Functions of multiple variables.

$f(x,y)$ x,y are real variables
either x,y can be all real variables, or defined on some domain in $\mathbb{R}^2$ (the plane).

ex.

$$f(x,y) = x^2 + y^2$$ 2 variables

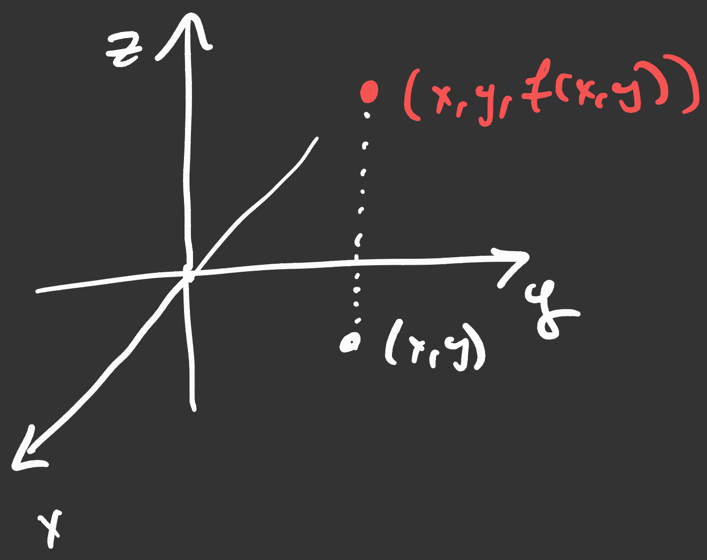
$$f(x,y) = \ln(x^2+y^2)$$

$$f(x,y) = e^{-\frac{1}{x^2-2y}}$$

ex

$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}$$ 3 variables

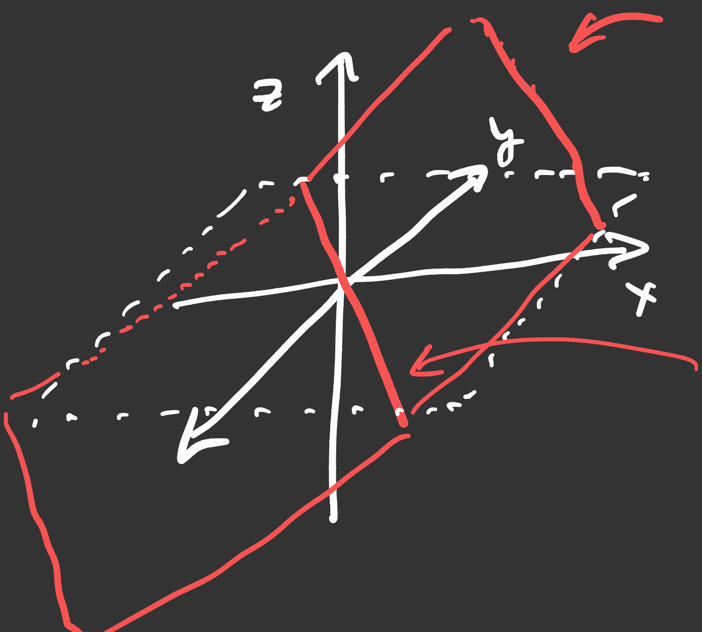
To represent graphically,



↖ if done for all points, you get a surface.

Ex: $f(x,y) = x+y$

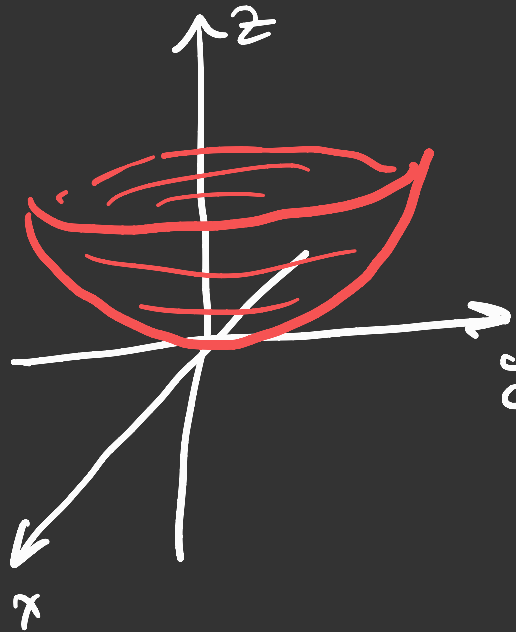
(x,y,z) axis form right triple



$(x, y, f(x,y))$ is a plane whose height is $x+y$.

$y=-x$

Ex: $f(x,y) = x^2 + y^2$



paraboloid of revolution

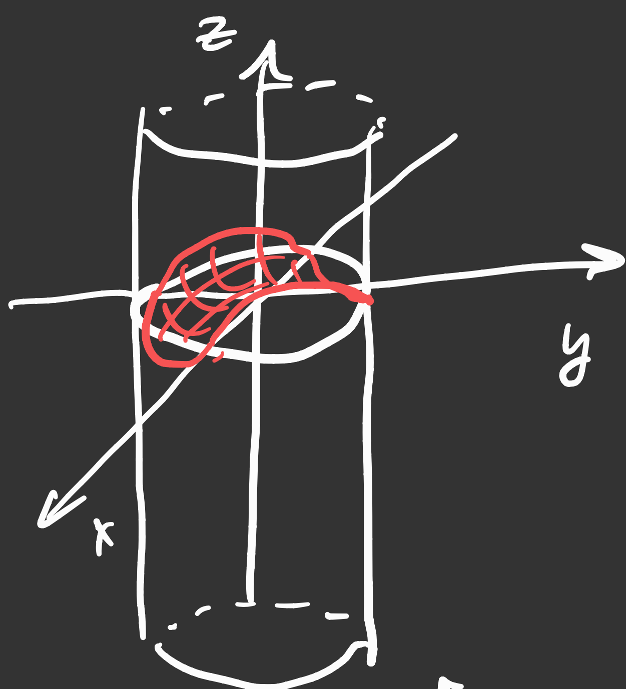
Ex: $f(x,y) = xy$

<u>Saddle</u>

sections along $x=y$,
upwards parabola $f=x^2$

along $x=-y$.
downwards $f=-x^2$

for $x,y>0$ $f>0$
$x>0, y<0$ $f<0$
$x,y<0$ $f>0$
$x>0$ $y<0$ $f<0$



draw piece of surface inside the cylinder

Another way to depict function of two variables: <u>level curves</u>

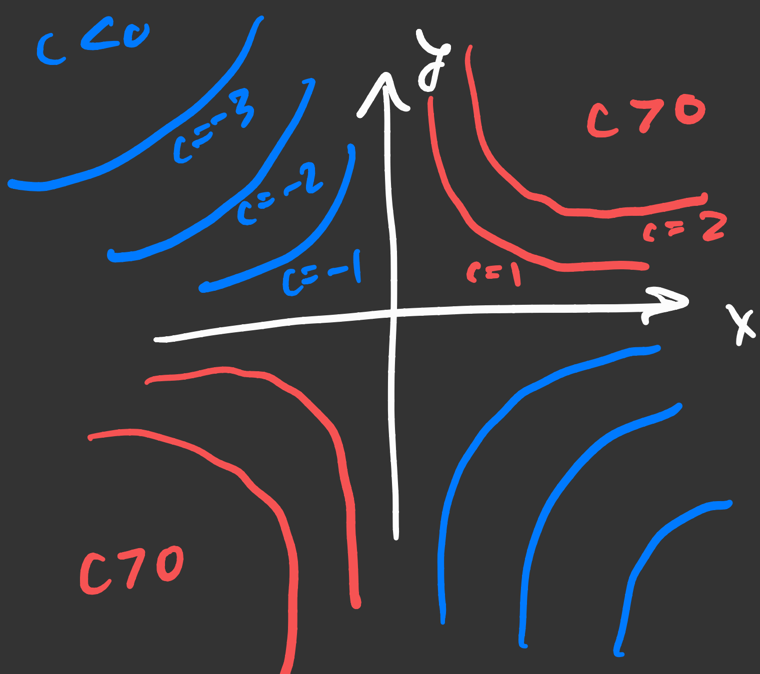
$f(x,y) = c$ defines a curve

different c, different curve

Ex:

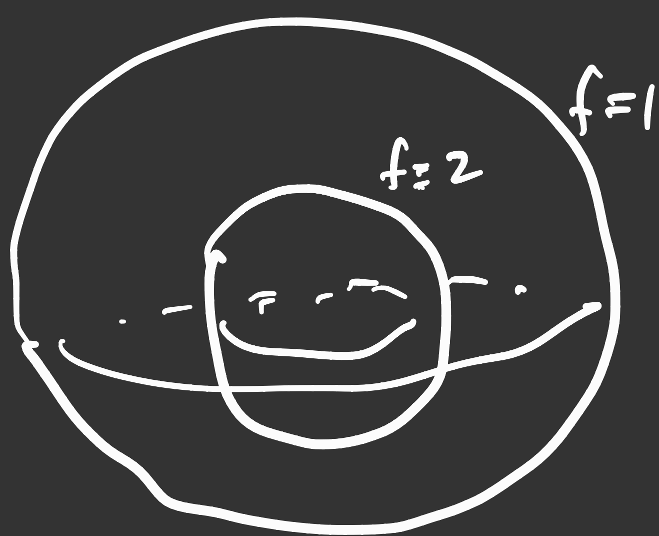
$f(x,y) = xy$

$xy = c$ defines hyperbolas.



gives a rough idea of the behavior of the function.

For a function of 3 variables, the graph is in the fourth dimension, so impossible to draw. Level curves are <u>level surfaces</u>. Sometimes these can be imagined (e.g. if spheres or tori) but in general it is difficult.



$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

Partial derivatives

$f(x)$ ← derivative studied extensively in calc I and II

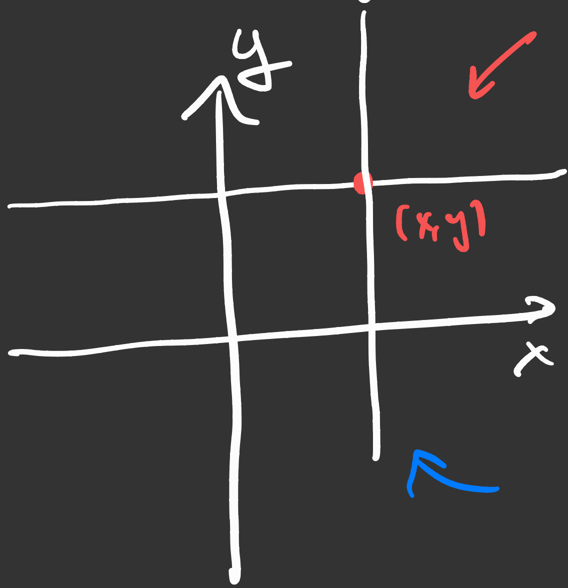
$f(x,y)$ ← 2 variable

Defn:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Geometrically



y = const.

y acts like a parameter, derivative is now as if was a function of 1 variable in "x"

x is parameter, differentiating now in variable y.

Ex $\quad f(x,y) = x^3 y^2$

y is constant $\quad \dfrac{\partial f}{\partial x} = 3x^2y^2$ $\qquad$ x is constant $\quad \dfrac{\partial f}{\partial y} = 2x^3y$

Notation: when differentiating a function of one variable, we write:

$$\frac{d}{dx} f(x)$$

For partials, we write

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

special symbol, not "d".

Ex: $\quad f(x,y) = e^{\frac{1}{x^2+y^2}}$

$$\frac{\partial f}{\partial x} = e^{\frac{1}{x^2+y^2}} \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

chain rule, holding y constant.

$$= e^{\frac{1}{x^2+y^2}} \left(-\frac{1}{(x^2+y^2)^2}\right) \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{-2x}{(x^2+y^2)^2} e^{\frac{1}{x^2+y^2}}$$

For functions of three variables we have

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

Ex:

$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

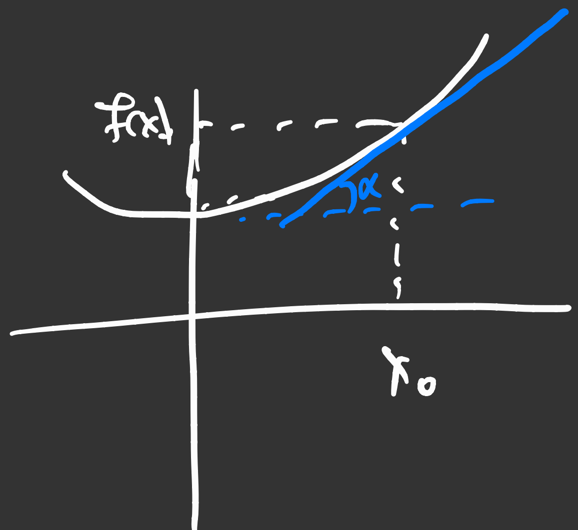
$$\frac{\partial}{\partial y} f(x,y,z) = -\frac{1}{2} \frac{1}{(x^2+y^2+z^2)^{3/2}} \cdot 2y$$

$$= \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial x} f = \frac{-x}{(x^2+y^2+z^2)^{3/2}} \qquad \frac{\partial f}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Linear approximation of a function

1-variable   Suppose $f(x)$ is differentiable and $f'(x)$ is continuous.



slope : $\tan \alpha = f'(x_0)$

$$f(x_0+h) \approx f(x_0) + f'(x_0)h$$

Approximation by linear function

means

$$f(x_0+h) + f(x_0) = f'(x_0) + R(h)$$

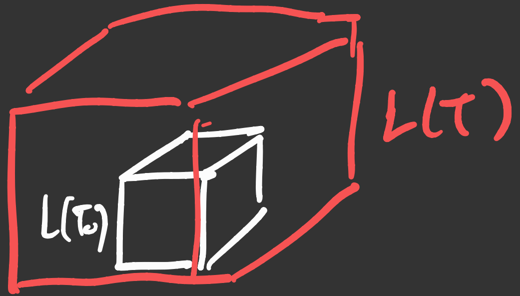
where

$$\frac{R(h)}{h} \to 0 \quad \text{as} \quad h \to 0.$$

Distance between the curve and tangent goes faster than distance b/w inputs

typically as square.

Expansion of solid bodies as you raise temperature



$L(T)$ = diameter as function of temperature

$$L(T_0+h) \approx L(T_0)(1+\alpha h)$$

α is coefficient of linear expansion (dependent on material)

Not only linear size increases, also volume

$$V(T_0+h) \approx V(T_0)(1+\beta h)$$

β is coefficient of volume expansion

Claim: $\beta = 3\alpha$

Is this an experimental fact?

No! It's a theorem using linear approximation

$$V(T_0+h) \approx V(T_0)(1+\alpha h)^3 \approx V(T_0)(1+3\alpha h)$$

$f(x) = x^3$ $\quad f(1) = 1$ $\quad f(1+\alpha h) = f(1) + f'(1)\alpha h$

$f'(x) = 3x^2$ $\quad f'(1) = 3$

<u>Two variables</u> : if derivatives exist and are cont.

$$f(x+h, y+k) \approx f(x,y) + \frac{\partial f}{\partial x}(x,y)\, h + \frac{\partial f}{\partial y}(x,y)\, k.$$

<u>Ex</u>: $f(x,y) = \ln(x^2+y^2)$ $\quad x=1,\ y=2$

$f(1,2) = \ln(5) \approx 1.609438\ldots$

$h = 0.1,\quad k = 0.1$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(1,2) = \frac{2}{5} = 0.4 \qquad \frac{\partial f}{\partial y}(1,2) = \frac{4}{5} = 0.8$$

$$f(1.1, 2.1) \approx \ln(5) + \frac{2}{5}(0.1) + \frac{4}{5}\cdot(0.1)$$

$$= 1.609\ldots + 0.12$$

$$\approx 1.729438 = y_{app}$$

$y = \ln((1.1)^2 + (2.1)^2) = 1.726331$

change in argument

$y_{app} - y = 0.003$ (order of square of 0.1)

$f(x,y,z)$ cont. differentiable

(x_0, y_0, z_0). If $|h|, |k|, |l|$ small, then

$$f(x_0+h, y_0+k, z_0+l) \approx f(x_0, y_0, z_0)$$
$$+ \frac{\partial f}{\partial x}(x_0, y_0, z_0)h + \frac{\partial f}{\partial y}(x_0, y_0, z_0)k + \frac{\partial f}{\partial z}(x_0, y_0, z_0)l$$

This is the linear approximation.

Often the most efficiant way to find the value of a complex function.

Notation

$$\frac{\partial f}{\partial x} = \partial_x f = f_x$$

$$\frac{\partial f}{\partial y} = \partial_y f = f_y$$

$$\frac{\partial f}{\partial z} = \partial_z f = f_z$$

Chain rule:

1) $x = x(t)$ $\quad f(x,y,z)$

$y = y(t)$

$z = z(t)$ $\quad F(t) = f(x(t), y(t), z(t))$

$t \to t + dt$ ← very small (infintesimal) change

$x(t) \to x(t+dt) = x(t) + x'(t)\,dt$ ← linear approximation

$y(t) \to y(t+dt) = y(t) + y'(t)\,dt$

$z(t) \to z(t+dt) = z(t) + z'(t)\,dt$

$$f(x,y,z) \to f(x+dx, y+dy, z+dz)$$
$$= f + f_x\,dx + f_y\,dy + f_z\,dz$$
$$= f + f_x\,x'\,dt + f_y\,y'\,dt + f_z\,z'\,dt$$

Thus

$$\boxed{\frac{d}{dt}F(t) = x'(t)\,f_x(x(t),y(t),z(t)) + y'(t)\,f_y(x(t),y(t),z(t)) + z'(t)\,f_z(x(t),y(t),z(t))}$$

Ex: $f(x,y,z) = x^2+y^2+z^2$

$x(t) = \cos t$
$y(t) = \sin t$
$z(t) = t$

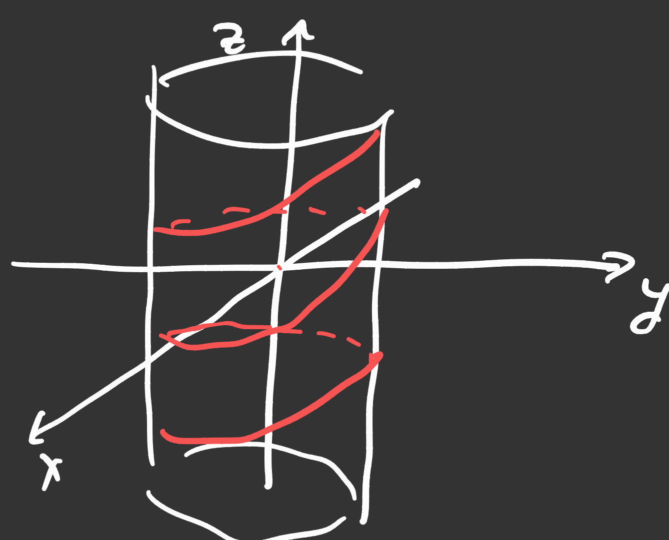
$F(t) = f(x(t), y(t), z(t))$
$F'(t) = f_x x' + f_y y' + f_z z'$

$f_x = 2x \qquad f_y = 2y, \qquad f_z = 2z$

$x' = -\sin t \qquad y' = \cos t \qquad z' = 1$

$F'(t) = -2\cos t \sin t + 2\sin t \cos t + 2t$

$= 2t.$



x^2+y^2 is constant on the Helix.

Ex: $f(x,y)$ but $x = x(u,v)$
$y = y(u,v)$

$$F(u,v) = f(x(u,v), y(u,v))$$

$$\frac{\partial F}{\partial u} = ? \qquad \frac{\partial F}{\partial v} = ?$$

$$(u,v) \rightarrow (u+du, v)$$

$$x(u,v) \rightarrow x(u,v) + \frac{\partial x}{\partial u}(u,v)\,du$$

$$y(u,v) \rightarrow y(u,v) + \frac{\partial y}{\partial u}(u,v)\,du$$

$x \rightarrow x+dx \quad y \rightarrow y+dy$

$$F(u,v) \rightarrow F(u,v) + F_x(u,v)\,dx + F_y(u,v)\,dy$$

$$= F(u,v) + F_x(u,v)\frac{\partial x}{\partial u}(u,v)\,du$$

Thus

$$+ F_y(u,v)\frac{\partial y}{\partial u}(u,v)\,du$$

$$\frac{\partial F}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x}(x(u,v), y(u,v)) + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y}(x(u,v), y(u,v)).$$

$$\frac{\partial F}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x}(x(u,v), y(u,v)) + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y}(x(u,v), y(u,v))$$

Ex:

$$f(x,y) = x^2 + 2y^2$$

$$x = u\cos v \qquad y = u\sin v$$

$$F(u,v) = f(x(u,v),\, y(u,v))$$

$$= u^2 \cos v^2 + 2u^2 \sin^2 v$$

$\partial_x f = 2x$ $\qquad \partial_u x = \cos v$ $\qquad \partial_u y = \sin v$

$\partial_y f = \cancel{2}4y$ $\qquad \partial_v x = -u\sin v$ $\qquad \partial_v y = u\cos v$

$$\frac{\partial F}{\partial u} = 2\cos v\,(u\cos v) + 4\sin v\,(u\sin v)$$

$$= 2u\cos v^2 + 4u\sin v^2$$

$$\frac{\partial F}{\partial v} = 2u\cos v\,(-u\sin v) + 4u\sin v\,(u\cos v)$$

$$= -2u^2\cos v\sin v + 4u^2\cos v\sin v$$

$$= 2u^2\cos v\sin v.$$