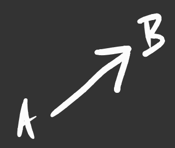
Vectors:

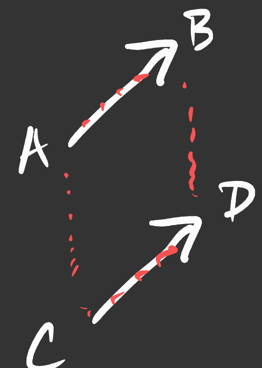
What is a vector? We can consider a vector as a segment with an arrow on one end:



The arrow is to show what is the origin, say point A, and what is the tip, say point B.

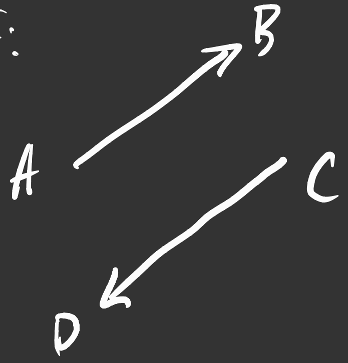
We can thus approximatly define a vector as an object having length (or magnitude) and direction.

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are <u>equal</u> if and only if the figure ABDC is a parallelogram:

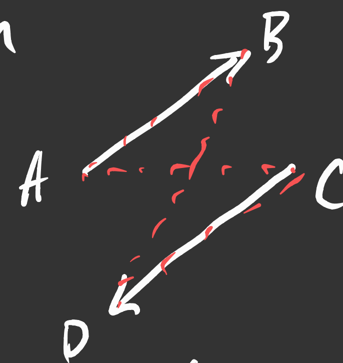


This means that they have equal lengths, are parallel and have the same direction.

If:



In this case ABDC is not a parallelogram



and $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are <u>anti-parallel</u>.

To describe vectors (which have no specified location) we may fix a point O as the origin and count all the vectors from this point.

$\overrightarrow{OA} = \mathbf{a}$ ← bold face "a"

vector notations: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \dots$

to distinguish, e.g. from scalars

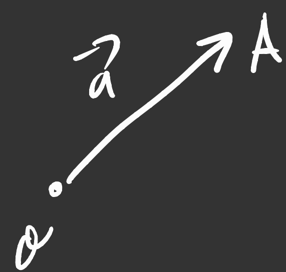
$\bar{a}, \bar{b}, \bar{c}, \bar{d}, \dots$ ← any

$\vec{a}, \vec{b}, \vec{c}, \vec{d}, \dots$

$\hat{a}, \hat{b}, \hat{c}, \hat{d}, \dots$

Some examples:

1) Position vector.

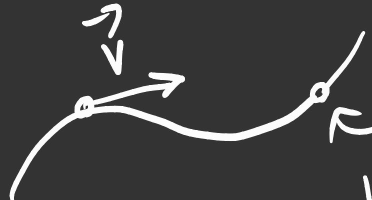


$\leftarrow$ position vector of the point A (relative to O).

2) Displacement vector

how much and in which direction □ was shifted

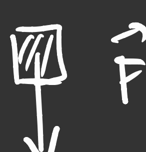
3) velocity Vector



$\leftarrow$ point moving along a curve, $\vec{v}$ is velocity vector.

4) Force vector:

force applied to a body



e.g. gravity force

$\vdots$

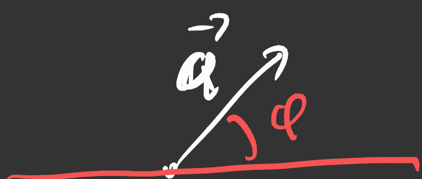
If you have a vector, it has magnitude and direction

<u>magnitude</u> is denoted $\|\vec{a}\|$

$\|\cdot\|$ is the "<u>norm</u>"

Depending on nature of the vector, magnitude is measured in different units.
e.g if $\vec{a}$ is a position vector, $\|\vec{a}\|$ is a length.

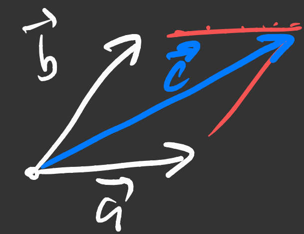
The direction of the vector is denoted by angle between, say, the positive direction of x-axis and the vector, call it φ.



Vector operations

1) <u>Addition</u>; $\vec{a} + \vec{b}$

parallelogram rule <u>Def 1</u>: complete to parallelogram

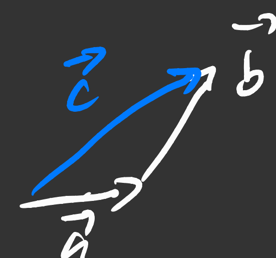


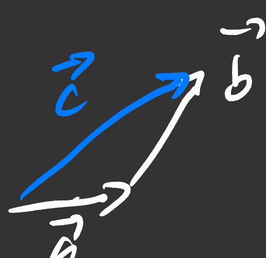
diagonal vector $\vec{c} = \vec{a} + \vec{b}$ (by definition)

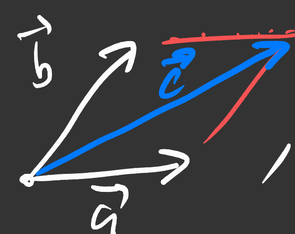
<u>Def 2</u>: take $\vec{a}$ and draw $\vec{b}$ starting from tip of $\vec{a}$:

triangle rule

third side of triangle
$\vec{c} = \vec{a} + \vec{b}$.



Since 

is half of 

,

The definitions are equivalent.

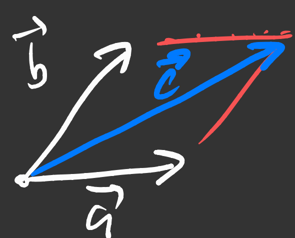
However, they make different aspects of vector addition conceptually clear.

Properties of vector addition:

a) for any two $\vec{a}, \vec{b}$,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative)}$$

Pf: Obvious from parallelogram rule



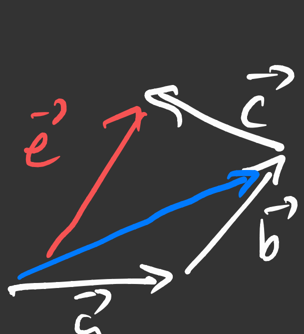
Since the construction of the parallelogram does not take into account the order of $\vec{a}$ and $\vec{b}$.

not hard to see from triangle rule, but obvious from parallelogram rule.

b) For three vectors $\vec{a}, \vec{b}, \vec{c}$:   associative

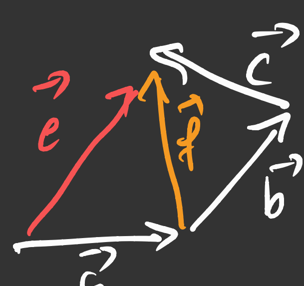
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Pf: triangle rule:



$\vec{d} = \vec{a} + \vec{b}$

$\vec{e} = \vec{d} + \vec{c}$



$\vec{f} = \vec{b} + \vec{c}$

$\vec{e} = \vec{a} + \vec{f}$

These two properties allow to define sum of arbitrary number of vectors.

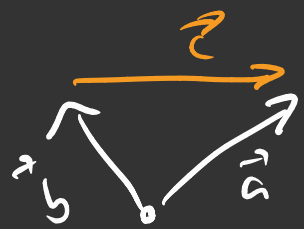
We may omit parentheses and add in arbitrary order.

Subtraction:

Def: $\vec{a} - \vec{b} =: \vec{c}$ is a vector such that $\vec{b} + \vec{c} = \vec{a}$.

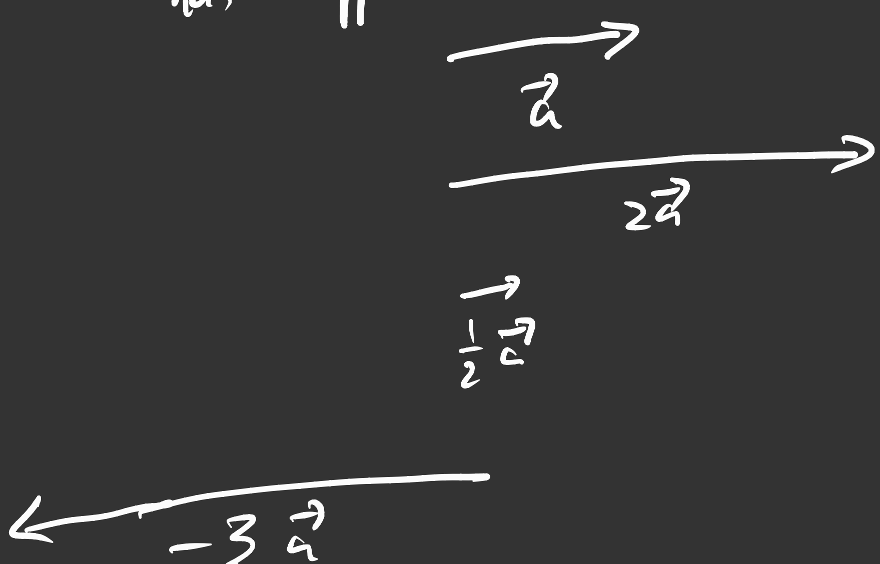
How to find $\vec{c}$?

$\vec{c}$ from tip of $\vec{b}$ to tip of $\vec{a}$.



Scalar Multiplication: $k \in \mathbb{R}$

- $k\vec{a}$ is parallel to $\vec{a}$
- $\|k\vec{a}\| = |k| \, \|\vec{a}\|$
- $k\vec{a}$ has the same direction as $\vec{a}$ if k is positive.
  has opposite direction if k is negative



Zero vector: $\vec{0} = \overrightarrow{AA}$ no definite direction

Def: $0 \cdot \vec{a} = \vec{0}$ for any vector $\vec{a}$.

Properties of Scalar Multiplication

a) $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$ (distributive)

b) $(k_1+k_2)\vec{a} = k_1\vec{a} + k_2\vec{a}$

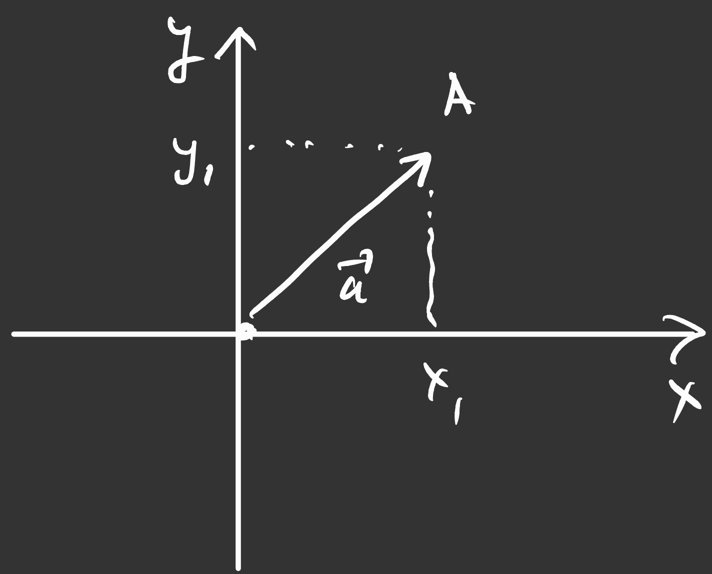
c) $(k_1 k_2)\vec{a} = k_1(k_2\vec{a}) = k_2(k_1 a)$

d) $0\vec{a} = \vec{0}$

e) $a - \vec{b} = a + (-1)b$

All are straightforward to prove.

In order to make some calculations we must introduce coordinate representation of vectors.



$\vec{a} = \overrightarrow{OA}$

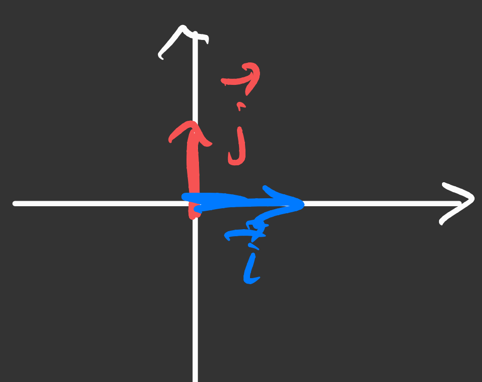
x_1, y_1 are <u>coordinates</u> of the point A.

we may write $\vec{a} = (x_1, y_1)$

x_1 and y_1 are <u>components</u> of vector $\vec{a}$.

<u>Standard Basis vectors</u>

$\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$



$\vec{a} = (x_1, y_1)$, $\vec{b} = (x_2, y_2)$, then

$$\vec{a} + \vec{b} = (x_1 + x_2, y_1 + y_2)$$
$$\vec{a} - \vec{b} = (x_1 - x_2, y_1 - y_2)$$
$$k\vec{a} = (kx_1, ky_1)$$

← vector operations in coordinates

$$\vec{a} = (x, y)$$

length defined by Pythagorian theorem

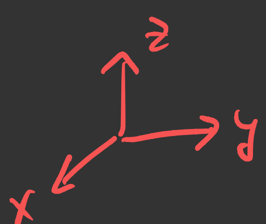
$$\|\vec{a}\| := \sqrt{x^2 + y^2}$$

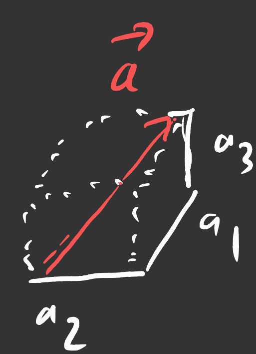
We can similarly define all operations in 3-dimensional (any d-dimenional) space.

length of main diagonal for rectangular parallelpiped

$$a = (a_1, a_2, a_3)$$

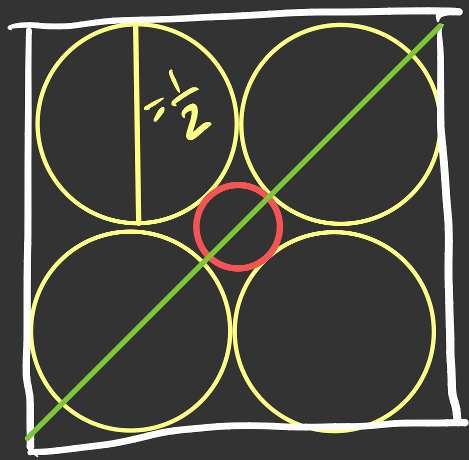
$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$ by the 3D Pythagorean theorem





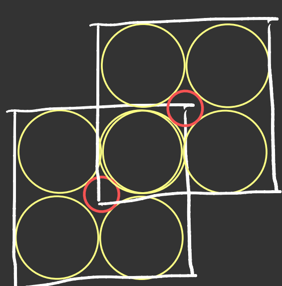
Digression about high dimensions

2D



↑ diagonal $= \sqrt{2}$

Pack 1x1 square with circles. Put disk in the middle. What is diameter?



$$\text{diam}(\bigcirc) = \frac{\sqrt{2}-1}{2} \approx 0.2 < 1$$

8 corners, 8 balls

3D

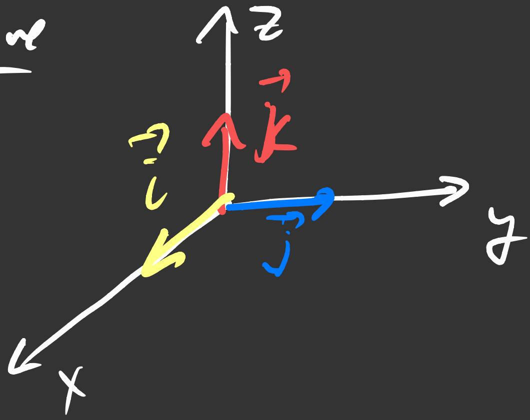
$$\text{diam}(\bigcirc) = \frac{\sqrt{3}-1}{2}$$

nD

$$\text{diam(middle ball)} = \frac{\sqrt{d}-1}{2} > 1 \text{ if } d > 9!$$

In dimension 10 the middle ball comes out of cube!

Picture



$$\vec{i} = (1,0,0)$$
$$\vec{j} = (0,1,0)$$
$$\vec{k} = (0,0,1)$$

$$\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

every vector $\vec{a} \in \mathbb{R}^3$ can be represented as linear combination of the standard basis vectors.

Example: $\vec{a} = (1,2,3)$ $\quad \vec{b} = (3,-1,2)$

$$a = \vec{i} + 2\vec{j} + 3\vec{k} \qquad \vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$$

$$\vec{a} + \vec{b} = 4\vec{i} + \vec{j} + 5\vec{k} = (4,1,5)$$

$$2\vec{a} = (2,4,6)$$

$$3\vec{a} - 2\vec{b} = (3\cdot 1 - 2\cdot 3,\ 3\cdot 2 - 2\cdot(-1),\ 3\cdot 3 - 2\cdot 2)$$

$$= (-3,\ 8,\ 5)$$

$$\|\vec{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\|\vec{b}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$$

__Dot product__: product of vector and vector

$\vec{a} = (a_1, a_2)$ $\quad\vec{b} = (b_1, b_2)$ in 2D

$\vec{a}\ (a_1, a_2, a_3)$ $\quad\vec{b} = (b_1, b_2, b_3)$

__Def__: Dot product (inner product) is

$$\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 \quad \text{in } 2D$$

$$\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{in } 3D$$

Properties: (Proofs)

a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (Obvious from definition)

b) $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ (write in coordinates RHS and LHS)

c) $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$

d) $\vec{a} \cdot \vec{a} = \|a\|^2$ (from definitions)

e) $\vec{i} \cdot \vec{i} = \|\vec{i}\|^2 = 1$

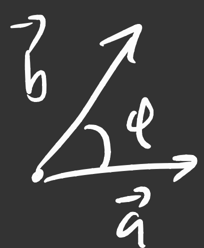
$\vec{j} \cdot \vec{j} = 1$

$\vec{k} \cdot \vec{k} = 1$

$\vec{i} \cdot \vec{j} = (1,0) \cdot (0,1) = 1 \cdot 0 + 0 \cdot 1 = 0$

First nontrivial theorem: geometrical meaning

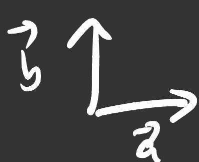
Theorem: Suppose $\vec{a}, \vec{b} \in \mathbb{R}^2$ or $\mathbb{R}^3$. Let φ be the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \varphi \le \pi)$



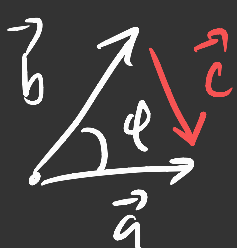
Then $\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos(\varphi)$

Note; that the dot product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is zero iff $\cos(\varphi) = 0$ iff $\varphi = \pi/2$.

e.g. The dot product is zero iff the vectors are perpendicular



Proof: Denote $\vec{c} = \vec{a} - \vec{b}$.



On one hand, by the Law of cosines

$$\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|a\|\|b\| \cos(\varphi)$$

On other hand:

$$\|\vec{c}\|^2 = \|a-b\|^2 = (a-b)\cdot(a-b)$$
$$= \|a\|^2 - b\cdot a - a\cdot b + \|b\|^2$$
$$= \|a\|^2 + \|b\|^2 - 2\,\vec{a}\cdot b$$

□

Angle between two vectors

$$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos \varphi$$

$$\Longleftrightarrow \quad \cos \varphi = \frac{\vec{a} \cdot \vec{b}}{\|a\| \|b\|}$$

$$\Longleftrightarrow \quad \varphi = \arccos \left(\frac{\vec{a} \cdot \vec{b}}{\|a\| \|b\|} \right)$$

Such a φ is necessarily $0 \leq \varphi \leq \pi$.

Example: $\vec{a} = (1,2,3)$ $\quad \vec{b} = (3,-1,4)$

$$\vec{a} \cdot \vec{b} = 3 - 2 + 12 = 13$$

$$\|a\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\|b\| = \sqrt{9+1+16} = \sqrt{26}$$

$$\frac{\vec{a} \cdot \vec{b}}{\|a\| \|b\|} = \frac{13}{\sqrt{14 \cdot 26}} = 0.681$$

$$\varphi = \arccos(0.681) = 0.821 \text{ radians}$$

$$= 47.05^\circ$$

degrees $= \left(\frac{\# \text{ radians}}{\pi} \right) \cdot 180$ degrees

Application: Cauchy-Schwarz inequality.

Given any two vectors $\vec{a}$ and $\vec{b}$, we have

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$$

with equality iff $\vec{a}$ is a scalar multiple of $\vec{b}$, or one of them is zero

Proof: From the geometric meaning of dot product,

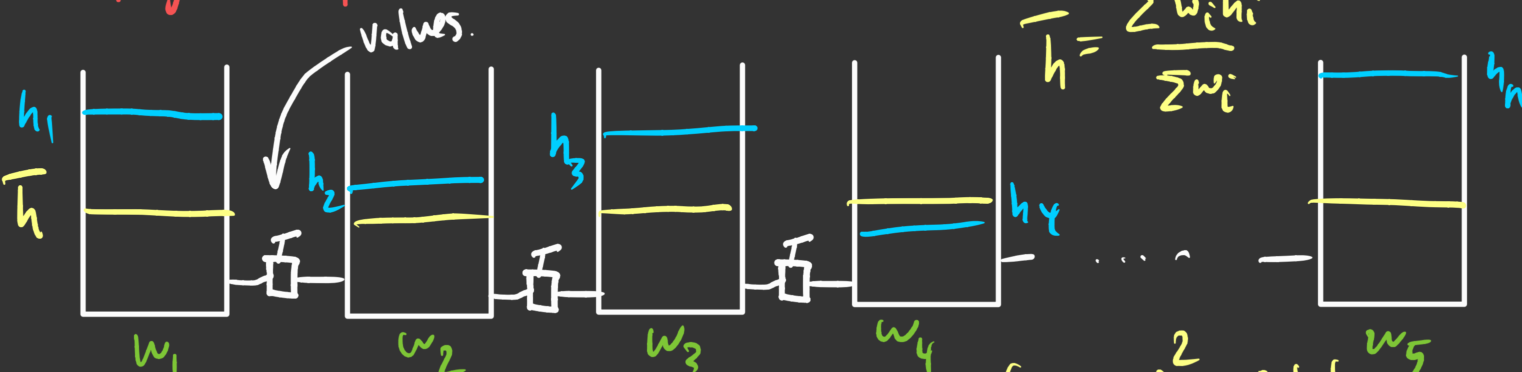
$$|\vec{a}\cdot\vec{b}| = \|\vec{a}\|\,\|\vec{b}\|\,|\cos\varphi| \leq \|\vec{a}\|\,\|\vec{b}\|$$

as $|\cos\varphi| \leq 1$ with equality iff $\varphi = 0$ or π, in which case $\vec{a}$ is a scalar multiple of $\vec{b}$.

If $\varphi \in (0,\pi)$, then $|\cos\varphi| < 1$ strictly. ∎

Mark Levi's Cool physical proof! water heights h_i in buckets of width w_i

$$\bar{h} = \frac{\sum w_i h_i}{\sum w_i}$$



Potential energy: $gmh = \frac{g}{2}\sum_i w_i h_i^2 \geq \frac{g}{2}\frac{\left(\sum w_i h_i\right)^2}{\sum w_i}$ ← Potential when equalized

Application: triangle inequality

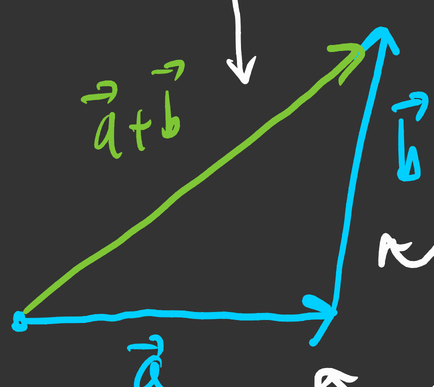
For vectors $\vec{a}$ and $\vec{b}$,

$$\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$$

Proof:

$$\begin{aligned}
\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \\
&= \|\vec{a}\|^2 + 2\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2 \\
&\le \|\vec{a}\|^2 + 2\|\vec{a}\|\,\|\vec{b}\| + \|\vec{b}\|^2 \quad \text{(Cauchy-Schwarz)} \\
&= \left(\|\vec{a}\| + \|\vec{b}\|\right)^2
\end{aligned}$$

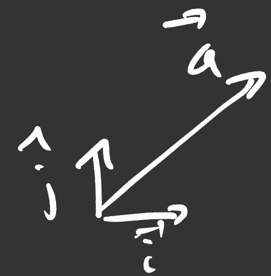
$$\Rightarrow \quad \|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$$



length of any one side of a triangle is less than the sum of other two.

Components of a vector

$\vec{a} = (a_1, a_2)$



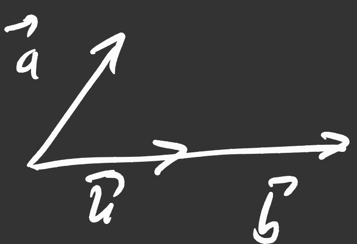
$a_1 = \vec{a} \cdot \vec{i}$ $\qquad$ $a_2 = \vec{a} \cdot \vec{j}$

components given by dot products

We can also consider $\vec{a}$ and $\vec{u}$ with $\|\vec{u}\| = 1$.

$$\mathrm{Comp}_{\vec{u}}\, \vec{a} = \vec{a} \cdot \vec{u}$$ (component of $\vec{a}$ on $\vec{u}$)

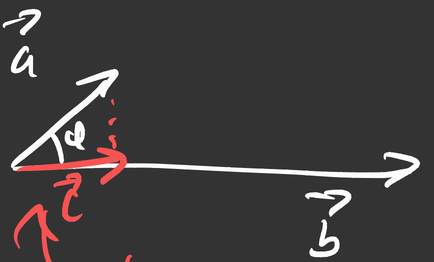
more generally, for any $\vec{a}, \vec{b}$, then



$$\vec{u} = \frac{\vec{b}}{\|\vec{b}\|}$$

$$\mathrm{Comp}_{\vec{b}}\, \vec{a} = \vec{a} \cdot \vec{u} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

Projection of $\vec{a}$ on direction of $\vec{b}$



orthogonal projection of endpoint of $\vec{a}$ on the direction of $\vec{b}$

$$\vec{c} = \mathrm{Proj}_{\vec{b}}\,\vec{a}$$

How to find a foormula?

$$\|c\| = \|a\| \cos\varphi$$

Thus $\vec{c} = \|\vec{c}\|\,\vec{u}$ where $\vec{u} = \dfrac{\vec{b}}{\|\vec{b}\|}$.

$$= \|\vec{a}\| \cos\varphi\ \vec{u}$$

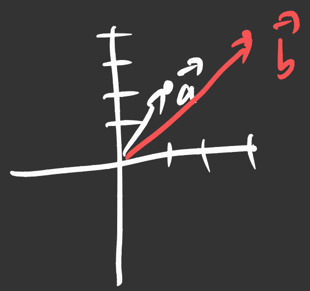
But $\|\vec{a}\| \cos\varphi = \|\vec{a}\|\,\|\vec{u}\| \cos\varphi = \vec{a}\cdot\vec{u}$.

Thus

$$\vec{c} = (\vec{a}\cdot\vec{u})\,\vec{u} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}.$$

$$\boxed{\mathrm{Proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}}$$

For example $\vec{a} = (1,2)$ $\vec{b} = (3,4)$



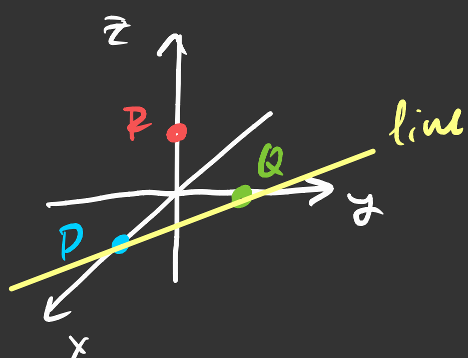
$$\text{Proj}_{\vec{b}}\vec{a} = \frac{3+8}{9+16}(3,4)$$

$$= \frac{11}{25}(3,4)$$

$$= \left(\frac{33}{25}, \frac{44}{25}\right).$$

All formulas work in any dimension.

<u>Problem:</u> Find the distance from the point $R = (0,0,1)$ to the line through points $P = (1,0,0)$ and $Q = (0,1,0)$.



<u>Solution:</u> Project $\vec{PR}$ onto the line via $\vec{PQ}$

$$\text{Proj}_{\vec{PQ}}\vec{PR} = \frac{(-1,0,1)\cdot(-1,1,0)}{\|(-1,1,0)\|\,\|(-1,0,1)\|}(-1,1,0)$$

$$= \frac{1}{2}(-1,1,0)$$

$\vec{PR} = R-P = (-1,0,1)$

$\vec{PQ} = Q-P = (-1,1,0)$ Vector we seek: (perpendicular to $\vec{PQ}$)

$$\vec{PR} - \text{Proj}_{\vec{PQ}}\vec{PR} = (-1,0,1) + \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right).$$

length of this vector is $\left\|\left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right)\right\| = \sqrt{3/2}$.