(1) For what values of a and b does the following system of equations

$$x + 2y + 3z = 4, \qquad x + 4y + 9z = 16, \qquad x + 8y + az = b,$$

(a) have a unique solution?
(b) have no solution?
(c) have infinitely many solutions?

Solution: (a) Row reduction can get us to

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & a & b \end{array}\right] \xrightarrow{\text{row ops.}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -8 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & a-21 & b-40 \end{array}\right].$$

If $a \neq 21$, then we can divide the 3rd row by $a - 12$ to get a (consistent) pivot in the 3rd column. This will give a unique solution as every column will have a pivot.
(b) If $a = 21$ *and* $b \neq 40$, then the constant column will have a pivot, meaning the system is inconsistent.
(c) If $a = 21$ *and* $b = 40$, the last row is all zeros, meaning that z is a non-pivot variable, which is free.

(2) Find a matrix A representing the linear transformation T in the following two cases:

(a) $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}$ (b) $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x+3y-7z\\ y-z\end{bmatrix}.$

Solution: (a) Let

$$A = \begin{bmatrix}0 & 2\\ 1 & 2\end{bmatrix}. \text{ Then, } A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \text{ and } A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}.$$

(b) Let

$$A = \begin{bmatrix}2 & 3 & -7\\ 0 & 1 & -1\end{bmatrix}. \text{ Then, } A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2x+3y-7z\\ y-z\end{bmatrix}.$$

(3) A square matrix A is called *nilpotent* if $A^k = 0$ for some positive integer k.
(a) Compute the determinant of any nilpotent matrix. **Solution:** Zero, since $0 = \det A^k = (\det A)^k$.
(b) Find a 3×3 matrix A such that $A^2 = 0$ but $A \neq 0$. **Solution:** Let

$$A = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \text{ Then, } A^2 = 0, \text{ i.e. the zero matrix.}$$

(c) Find a 3×3 matrix A such that $A^3 = 0$ but $A^2 \neq 0$. **Solution:** Let

$$A = \begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}. \text{ Then, } A^2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and } A^3 = AA^2 = 0.$$

(4) *True or False.* Answer the following questions concerning systems (i) and (ii) below. Here, x, y, z are the unknowns and $a_i, b_i, c_i, d_i \in \mathbb{R}$. *Explain your answers. If a statement is false, give a counterexample.*

$$\text{(i)} \begin{array}{rcl} a_1x + b_1y + c_1z &=& d_1 \\ a_2x + b_2y + c_2z &=& d_2 \\ a_3x + b_3y + c_3z &=& d_3 \end{array} \qquad \text{(ii)} \begin{array}{rcl} a_1x + b_1y + c_1z &=& d_1 \\ a_2x + b_2y + c_2z &=& d_2 \\ a_3x + b_3y + c_3z &=& d_3 + 1 \end{array}$$

(a) If (i) has exactly one solution, then the same is true for (ii). **True**. Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} d_1 \\ d_2 \\ d_3 + 1 \end{bmatrix}$$

If (i) has exactly one solution then $\mathrm{rref}([A \mid \vec{d}])$ has a pivot in in every one of the first three columns. Since $\mathrm{rref}([A \mid \vec{d}])$ and $\mathrm{rref}([A \mid \vec{d}'])$ differ only in the last column, we see that (ii) must have a unique solution. Geometrically, the three row vectors of A define three intersecting planes. A unique solution to (i) means that the first two planes intersect in a line and the third plane intersects that line transversely (i.e. doesn't contain the line). The third plane in (ii) is simply a shift of the third plane in (i) in the direction of $\vec{r}_3$, so the intersection remains a point.

(b) If the solution set of (i) is a line, then the same is true for (ii). **False**. A counterexample is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where the solution set of (i) is $\mathrm{span}\{[0\ 0\ 1]^t\}$, while (ii) is inconsistent.

(c) If (i) has no solutions, then the same is true for (ii). **False**. A counterexample is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where (i) is inconsistent, while $\mathrm{span}\{[0\ 0\ 1]^t\}$ is the solution set of (ii).

(5) * Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $f_0 = 0$, $f_1 = 1$, $f_2 = 1$ and, for $n \geq 1$, $f_{n+2} = f_{n+1} + f_n$ be the Fibonacci sequence.

(a) Show that $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and prove, using induction, that

$$A^{2n} = \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}, \qquad n = 1, 2, 3, \ldots$$

Solution: It hold manifestly for A^2 (e.g. $n = 1$), By the induction hypothesis (that the identity holds for A^{2n}) , we have

$$\begin{aligned} A^{2(n+1)} = A^{2n} A^2 &= \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} f_{2n-1} + f_{2n} & f_{2n+1} + f_{2n} \\ f_{2n-1} + 2f_{2n} & 2f_{2n+1} + f_{2n} \end{pmatrix} = \begin{pmatrix} f_{2(n+1)-1} & f_{2(n+1)} \\ f_{2(n+1)} & f_{2(n+1)+1} \end{pmatrix} \end{aligned}$$

using the fact that Fibonacci numbers are defined by the recurrence $f_n = f_{n-1} + f_{n-2}$. E.g. $f_{2(n+1)+1} = f_{2n+3} = f_{2n+2} + f_{2n+1} = 2f_{2n+1} + f_{2n}$.

(b) Compute the determinant of A^2, and use it, together with the formula for A^{2n} (relating the determinants), to prove the identity

$$f_{2n-1} f_{2n+1} - f_{2n}^2 = 1, \qquad \text{for} \qquad n = 1, 2, 3, \ldots.$$

Solution: The determinant is, on one hand,

$$\det\left(A^2\right) = 1, \qquad \det\left(A^{2n}\right) = \det\left(A^2\right)^n = 1.$$

On the other hand, by part (a), we have

$$\det\left(A^{2n}\right) = f_{2n-1} f_{2n+1} - f_{2n}^2.$$

The claimed relation follows.

(6) * Stony Brook University's Board of Trustees, which consists of 20 members, recently had to elect a President. There were three candidates on the shortlist (A, B, and C). On each ballot, the three candidates were listed in order of preference, with no abstentions. The vote outcome is as follows:

- 11 members, a majority, preferred A to B, thus 9 preferred B to A
- 12 members preferred C to A.

Given this, it was suggested that B should withdraw, to enable a direct comparison between A and C. However, B's proponents objected. It turned out that 14 members preferred B to C. Suppose *every possible order* of A, B, and C appeared on at least one ballot, how many members voted for B as their first choice? *Argue how, by eliminating by first comparing a given two of the candidates head to head, and then comparing the remaining two, you could "fairly" elect either A, B or C in this election. (!!)*

Solution: There are 6 possible ways to fill out a ballot: ABC, ACB, BAC, BCA, CAB, CBA. Suppose each of these received a, b, c, d, e, f votes, respectively. The information in the problem tells us that

$$\begin{aligned} a, b, c, d, e, f &\geq 1 \\ a+b+c+d+e+f &= 20 \\ a+b+e &= 11 \\ d+e+f &= 12 \\ a+c+d &= 14 \end{aligned}$$

The last four equations give a linear system that we can solve.

$$\operatorname{rref}\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 20 \\ 1 & 1 & 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 & 1 & 1 & 12 \\ 1 & 0 & 1 & 1 & 0 & 0 & 14 \end{array}\right] = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 12 \end{array}\right]$$

So

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -3 \\ 12 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

We must find values for e, f such that $a, b, c, d, e, f \geq 1$. The only such values are $e = 4$ and $f = 1$, giving

$$\boxed{a = 6, b = 1, c = 1, d = 7, e = 4, f = 1}$$

Therefore $c + d = 1 + 7 = 8$ had B as their first choice.

Voting paradoxes have interesting ramifications in political elections. When there are more than two candidates, the election authority can always manipulate the voters to adopt a voting system that elects a particular candidate as the ultimate winner. In our example, to elect A, the board is asked to choose between B and C, and then B and A; to elect B, the board is asked to choose been A and C, and then B and C; to elect C, the board is to choose between A and B, then C and A. Professor Kenneth Arrow won the Nobel Prize in economics in part for his work to prove that no voting system is entirely fair, i.e. satisfy a set of reasonable criteria for fairness.