(1) Let $\vec{r}(t)$ be a parametrized curve representing the trajectory of a particle. Let $\vec{v}(t) = \vec{r}'(t)$ be its velocity and $\vec{a}(t) = \vec{r}''(t)$ be its acceleration. Prove that the rate of change of the speed of the particle is the component of its acceleration along its velocity:

$$\frac{d}{dt}\|\vec{v}(t)\| = \mathrm{Comp}_{\vec{v}(t)}\vec{a}(t).$$

Give an example of a particle motion that *does* accelerate, but whose speed never changes.

Solution: We compute

$$\|\vec{v}(t)\|\frac{d}{dt}\|\vec{v}(t)\| = \frac{1}{2}\frac{d}{dt}\|\vec{v}(t)\|^2 = \vec{a}(t)\cdot\vec{v}(t).$$

Rearranging and using the definition of the component establishes the identity. For the example, just consider uniform circular motion.

(2) Fix $\omega > 0$ and $a \neq 0$. Consider a particle with position $\vec{r}(t)$ given by the helix

$$\vec{r}(t) = (a\cos\omega t, a\sin\omega t, b\omega t).$$

(a) Show the speed is constant, and the velocity maintains a constant angle with the z-axis.

Solution: First we compute $\vec{r}'(t) = \omega(-a\sin(\omega t), a\cos(\omega t), b)$ so that

$$\|\vec{r}'(t)\| = \omega\sqrt{a^2(\sin^2(\omega t)+\cos^2(\omega t))+b^2} = \omega\sqrt{a^2+b^2}.$$

Since $\vec{r}'(t)\cdot\hat{k} = b\omega$ is constant, the angle $\cos\theta = \frac{b\omega}{\sqrt{a^2+b^2}\omega} = \frac{b}{\sqrt{a^2+b^2}}$ is constant.

(b) Find the arclength parametrization of the helix and compute the curvature and principle normal vector. Show that the acceleration vector of the particle is always parallel to the xy plane.

Solution: From part (a), we have $s(t) = \omega\sqrt{a^2+b^2}t$. The inverse function is $t(s) = \frac{s}{\omega\sqrt{a^2+b^2}}$. The arc-length parametrization is therefore given by composing $\vec{r}(t)$ with $t(s)$, i.e.

$$\vec{r}(s) = \left(a\cos\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), \tfrac{bs}{\sqrt{a^2+b^2}}\right).$$

Unit tangent vector to the helix is therefore

$$\vec{T}(s) = \vec{r}'(s) = \tfrac{1}{\sqrt{a^2+b^2}}\left(-a\sin\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), a\cos\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), b\right).$$

Therefore, the acceleration vector is

$$\frac{d}{ds}\vec{T}(s) = \vec{r}''(s) = -\tfrac{a}{a^2+b^2}\left(\cos\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), 0\right).$$

This shows that the acceleration vector of the particle is always parallel to the xy plane (since conversion between s and t derivatives modifies this expression by constants). The curvature is

$$\kappa(s) = \left\|\frac{d}{ds}\vec{T}(s)\right\| = \frac{|a|}{a^2+b^2}.$$

The principle normal vector is

$$\vec{N}(s) = \frac{\frac{d}{ds}\vec{T}(s)}{\left\|\frac{d}{ds}\vec{T}(s)\right\|} = -\mathrm{sign}(a)\left(\cos\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\tfrac{s}{\sqrt{a^2+b^2}}\right), 0\right).$$

(c) Let $P = \vec{r}(0) = (a,0,0)$ and $Q = \vec{r}(\frac{2\pi}{\omega}) = (a,0,2\pi b)$. Note $\overrightarrow{PQ}$ is vertical. Show that

$$\vec{r}(\tfrac{2\pi}{\omega}) - \vec{r}(0) = \tfrac{2\pi}{\omega}\vec{r}'(s)$$

cannot hold for any $s \in (0, \frac{2\pi}{\omega})$. Thus, the Mean Value Theorem does not hold for vector functions.

Solution: Since $\overrightarrow{PQ}$ is vertical, and $\vec{r}'$ is never vertical, $\overrightarrow{PQ}$ can not be $\frac{2\pi}{\omega}\vec{r}'(s)$ for any s.

(3) Compute the curvature and draw a (rough) picture of the following spirals:

(a) the Archimedean spiral $\vec{r}(t) = (t\cos(t), t\sin(t), 0)$,

Solution: Here we cannot easily compute the arc-length explicitly, so we instead use the formulae in terms of an arbitrary parametrization. We require

$$\vec{r}'(t) = (\cos(t) - t\sin(t), t\cos(t) + \sin(t), 0), \qquad \vec{r}''(t) = -(t\cos(t) + 2\sin(t), 2\cos(t) + t\sin(t), 0).$$

We require also

$$\|\vec{r}'(t)\| = \sqrt{(\cos(t) - t\sin(t))^2 + (t\cos(t) + \sin(t))^2} = \sqrt{1+t^2}$$

so that $\|\vec{r}'(t)\|^3 = (1+t^2)^{3/2}$. Finally, we need

$$\vec{r}'(t) \times \vec{r}''(t) = (2+t^2)(0,0,1), \qquad \|\vec{r}'(t) \times \vec{r}''(t)\| = 2+t^2.$$

Then we assemble the curvature

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{2+t^2}{(1+t^2)^{3/2}}.$$

As time tends to infinity in either way, the curvature tends to zero. This corresponds to two "arms" of the Archimedean spiral: for $t > 0$ it spirals out and for $t < 0$. The maximum curvature occurs at $t = 0$, which is the value 2.

(b) the logarithmic spiral $\vec{r}(t) = (e^t\cos(t), e^t\sin(t), 0)$.

Solution: Again, we cannot easily compute the arc-length explicitly, so we instead use the formulae in terms of an arbitrary parametrization. We require

$$\vec{r}'(t) = e^t(\cos(t) - \sin(t), \cos(t) + \sin(t), 0), \qquad \vec{r}''(t) = 2e^t(-\sin(t), \cos(t), 0).$$

We require also

$$\begin{aligned}\|\vec{r}'(t)\| &= e^t\sqrt{(\cos(t) - \sin(t))^2 + (\cos(t) + \sin(t))^2} \\ &= \sqrt{2}e^t\sqrt{\cos^2(t) + \sin^2(t)} = \sqrt{2}e^t\end{aligned}$$

so that $\|\vec{r}'(t)\|^3 = 2^{3/2}e^{3t}$. Finally, we need

$$\vec{r}'(t) \times \vec{r}''(t) = 2e^{2t}(0,0,1), \qquad \|\vec{r}'(t) \times \vec{r}''(t)\| = 2e^{2t}.$$

Then we assemble the curvature

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{e^{-t}}{\sqrt{2}}.$$

As time tends to infinity, the curvature tends to zero, in accord with it becoming flatter as it gets farther out. As time goes to minus infinity, the curve coils around the origin, the curvature diverges.

(4) Let $\vec{r}(t)$ be the vector from the origin to the position of an object of mass $m > 0$, $\vec{v}(t) := \vec{r}'(t)$ be the velocity and $\vec{a}(t) := \vec{r}''(t)$ be the acceleration. Suppose that $\vec{F}(t) = m\vec{a}$ is the force acting at time t.

(a) Prove that $\frac{d}{dt}(m\vec{r} \times \vec{v}) = \vec{r} \times \vec{F}$. What do you conclude if $\vec{F}$ is parallel to $\vec{r}$?

Solution: Note $\frac{d}{dt}(m\vec{r}(t)\times\vec{v}(t)) = m\vec{r}'(t)\times\vec{v}(t)+m\vec{r}(t)\times\vec{v}'(t) = m\vec{v}(t)\times\vec{v}(t)+\vec{r}(t)\times(m\vec{a})(t) = \vec{r}\times\vec{F}$, since $\vec{F} = m\vec{a}$ and $\vec{v}\times\vec{v} = 0$. If $\vec{F}$ is parallel to $\vec{r}$, then $\vec{r}\times\vec{F}$ and therefore the quantity $m\vec{r}\times\vec{v}$, the angular momentum, is preserved in time.

(b) Prove that a planet (say, of mass m) moving about the Sun (say, of mass M) does so in a fixed plane. Recall, Newton's universal law of gravity says that $\vec{F} = -\frac{GmM}{\|\vec{r}\|^3}\vec{r}$.

Solution: Since angular momentum is conserved, $\vec{\ell} = \vec{r}(t)\times\vec{v}(t)$ for some fixed vector $\vec{\ell}\in\mathbb{R}^3$. Moreover, $\vec{r}(t)\cdot\vec{\ell} = 0$ since $\vec{r}(t)\cdot(\vec{r}(t)\times\vec{a}) = 0$ for any $\vec{a}\in\mathbb{R}^3$. This implies that the position vector $\vec{r}(t)$ is confined to a plane with normal vector $\vec{\ell}$.

(c) Kepler's First Law says that the orbit of a planet around the sun is an ellipse. Derive Kepler's Second Law which states that the rate at which the area swept out by the planet around the sun is constant.

Solution: From (b), $\vec{r}(t)$ and $\vec{v}(t)$ must lie in the plane through the origin (sun) with normal $\vec{\ell}$. Note

$$\frac{1}{2}\|\vec{r}(t) \times \vec{v}(t)\|dt = \frac{1}{2}\|\vec{\ell}\|dt$$

is the area swept out by the planet during time interval $[t, t + dt]$. The rate that the area swept out by the planet around the sun is therefore constant at $\frac{1}{2}\|\vec{\ell}\|$.

(5) (a) Let $\vec{r}(t)$ be a differentiable vector valued function of t. Show that, at a local maximum or minimum of $\|\vec{r}(t)\|$, the vector $\vec{r}'(t)$ is perpendicular to $\vec{r}(t)$.

Solution: $0 = \frac{1}{2}\frac{d}{dt}\|\vec{r}(t)\|^2 = \vec{r}'(t) \cdot \vec{r}(t)$.

(b) Prove that if the curvature of a curve is identically zero, then the curve is straight.

Solution: This is intuitively obvious. To prove it, suppose that the curve is parametrized by arc-length. Then $\vec{v}(s) = \vec{T}(s)$ and $\vec{T}'(s) = 0$ since the curvature is zero, so $\vec{v}(s)$ is constant in s and therefore the same on the entire curve. Therefore the curve is a straight line, the only type of curve with equal tangent vector at all points.

(6) Suppose that $\gamma(s) : [0, L] \to \mathbb{R}^d$, $d \geq 2$ is an arc length parametrization of a closed curve Γ of length L. Suppose Γ lies entirely within a sphere of radius R. Prove that average of the absolute curvature $|\kappa(s)| = |\gamma''(s)|$ enjoys the lower bound

$$\frac{L}{R} \leq \int_\Gamma |\kappa(s)| ds.$$

Use this result, together with some typical numbers found searching the internet, to lower bound the net absolute curvature of a strand of your DNA. Hint: use $\gamma'(s) \cdot \gamma'(s) = 1$ and integrate! Bonus: determine precisely which curves saturate the above inequality.

Solution: Since γ is arclength parametrized, we have

$$1 = \gamma'(s) \cdot \gamma'(s) = \frac{d}{ds}\Big(\gamma(s) \cdot \gamma'(s)\Big) - \gamma(s) \cdot \gamma''(s).$$

Thus, using that $|\gamma(s)| \leq R$, upon integration we find

$$L = -\int_0^L \gamma(s) \cdot \gamma''(s) ds \leq R \int_0^L |\ddot{\gamma}(s)| ds = R \int_\Gamma |\kappa(s)| ds.$$

The inequality follows.

Let us reflect on what you showed: a cell in your body is roughly 10^{-5} meters, whereas a strand of your DNA is on the order of one meter long. So the bound implies the net (or, even, the maximum) absolute curvature of your DNA is 10^5, which is probably an enormous underestimate!

We remark that equality holds above if and only if $|\gamma(s)| = R$ and $\gamma''(s) = \lambda(s)\gamma(s)$ for some scalar function $\lambda(s)$. In fact, γ is a geodesic on the sphere and λ is the Lagrange multiplier to enforce the constraint of remaining on the sphere. We can find and expression for λ be differentiating the constraint $|\gamma(s)|^2 = R^2$ twice:

$$0 = \gamma(s) \cdot \gamma''(s) + |\gamma'(s)|^2 = \lambda|\gamma(s)|^2 + 1 = \lambda R^2 + 1.$$

As such, γ must satisfy $\gamma''(s) = -\frac{\gamma(s)}{R^2}$. Without loss of generality, consider initial data $\gamma(0) = (R, 0, \ldots, 0)$ and $\gamma'(0) = (0, 1, \ldots, 0)$. Then the solution of the above is

$$\gamma(s) = R\big(\cos(s/R), \sin(s/R), 0, \ldots, 0\big).$$

So the curves that satisfy the inequality with equality are circles (great circles) on the sphere which are traversed the appropriate number of times.