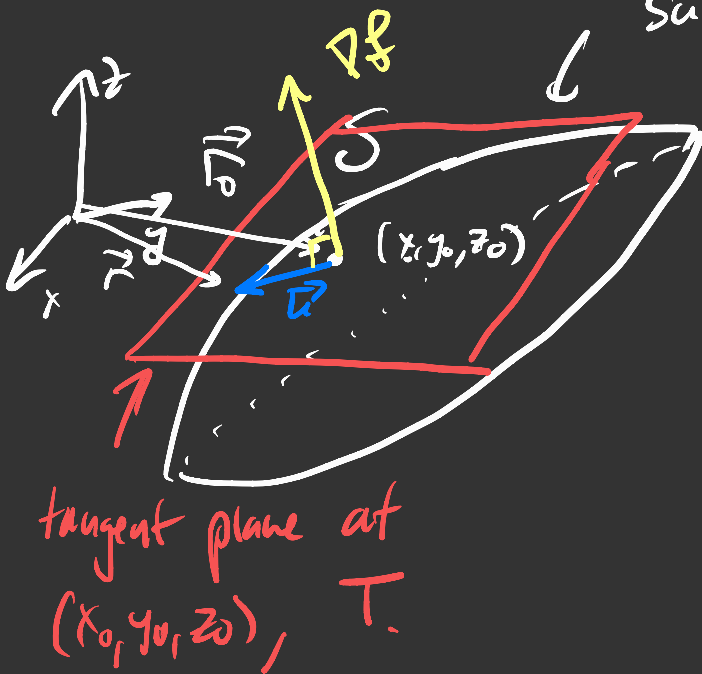
$\nabla f(x_0, y_0) = (16, 32)$

$\vec{r} = \vec{r}_0 + t\,\nabla f(\vec{r}_0)$

$x - 1 = 16t$
$y - 1 = 32t$

or $\frac{x-1}{16} - \frac{y-1}{32} = 0$

Situation in 3D: Tangent plane



Surface in 3D space given by

$$S: \; F(x, y, z) = c$$

For $\vec{u}$ in the plane, we have

$$D_{\vec{u}} f = 0.$$

This means that

$$\nabla f(\vec{r}_0) \cdot \vec{u} = 0$$

$$\nabla f(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) = 0$$

Condition to be on tangent plane T.

Ex: $\quad S: \quad xyz = 1.$

$$F(x,y,z) = xyz - 1$$

$\vec{r}_0 = (1, 2, \frac{1}{2}).$ (note $\vec{r}_0$ on S)

$$\nabla F = (\partial_x F, \partial_y F, \partial_z F)$$
$$= (yz, xz, xy)$$

$$\nabla F(\vec{r}_0) = (1, \tfrac{1}{2}, 2)$$

Tangent plane T has equation

$$1 \cdot (x-1) + \frac{1}{2}(y-2) + 2\left(z - \frac{1}{2}\right) = 0$$

or

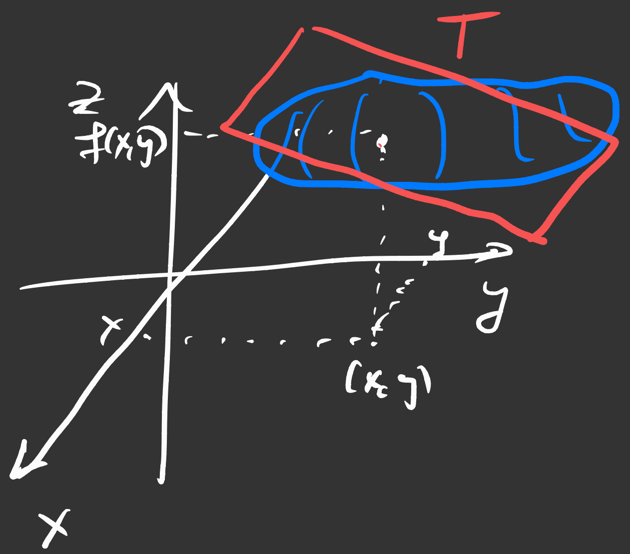
$$x + \frac{1}{2}y + 2z = 3$$

Equation for the normal line is

$$\begin{cases} x = 1 + t \\ y = 2 + \frac{1}{2}t \\ z = \frac{1}{2} + 2t \end{cases}$$

Graph of a function

$$z = f(x,y) \qquad F(x,y,z) = z - f(x,y)$$



$$\boxed{F=0 \iff z = f(x,y)}$$

$$F_x = -f_x, \quad F_y = -f_y, \quad F_z = 1.$$

$$\nabla F(x,y,f(x,y)) = (-f_x, -f_y, 1)$$

tangent plane T. at $(x_0, y_0, f(x_0,y_0))$

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0$$

rearranging:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

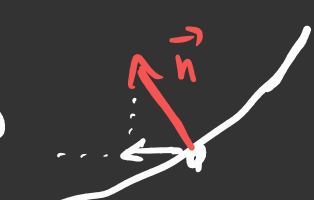
Tangent plane T at point (x_0,y_0)

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

This is just equation for linear approximation!

Normal line

$$x = x_0 - t f_x(x_0,y_0,z_0)$$
$$y = y_0 - t f_y(x_0,y_0,z_0)$$
$$z = f(x_0,y_0) + t$$



Higher order partial derivatives

$f(x,y)$, smooth (all derivatives exist)

$f_x = \frac{\partial f}{\partial x}$ $\quad f_y = \frac{\partial f}{\partial y}$ are also two variable functions.

Four second order derivatives

$f_{xx} = \frac{\partial f_x}{\partial x}$ $\quad f_{xy} = \frac{\partial f_y}{\partial x}$ $\quad f_{yx} = \frac{\partial f_x}{\partial y}$, $\quad f_{yy} = \frac{\partial f_y}{\partial y}$

Example: $f(x,y) = \sin(xy^2)$

$f_x = y^2 \cos(xy^2)$ $\qquad f_y = 2yx\cos(xy^2)$

$f_{xx} = -y^4 \sin(xy^2)$ $\qquad f_{yy} = 2x\cos(xy^2) - 4y^2x^2\sin(xy^2)$

$f_{yx} = 2y\cos(xy^2) - 2y^3x\sin(xy^2)$

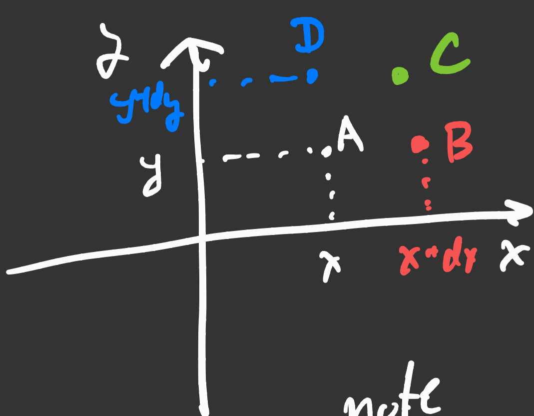
$f_{xy} = 2y\cos(xy^2) - 2y^3x\sin(xy^2)$

Notice, in our example, $f_{xy} = f_{yx}$.

This is a <u>general fact</u>.

> <u>Theorem</u>: (Clairot, Schwarz) Suppose $f, f_x, f_y, f_{xy}, f_{yx}$ exist and are continuous. Then
> $$f_{xy} = f_{yx}$$

Idea of Proof:



$A = (x, y)$

$B = (x+dx, y)$

$C = (x+dx, y+dy)$

$D = (x, y+dy)$

note $f(B) - f(A) \approx f_x(A)\,dx$

etc

$$\Rightarrow \quad f(B) \approx f(A) + f_x(A)\,dx$$

$$f(C) \approx f(D) + f_x(D)\,dx$$

$$f(D) \approx f(A) + f_y(A)\,dy$$

$$f(C) = f(B) + f_y(B)\,dy$$

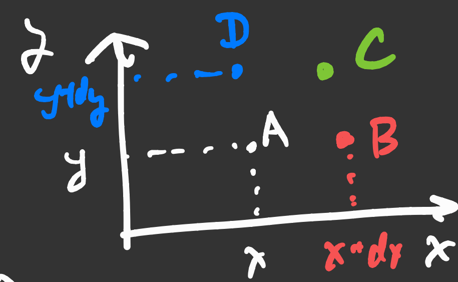
Consider

$$\big(f(C)-f(D)\big) - \big(f(B)-f(A)\big)$$

$$\approx f_x(D)\,dx - f_x(A)\,dx$$

$$= \big(f_x(D) - f_x(A)\big)\,dx$$

$$\approx f_{yx}(A)\,dy\,dx$$



$$\big(f(C)-f(B)\big) - \big(f(D)-f(A)\big)$$

$$\approx f_y(B)\,dy - f_y(A)\,dy$$

$$= \big(f_y(B) - f_y(A)\big)\,dx$$

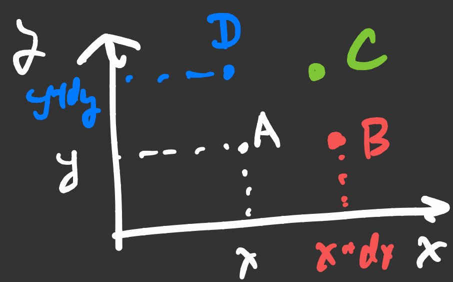
$$\approx f_{xy}(A)\,dy\,dx$$

Thus we have:

$$f_{yx}(A)\,dy\,dx \approx f(A) - f(B) - f(D) + f(C)$$
$$f_{xy}(A)\,dy\,dx \approx f(A) - f(D) - f(B) + f(C)$$

So $\quad f_{yx}(A) \approx f_{xy}(A)$

To evaluate mixed derivatives f_{xy} or f_{yx} numerically, we use the approximate formulae:



$$f_{yx}(A)\,dy\,dx \approx f(A) - f(B) - f(D) + f(C)$$
$$f_{xy}(A)\,dy\,dx \approx f(A) - f(D) - f(B) + f(C)$$

In 3D, $f(x, y, z)$

$$f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y} \quad f_z = \frac{\partial f}{\partial z}$$

$$f_{xx} = \frac{\partial f_x}{\partial x} \quad f_{xy} \; \frac{\partial f_y}{\partial x} \quad f_{xz} = \frac{\partial f_z}{\partial x} \quad \ldots$$

There are 9 different derivatives, but mixed derivatives are equal

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}.$$

No new proof required... reduces to 2D.