MAT 307, Multivariable Calculus with Linear Algebra, Fall 2024

- (1) Using the dot product, prove the Pythagorean theorem. Namely, show that if the lengths a, b, c of the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle and visa verse.
- (2) Let \vec{v} and \vec{u} be vectors in \mathbb{R}^2 and recall that the area of the triangle is defined by \vec{v} and \vec{u} is given by $A(\vec{v},\vec{u}) = \frac{1}{2} |\vec{v}^{\perp} \cdot \vec{u}|$ where $\vec{v}^{\perp} = (-v_2, v_1)$ denotes counterclockwise rotation of the vector \vec{v} by 90°.
 - (a) Show that $(\vec{v}^{\perp} \cdot \vec{u})^2 = \|\vec{v}\|^2 \|\vec{u}\|^2 (\vec{v} \cdot \vec{u})^2$. Explain what this means geometrically.
 - (b) [†] Use this fact to establish *Heron's formula* (most probably due to Archimedes^{*}), namely that a triangle with sides lengths a, b and c has area

$$\operatorname{Area}(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s := \frac{1}{2}(a+b+c).$$

To do this, you will need to relate to $(\vec{v} \cdot \vec{u})^2$ with magnitudes of \vec{v} , \vec{u} and $\vec{v} - \vec{u}$ (which should describe the sides of the triangle). There is also some algebra.

- (c) Show that, for fixed side lengths a and b, the triangle with the largest area is right.
- (d) [†] Show that, for a fixed perimeter, the triangle with the largest area is equilateral.
- (3) (a) Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that they are not parallel, describe the set of vectors $s\vec{u} + t\vec{v}$ where s + t = 1. Where are the vectors when s and t are both non-negative?
 - (b) Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ do not lie in the same plane. Describe the set of vectors $r\vec{u} + s\vec{v} + t\vec{w}$ where $r, s, t \in \mathbb{R}$ and r + s + t = 1. Where are the vectors when r, s and t are all non-negative?
- (4) Four vectors are erected perpendicularly to the four faces of a general tetrahedron, each vector is pointing outward and has length equal to the area of the face. Show that the sum of these four vectors is $\vec{0}$. This fact is a precursor to Stokes' theorem.
- (5) (a) Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors that are not co-planar, namely $\alpha \vec{u} + \beta \vec{v} + \gamma \vec{w} = \vec{0}$ only if $\alpha = \beta = \gamma = 0$. Show that $\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}$ are not co-planar.
 - (b) Suppose $a, b, c \in \mathbb{R}$, find the point of intersections of the three planes

$$\vec{u} \cdot (x, y, z) = a, \qquad \vec{v} \cdot (x, y, z) = b, \qquad \vec{w} \cdot (x, y, z) = c.$$

Express the solution (x, y, z) as $(x, y, z) = \alpha \vec{v} \times \vec{w} + \beta \vec{w} \times \vec{u} + \gamma \vec{u} \times \vec{v}$ for some α, β, γ .

(6) This problem illustrates some counterintuitive features of high dimensional space. Consider a pyramid in \mathbb{R}^n with vertices at the origin O and $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n$ ($e_1 = (1, 0, \dots, 0)$ through $e_n = (0, 0, \dots, 1)$ are the standard Euclidean basis vectors). The base of the pyramid is the (n-1)-dimensional object B defined by

$$B = \left\{ (x_1, x_2, \cdots, x_n) \middle| \sum_{i=1}^n x_i = 1, x_i \ge 0 \text{ for all } i \right\}.$$

- (a) Find the coordinates of the point (called the centroid of B) C in the base B which is equidistant from each vertex of B and calculate the length $\|\overrightarrow{OC}\|$.
- (b) Use the Cauchy-Schwarz inequality to show that C is the closest point in B to the origin O.
- (c) Calculate the angle θ between \overrightarrow{OC} and any edge $\overrightarrow{OV_i}$, where V_i is the vertex corresponding to \hat{e}_i . What happens to this angle θ and the length $\|\overrightarrow{OC}\|$ as $n \to \infty$?

(Extra) [†] Marsden & Tromba: §1.1: #6, 14, 18, 34; §1.2: #6, 20, 26, 38; §1.R: #14, 20.

^{*}See Vol. II, 321 of T. Heath. History of Greek Mathematics. Oxford University Press, 1921.

[†]Not to appear on quiz.