Celestial Mechanics and Kepler's laws

Theorem: (Newton) Suppose, for constants G, m, and M, that planetary motion about thus is governed by

$$m\vec{r}''(t) = -\frac{GMm}{\|\vec{r}\|^3}\vec{r}(t). \tag{1}$$

Then the planet's orbit, traced out by $\vec{r}(t)$, is a conic section.

(1) Show conservation energy. Namely, $E = \frac{m}{2}\|\vec{v}\|^2 - \frac{GMm}{\|\vec{r}\|}$ is a constant of motion.
First note that, dotting Newton's equation $m\vec{a} = -\frac{GMm}{\|\vec{r}\|^3}\vec{r}$ with $\vec{v}$ and using the product rule we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m}{2}\|\vec{v}(t)\|^2 + \frac{GMm}{\|\vec{r}\|^3}(\vec{v}\cdot\vec{r}) = 0. \tag{2}$$

On the other hand, computing the evolution of the potential energy $U(r) = -\frac{GMm}{\|\vec{r}\|}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}U(r(t)) = \frac{GMm}{\|\vec{r}\|^3}\vec{v}\cdot\vec{r}. \tag{3}$$

The we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\|\vec{v}(t)\|^2 - \frac{GMm}{\|\vec{r}\|}\right) = 0. \tag{4}$$

(2) Deduce from conservation of energy that, provided $E_0 := \frac{m}{2}\|\vec{v}(0)\|^2 - \frac{GMm}{\|\vec{r}(0)\|} < 0$, the planet's orbit $\vec{r}(t)$ is bounded for all time. In fact, $\|\vec{r}(t)\| \leq \frac{GMm}{|E_0|}$.
The claim follows immediately from energy conservation:

$$-E_0 = \frac{GMm}{\|\vec{r}(t)\|} - \frac{m}{2}\|\vec{v}(t)\|^2 \leq \frac{GMm}{\|\vec{r}(t)\|}. \tag{5}$$

(3) Show conservation angular momentum. Namely, $\vec{L} = \vec{r}\times\vec{v}$ is a constant of motion. Argue that the the motion is confined for all time to the plane $\Pi_{\vec{L}}$ that passes through the origin and is orthogonal to $\vec{L}$.
We cross the equations of motion with $\vec{r}$ to find $\vec{r}\times\vec{a} = 0$. But this implies $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}\times\vec{v}) = \vec{v}\times\vec{v} = 0$.

(4) The above is equivalent to Kepler's second law: "*The line segment from the sun to the planet sweeps out equal areas in equal times.*" That is, the vector $\vec{r}(t)$ sweeps out area $A(t)$ in the plane $\Pi_{\vec{L}}$ at a constant rate. Explain why by proving $A'(t) = \frac{1}{2}\|\vec{L}\|$.
We claim that the rate of change of the swept area is $A'(t) = \frac{1}{2}|\vec{r}(t)\times\vec{v}(t)|$, whence the claim follows. To see it, note that over short time intervals, this area $A(t+dt) - A(t)$ can be approximated as half the area of the parallelogram made from vectors $\vec{r}(t)$ and $\vec{v}(t)dt$. Namely, $A(t+dt) - A(t) \approx \frac{1}{2}|\vec{r}(t)\times\vec{v}(t)|dt$.

(5) Prove conservation of the Laplace–Runge–Lenz vector $\vec{d} := \vec{v}\times\vec{L} - GM\frac{\vec{r}}{\|r\|}$.
The claim follows by the following computation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{v}\times\vec{L}) &= \vec{a}\times\vec{L} = -\frac{GM}{\|\vec{r}\|^3}\vec{r}\times\vec{L} = -\frac{GM}{\|\vec{r}\|^3}\vec{r}\times(\vec{r}\times\vec{v}) \\ &= -\frac{GM}{\|\vec{r}\|^3}\Big((\vec{v}\cdot\vec{r})\vec{r} - \|\vec{r}\|^2\vec{v}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(GM\frac{\vec{r}}{\|r\|}\Big).\end{aligned}$$

(6) Argue that $\vec{d}$ is in the plane spanned by $\vec{r}$ and $\vec{v}$.

Since $\vec{v}\times\vec{L}$ is perpendicular to $\vec{L}$, then $GM\frac{\vec{r}}{\|r\|} + \vec{d}$ is also perpendicular to $\vec{L}$. In particular, it is in the plane spanned by $\vec{r}$ and $\vec{v}$. As such, $\vec{d}$ is also in the same plane spanned by $\vec{v}$ and $\vec{r}$.

(7) Let θ be the angle between $\vec{d}$ and $\vec{r}/\|\vec{r}\|$. Let $L = \|\vec{L}\|$ and $d = \|\vec{d}\|$, then

$$\|r\| = \frac{p}{1+e\cos\theta}, \qquad p = \frac{L^2}{GM}, \qquad e = \frac{d}{GM}.$$

We compute:

$$\begin{aligned} \|\vec{L}\|^2 &= \vec{L}\cdot\vec{L} = (\vec{r}\times\vec{v})\cdot\vec{L} \\ &= (\vec{v}\times\vec{L})\cdot\vec{r} \\ &= (GM\frac{\vec{r}}{\|\vec{r}\|} + \vec{d})\cdot\vec{r} \\ &= GM\|r\| + d\|r\|\cos\theta. \end{aligned}$$

Therefore $\|r\| = \frac{L^2}{GM+d\cos\theta}$. The conclusion follows.

(8) Rotating the plane containing $\vec{v}$ and $\vec{r}$ so that $\vec{d}$ coincides with the positive x-axis. Show the result of (c), in Cartesian coordinates $x = r\sin\theta$, $y = r\cos\theta$, is

$$(1-e^2)x^2 + 2pex + y^2 = p^2.$$

Show that the curve $\{(x,y)\in\mathbb{R}^2 \mid (1-e^2)x^2+2pex+y^2=p^2\}$ is an

- ellipse if $|e| < 1$,
- parabola if $|e| = 1$,
- hyperbola if $|e| > 1$.

First, by the equation in Cartesian coordinates, note that (x, y) satisfy

$$p = r + er\cos\theta = r + ex, \qquad \text{or} \qquad r = p - ex.$$

Therefore, we have $r^2 = x^2 + y^2 = (p-ex)^2$ and, expanding, we obtain

$$(1-e^2)x^2 + 2pex + y^2 = p^2.$$

The curve

$$\{(x,y)\in\mathbb{R}^2 \mid (1-e^2)x^2+2pex+y^2=p^2\}$$

is a conic section of the type claimed in (d). Indeed,

- if $|e| < 1$, we get
$$\left(\frac{x+\frac{pe}{1-e^2}}{\frac{p}{1-e^2}}\right)^2 + \left(\frac{y}{\frac{p}{\sqrt{1-e^2}}}\right)^2 = 1.$$
The center of the ellipse is at $(-\frac{pe}{1-e^2}, 0)$. $\sqrt{a^2-b^2} = \frac{pe}{1-e^2}$, the foci are at the $(0,0)$ and $\left(-2\frac{pe}{1-e^2}, 0\right)$,
- if $|e| = 1$, we get $y^2 = p^2 \pm 2px$, the equation for a parabola,
- if $|e| > 1$, we have
$$\left(\frac{x+\frac{pe}{1-e^2}}{\frac{p}{1-e^2}}\right)^2 - \left(\frac{y}{\frac{p}{\sqrt{1-e^2}}}\right)^2 = 1,$$
the equation for a hyperbola in its canonical form.

(9) Prove Kepler's third law for elliptical orbits: "*The square of the period is proportional to the cube of the major axis of the ellipse.*"

For elliptical orbit, we see that the area of the ellipse is given by πab, where a is the length of the major axis $\frac{p}{1-e^2}$, and b the length of the minor axis $\frac{p}{\sqrt{1-e^2}}$. Recall the area swept out in time t is $A(t) = \frac{L}{2}t$ Let T be the period of the orbit, then area swept after time T is given by $A(T) = \pi ab$. Hence $\frac{L}{2}T = \pi ab$ and thus $\frac{1}{4}L^2 2T^2 = \pi^2 a^3 p$. Since $\frac{p}{L^2} = \frac{1}{GM}$, we have

$$T^2 = \frac{(2\pi)^2 a^3}{GM}.$$