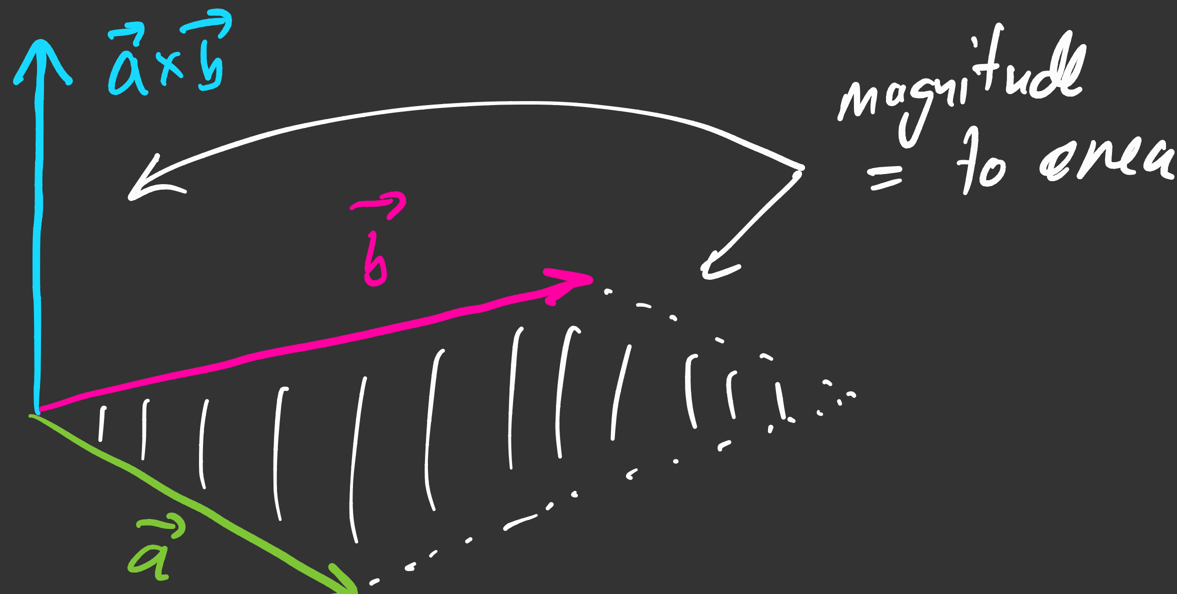
Why is $\vec{a}^{\perp} \cdot \vec{b}$ the cross product in 2D?

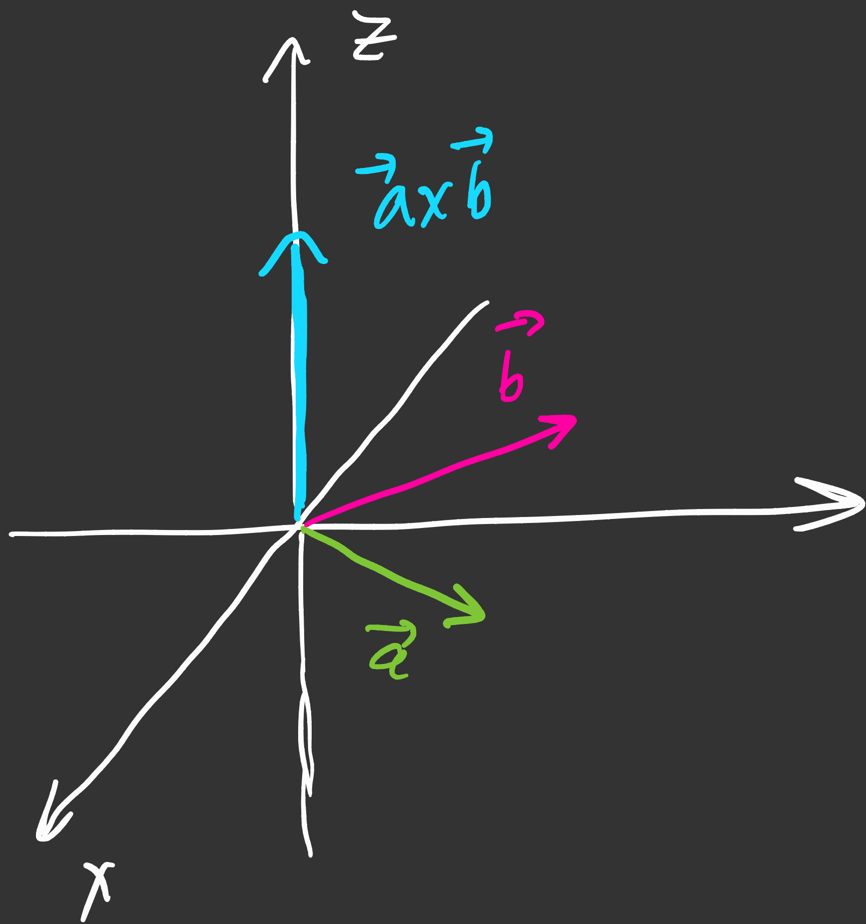
First, given a pair of 3D vectors $\vec{a}$ and $\vec{b}$,

$\vec{a} \times \vec{b}$ is the vector that is

1. orthogonal to $\vec{a}$ and $\vec{b}$
2. whose magnitude is equal to the area of the parallelogram made by $\vec{a}$ and $\vec{b}$
3. whose direction is such that $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ form a right triple.

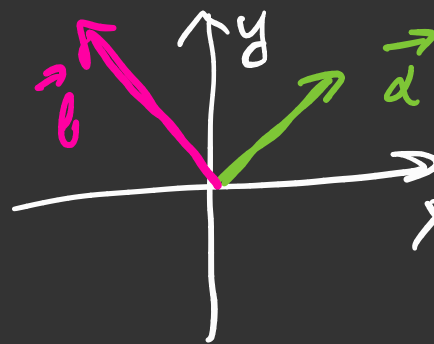
Here is the picture:





Suppose $\vec{a}$ and $\vec{b}$ live in the x-y plane.

$$\vec{a} = (\vec{\alpha}, 0)$$
$$\vec{b} = (\vec{\beta}, 0)$$



Then, according to the definition of $\vec{a} \times \vec{b}$, we should have

since $(\vec{\alpha}, \vec{\beta})$ is a right pair, $A(\vec{\alpha}, \vec{\beta})$ = area

$$\vec{a} \times \vec{b} = A(\vec{\alpha}, \vec{\beta})\, \vec{k}$$
$$= \vec{\alpha}^{\perp} \cdot \vec{\beta}\, \vec{k}$$

makes $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ a right triple.

Thus, $\vec{\alpha}^{\perp} \cdot \vec{\beta}$ is called the cross product in 2D, since it represents the "scalar part" of the cross product of 2 planar vectors in 3D.

Indeed, from the algebraic definition:

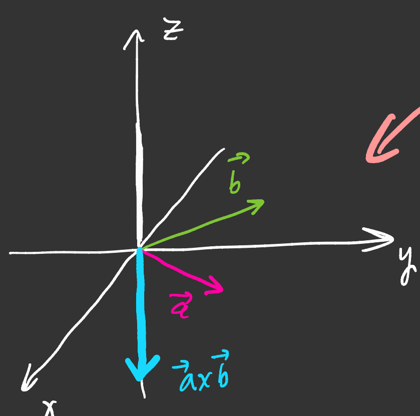
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \end{vmatrix}$$

$$= (\alpha_1 \beta_2 - \alpha_2 \beta_1)\,\vec{k}$$

$$= (\vec{\alpha}^{\perp} \cdot \vec{\beta})\,\vec{k}$$

Note that if $(\vec{\alpha}, \vec{\beta})$ were a left pair, then $A(\vec{\alpha}, \vec{\beta}) = -\text{area}$ and

$$\vec{a} \times \vec{b} = \overbrace{\left(-A(\vec{\alpha}, \vec{\beta})\right)}^{\text{area}} \cdot \overbrace{(-\vec{k})}^{\text{direction}}$$



$(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ is a left triple.

To summarize, if

$$\vec{a} = (\vec{\alpha}, 0) \quad \text{and} \quad \vec{b} = (\vec{\beta}, 0)$$

are two 3D vectors that lye in the x-y plane, then the scalar $\vec{\alpha}^{\perp} \cdot \vec{\beta}$ has

1) magnitude equal to $\|\vec{a} \times \vec{b}\|$

2) sign which determines the orientation:
   - If $\vec{\alpha}^{\perp} \cdot \vec{\beta} > 0$, then $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ is a right triple
   - If $\vec{\alpha}^{\perp} \cdot \vec{\beta} < 0$, then $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ is a left triple.

Thus from the single number $\vec{\alpha}^{\perp} \cdot \vec{\beta}$, one recovers all information about $\vec{a} \times \vec{b}$, justifying the term "cross product in 2D".