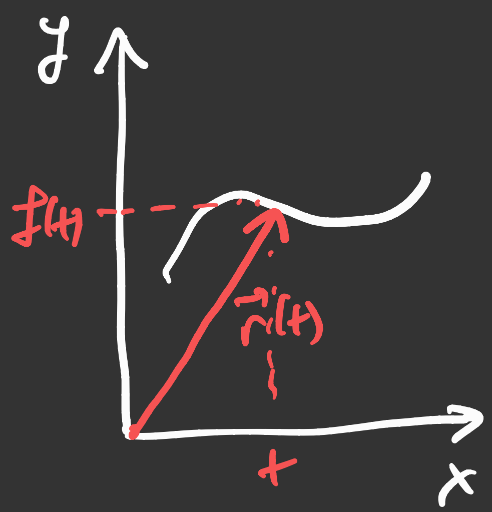
Plane curves

$$\vec{r}(t) = (x(t), y(t), 0)$$

keep since some objects are visualized in 3D.

Let us restrict further to consider our curve to be the graph of a function



$$y = f(x)$$

$$\vec{r}(t) = (t, f(t), 0)$$

$$\vec{v}(t) = \vec{r}\,'(t) = (1, f'(t), 0)$$

$$\vec{a}(t) = \vec{r}\,''(t) = (0, f''(t), 0)$$

$$v(t) = \|\vec{v}(t)\| = \sqrt{1 + (f'(t))^2}$$

Unit tangent vector:

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{1}{\sqrt{1+(f')^2}}(1, f', 0)$$

Curvature

$$\kappa(t) = \frac{\|\vec{v}\times\vec{a}\|}{v^3} = \frac{|f''(t)|}{(1+(f')^2)^{3/2}}$$

Since

$$\vec{v}\times\vec{a} = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = f''\vec{k}.$$

$$= (0, 0, f'')$$

Binormal vector

remember $\vec{a}(t) = \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N}$

Thus

$$\vec{v}\times\vec{a} = \kappa v^3 \vec{B} \implies \vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}$$

Thus $\vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \frac{f''(t)}{|f''(t)|}\hat{k}$

Normal vector

$$\vec{N} = \vec{B} \times \vec{T}$$

$$= \frac{f''}{|f''|} \hat{k} \times \vec{T} \quad = \mathrm{sgn}(f'') \vec{k} \times \frac{(1, f', 0)}{\sqrt{1+(f')^2}}$$

$$= \frac{\mathrm{sgn}(f'')}{\sqrt{1+(f')^2}} \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & f' & 0 \end{vmatrix}$$

$$= \frac{\mathrm{sgn}(f'')}{\sqrt{1+(f')^2}} (-f'(t), 1, 0)$$

What is this?

$y = f(x)$

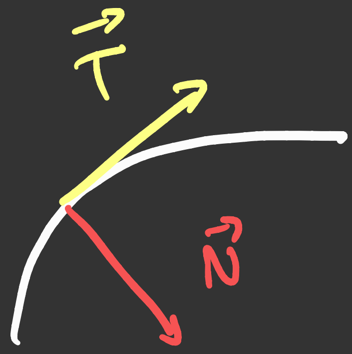
$$\vec{T} = \frac{(1, f', 0)}{\sqrt{1+(f')^2}}$$

rotation ccw $90°$

$$\vec{N} = \mathrm{sgn}(f'') \frac{(-f', 1, 0)}{\sqrt{1+(f')^2}}$$

directed in concave direction $f'' > 0$

In opposite case



$f'' < 0$

Thus we found:

$$\vec{r}(t) = (t, f(t), 0)$$

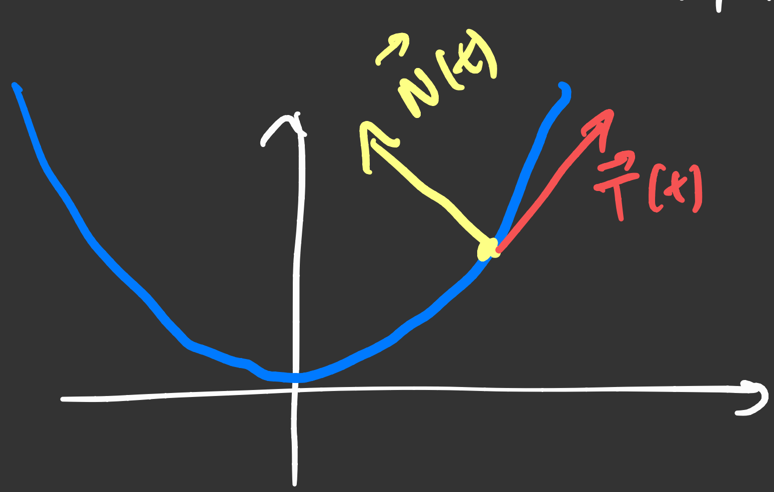
$$k(t) = \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}$$

$$\vec{N}(t) = \frac{\operatorname{sgn}(f''(t))}{\sqrt{1 + (f')^2}} (-f'(t), 1, 0)$$

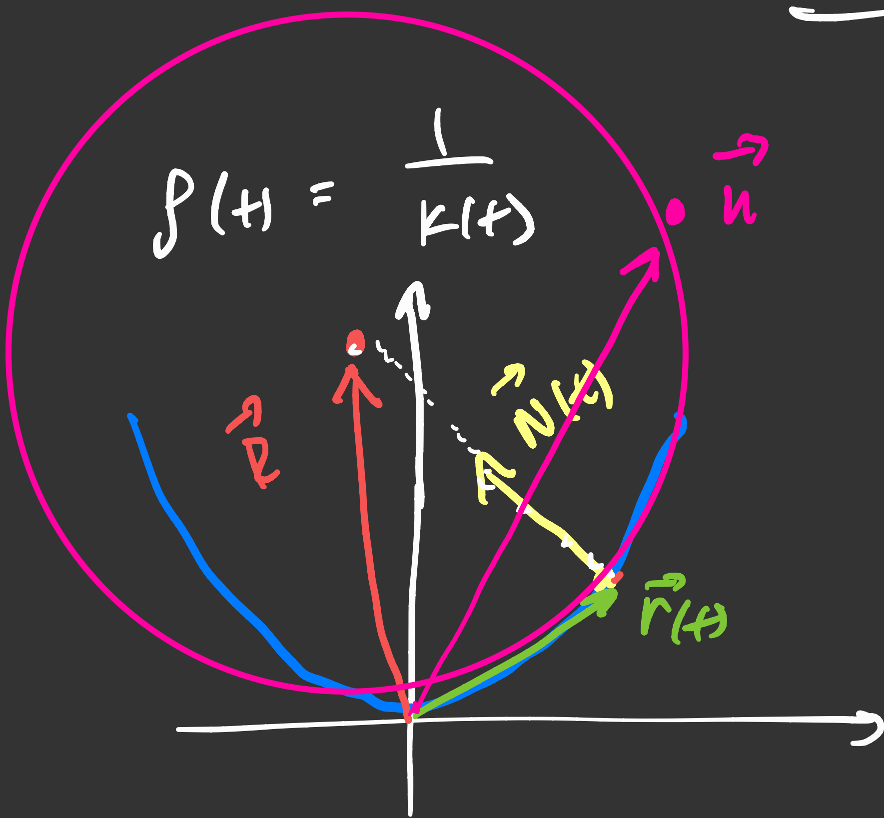
Example: $f(t) = t^2$

$$k(t) = \frac{2}{(1 + 4t^2)^{3/2}}$$

$$\vec{N}(t) = \frac{(-2t, 1, 0)}{\sqrt{1 + 4t^2}}$$



Oscular circle



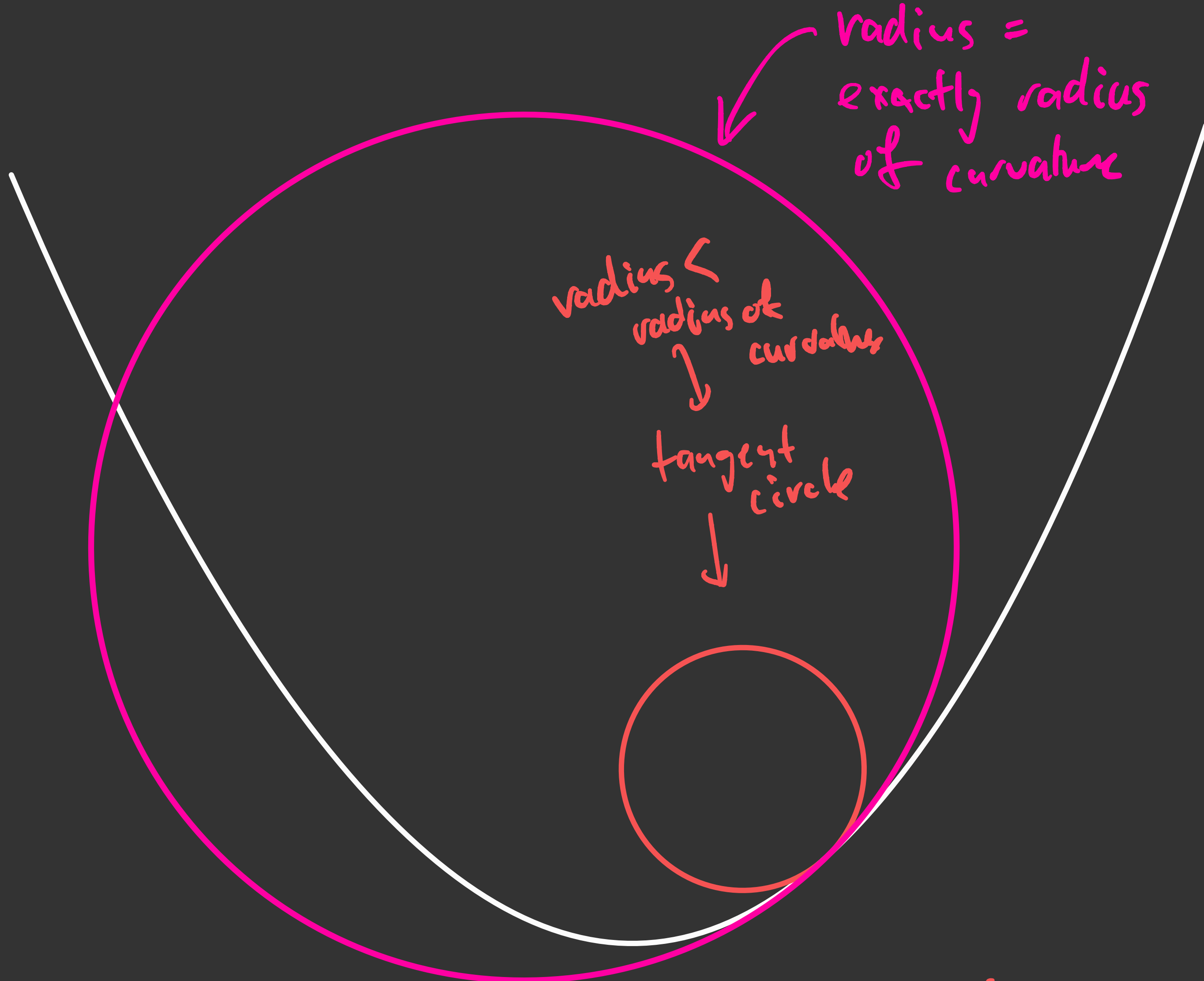
Oscular circle: circle whose center $\vec{R}(t)$ is on the line through $\vec{r}(t)$ in the direction of $\vec{N}(t)$, i.e.

$$\vec{R}(t) = \vec{r}(t) + \rho\,\vec{N}(t)$$

The equation

$$\|\vec{u} - \vec{R}(t)\| = \rho(t)$$

equation for osculating circle.



dist between parabola and circle, decreases like square of dist. entirely above parabola.

osculating circle: on one side, it is over parabola. On other, it is under.

$\boxed{\text{distance to parabola behaves as the cube of the distance}}$