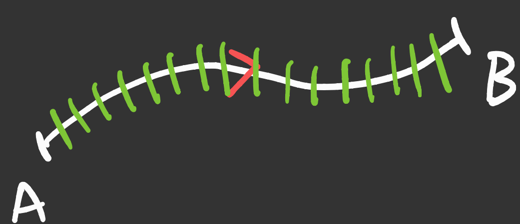
Curvature

Suppose we have a parametized curve

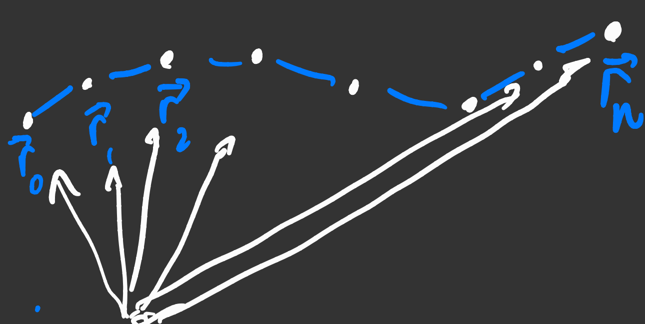
$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$a \leq t \leq b$$

What is the length of the curve?



$$A = t_0 < t_1 < \dots < t_N = B$$



length:

$$\|\vec{r}_1 - \vec{r}_0\| + \|\vec{r}_2 - \vec{r}_1\| + \dots + \|\vec{r}_{N-1} - \vec{r}_N\|.$$

How to estimate?

$$\|\vec{r}_1-\vec{r}_0\| + \|\vec{r}_2-\vec{r}_1\| + \dots + \|\vec{r}_{N-1}-\vec{r}_N\|$$

$$= \|\vec{r}(t_1)-\vec{r}(t_0)\| + \dots + \|\vec{r}(t_{N-1})-\vec{r}(t_N)\|$$

assume differentiable

$$\approx \|\vec{r}\,'(t_0)(t_1-t_0)\| + \|\vec{r}\,'(t_1)(t_2-t_1)\|$$

$$+ \dots + \|\vec{r}\,'(t_{N-1})(t_N-t_{N-1})\|$$

$$= \|\vec{r}\,'(t_0)\|(t_1-t_0) + \|\vec{r}(t_1)\|(t_2-t_1) + \dots \|\vec{r}\,'(t_{N-1})\|(t_N-t_{N-1})$$

But this is just an approximation of an integral!

$$\xrightarrow{\max_i |t_i-t_{i-1}| \to 0} \int_{t_0}^{t_N} \|\vec{r}\,'(\tau)\|\, d\tau.$$

Thus, for curves such that $\vec{r}(t)$ has continuous derivative, we have

$$\boxed{\text{length of curve } \vec{r}(t)\ (a \le t \le b) = \int_a^b \|\vec{r}\,'(\tau)\|\, d\tau.}$$

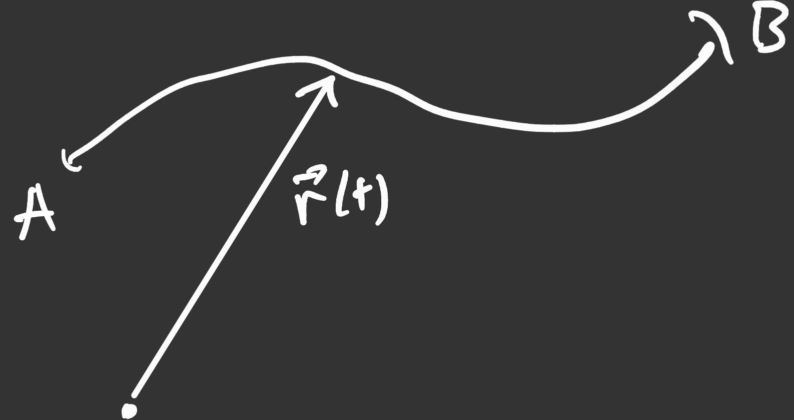
In coordinates: $\vec{r}(t) = (x(t), y(t), z(t))$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

The problem with this is that rarely can you evaluate this integral in terms of elementary functions, even for simple curves like ellipse or parabola!

However, in principle, the problem is solved.

Length parameter on a curve



A curve can be parametrized in many ways, e.g. replace t by t^2.

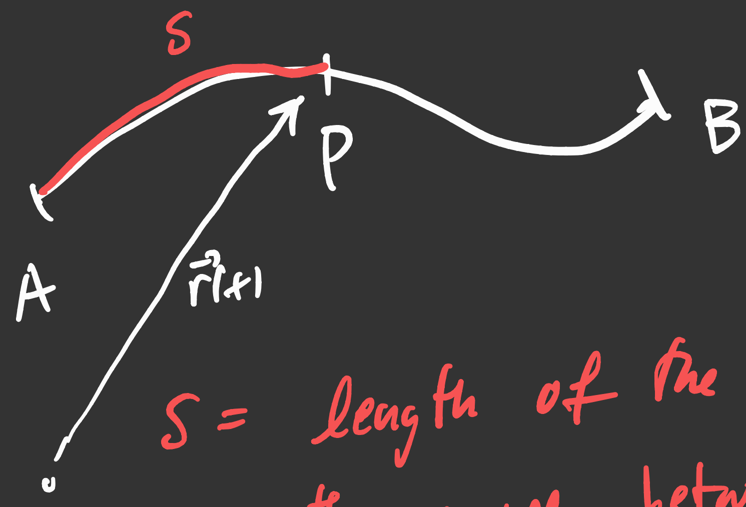
Sometimes, changing parametrization can help to evaluate, say, the length (a quantity independent of parametrization):

$$\int_a^b \|\vec{r}_1{}'(t)\|\,dt = \int_a^b \|\vec{r}_2{}'(h(t))\|\,\underbrace{|h'(t)|\,dt}_{dh(t)}$$

$$\vec{r}_1(t) = \vec{r}_2(h(t)) \qquad = \int_{h(a)}^{h(b)} \|\vec{r}_2{}'(h)\|\,dh.$$

$$\frac{d}{dt}\vec{r}_1(t) = \frac{dh}{dt}\frac{d\vec{r}_2}{dh}(h(t))$$

There is one privaledged parameterization
Arc-length parametrization



$s =$ length of the piece of the curve between points A and P.

If $\vec{r}(t)$ is given by another parameter, t, then s is a function of that parameter.

$\vec{r}(t) \quad a \le t \le b \qquad \vec{r}(a) = A \qquad \vec{r}(b) = B$

Define

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\, d\tau = \text{length of curve from point A to point } P = \vec{r}(t).$$

$s(t)$ is increasing, consider the inverse $t = t(s)$

Assume that s is known and

$\vec{r} = \vec{r}(s)$ is such that

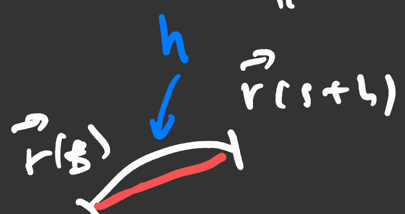
length of curve between $\vec{r}(s_1)$ and $\vec{r}(s_2)$ is $|s_2 - s_1|$.

Consider derivative

$$\vec{T}(s) = \frac{d\,\vec{r}(s)}{ds}$$

Note that $\|\vec{r}(s+h) - \vec{r}(s)\| \approx h$

$\uparrow$ length of segment if $\vec{r}$ is smooth



$\nwarrow$ angle between curve and line $\to 0$ as $h \to 0$.

Thus

$$\frac{\|\vec{r}(s+h) - \vec{r}(s)\|}{h} \to 1 \quad \text{as } h \to 0$$

Thus

$$\|\vec{T}(s)\| = \lim_{h\to 0} \frac{\|\vec{r}(s+h) - \vec{r}(s)\|}{h} = 1.$$

Consider $\dfrac{d\vec{T}}{ds} = \dfrac{d^2}{ds^2}\vec{r}(s).$

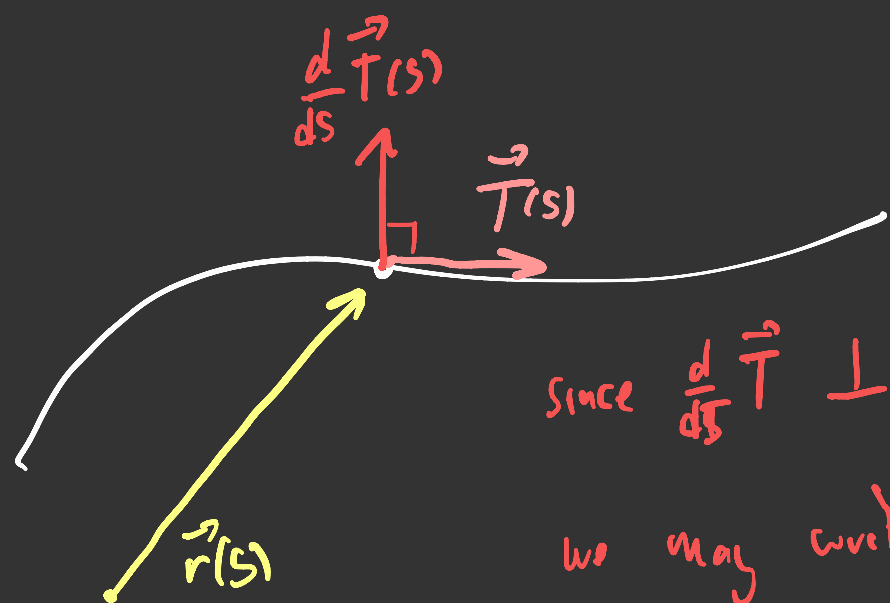
$$\boxed{\text{Theorem: } \quad \frac{d\vec{T}}{ds}\cdot\vec{T} = 0, \quad \text{i.e. } \frac{d}{ds}\vec{T} \perp \vec{T}.}$$

<u>Proof</u>: We know $\|\vec{T}(s)\| = 1$. Thus

$$1 = \|\vec{T}(s)\|^2 = \vec{T}(s)\cdot\vec{T}(s).$$

Thus
$$0 = \frac{d}{ds}\vec{T}(s)\cdot\vec{T}(s) + \vec{T}(s)\cdot\frac{d}{ds}\vec{T}(s)$$

$$\Rightarrow \quad 2\,\frac{d}{ds}\vec{T}(s)\cdot\vec{T}(s) = 0. \qquad \blacksquare$$



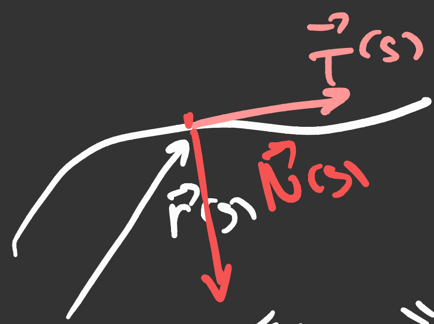
Since $\frac{d}{ds}\vec{T} \perp \vec{T}(s)$, we may write

only sensible when $\frac{d}{ds}\vec{T} \neq 0$.

$$\frac{d}{ds}\vec{T}(s) = \left\|\frac{d\vec{T}}{ds}\right\| \vec{N}(s)$$

with $\|\vec{N}(s)\| = 1$

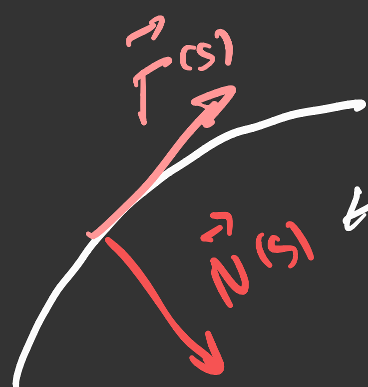
The vector $\vec{N}(s)$ is called the <u>principal normal</u> (vector) at the point $\vec{r}(s)$



"kappa" is the <u>curvature</u>.

$$\frac{d}{ds}\vec{T}(s) = \kappa(s)\,\vec{N}(s)$$

$\kappa(s)$ can be thought of as the <u>rotation rate</u> of the vector $\vec{T}(s)$ as one varies s (moves along the curve).

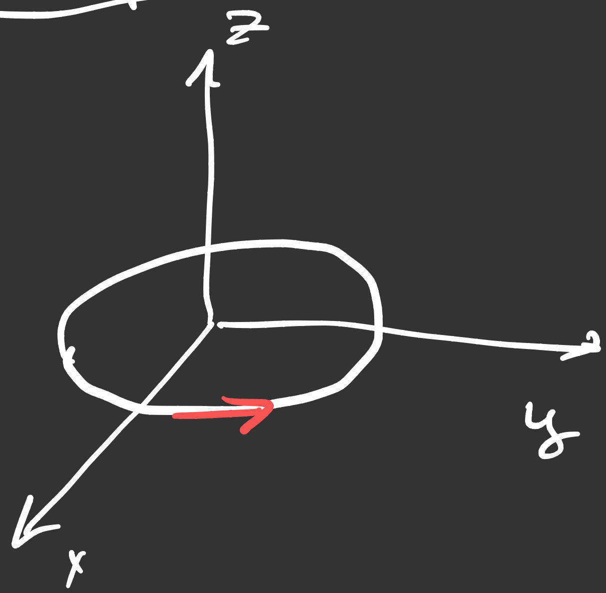


directed in direction that curve is concave

Example:

$$\vec{r}(t) = (\rho\cos t, \rho\sin t, 0)$$

$$\vec{r}\,'(t) = (-\rho\sin t, \rho\cos t, 0)$$



To find the curvature, we must change to the arc length parametrization.
"t" is not this parametrization.

To this end, we find

$$s(t) = \int_0^t \|\vec{r}\,'(t)\|\,dt = \int_0^t \sqrt{\rho^2\sin^2 t + \rho^2\cos^2 t}\,dt = \rho\int_0^t dt = t\rho.$$

$\uparrow$ length from 0 to t.

Thus $t(s) = \dfrac{s}{\rho}$ (inverse function).

Now parametrization

$$\vec{r}(s) = \vec{r}(t(s)) = \left(\rho\cos\left(\frac{s}{\rho}\right), \rho\sin\left(\frac{s}{\rho}\right), 0\right).$$

$$\frac{d\vec{r}(s)}{ds} = \left(-\sin\left(\frac{s}{\rho}\right), \cos\left(\frac{s}{\rho}\right), 0\right)$$

$$\left\| \frac{d\vec{r}(s)}{ds} \right\| = 1 \qquad \frac{d\vec{r}}{ds} = \vec{T}(s)$$

$$\underbrace{\frac{d}{ds}\vec{T}(s)}_{\vec{r}''(s)} = \left(-\frac{1}{\rho}\cos\left(\frac{s}{\rho}\right), -\frac{1}{\rho}\sin\left(\frac{s}{\rho}\right), 0\right)$$

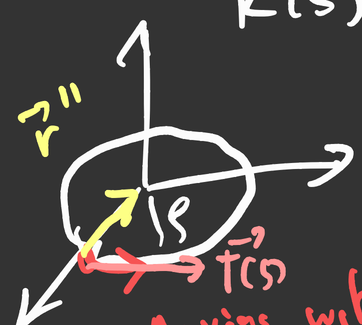
$$= \frac{1}{\rho}\vec{N}(s) = -\frac{1}{\rho^2}\vec{r}(s)$$

where $\vec{N}(s) = \left(-\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0\right)$

Note that $\frac{d\vec{T}}{ds} \cdot \vec{T} = 0$. ✓

And

$\kappa(s) = \frac{1}{\rho}$ by definition.



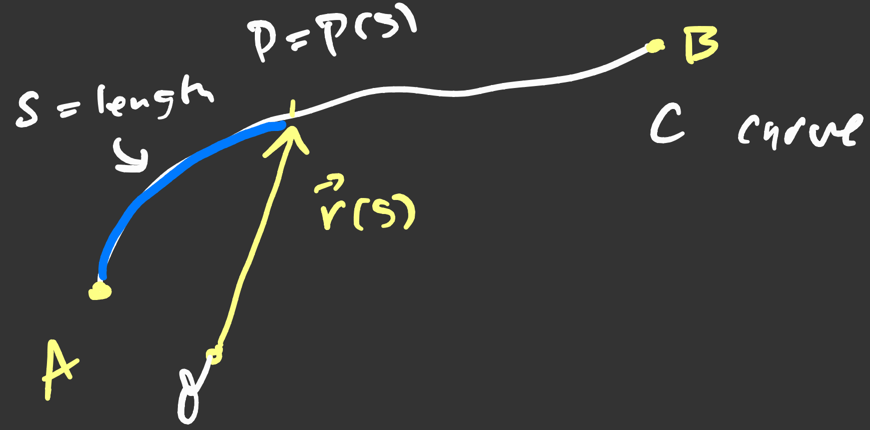
moving with units speed

$$\frac{d^2}{ds^2}\vec{r}(s) = -\frac{1}{\rho^2}\vec{r}(s)$$

curvature $\to 0$ as $\rho \to 0$, since a large circle looks approximately straight.

Generally, such straightforward computations not possible...

Decomposition of acceleration



$$\frac{d}{ds}\vec{r}(s) = \vec{T}(s), \qquad \|\vec{T}(s)\| = 1$$

(velocity along curve with speed 1)

$$\frac{d}{ds}\vec{T}(s) \cdot \vec{T}(s) = 0$$

orthogonal to $\vec{T}$

$$\frac{d}{ds}\vec{T}(s) = \kappa(s)\,\vec{N}(s)$$

curvature

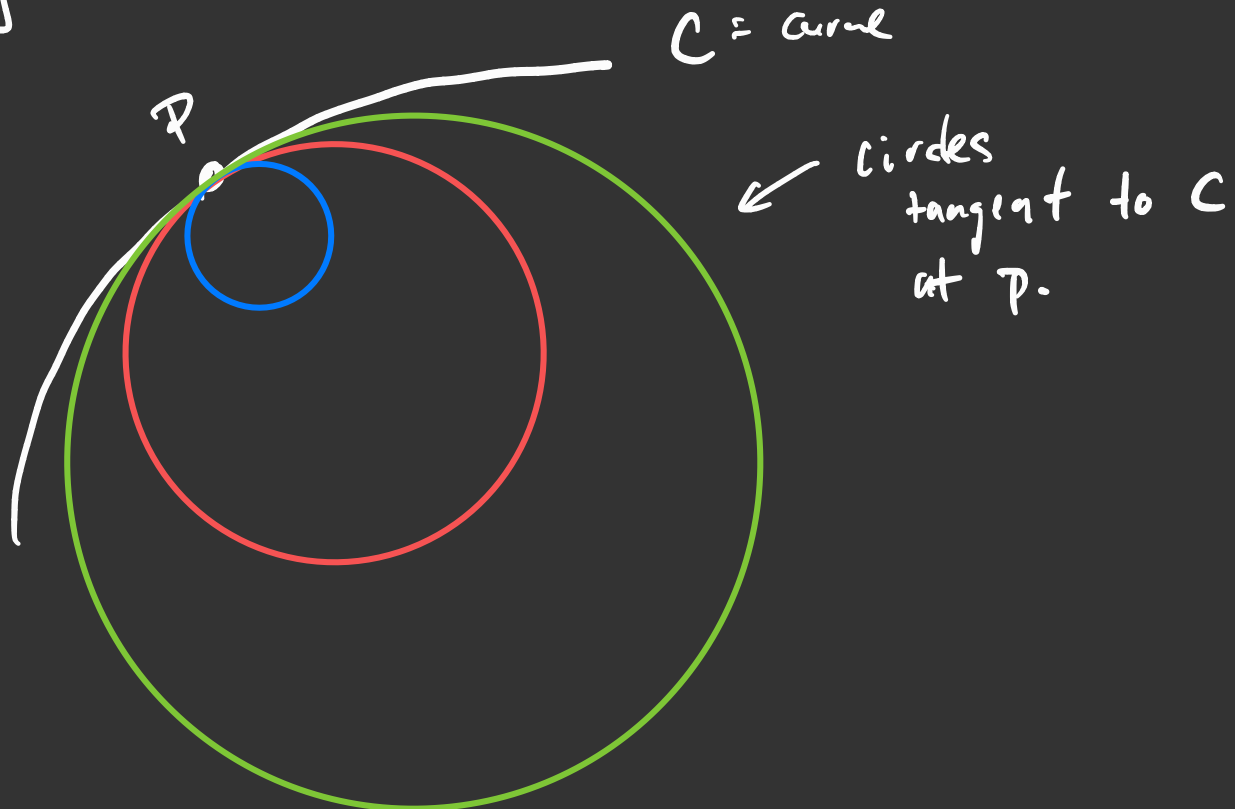
principal normal at $\vec{r}(s)$

$\|\vec{N}(s)\| = 1.$

$\kappa(s)$ shows how fast the direction of $\vec{T}(s)$ change as s grows.

$\rho(s) = \dfrac{1}{\kappa(s)}$ is <u>radius of curvature</u>

Meaning of radius of curvature:



each circle is an approximation to the curve C at p. For different circles, the quality of the approximation depends on the <u>radius</u> of the curve.

In 3d, it can be in different planes.

Among all the circles, there is one that fits the curve C the best.

(dist between curve and circle $\to 0$ as approach p. quality of approximation is the rate.) for plane curve, $(\text{dist})^2$.

Its radius is exactly ρ, radius of curvature.

Dynamically, as you move along the curve, you feel acceleration which feels like centrifugal acceleration which matches the centripetal force as if you move along circle of <u>best approximation</u>.

If you compare to the force for smaller circles, it is smaller. compared to larger circles it is larger.

This circle is called

<u>Osculating circle</u>

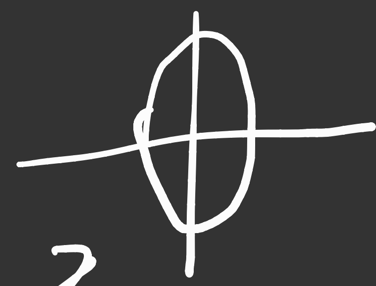
↑

means kissing. the circle is so close to the curve, it "kisses".

We return to this later.

Given a simple curve

$$x(t) = \cos t$$
$$y(t) = 2\sin t$$



ellipse.

Already, it's impossible to express the length of the curve in terms of elementary functions! So we must adopt a more robust approach.

Consider a generic curve

$$\vec{r}(t) = (x(t), y(t), z(t))$$

In principle, we can define the length

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(\tau)\|\, d\tau .$$

difficult to compute

$$\frac{ds}{dt} = \|\vec{r}\,'(t)\| = \|\vec{v}(t)\| =: v(t).$$

this is known.

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{dt}{ds}\frac{d\vec{r}}{dt} \qquad \text{(chain rule)}$$

$$= \frac{1}{v(t)}\frac{d\vec{r}}{dt} \qquad \left(\text{Since } \frac{ds}{dt} = v(t)\right)$$

Thus we can express unit tangent vector by velocity divided by abs. value of velocity

$$\frac{d\vec{T}}{ds} = \frac{1}{ds/dt}\frac{d\vec{T}}{dt} = \frac{1}{v(t)}\frac{d}{dt}\left(\frac{1}{v(t)}\frac{d\vec{r}}{dt}\right)$$

$$= \frac{1}{v}\left[-\frac{v'(t)}{|v(t)|^2}\frac{dr}{dt} + \frac{1}{v(t)}\frac{d^2}{dt^2}\vec{r}(t)\right] \qquad \text{(Leibnitz rule)}$$

$$= -\frac{v'}{v^3}\vec{v} + \frac{1}{v^2}\vec{a}$$ on one hand...

on other hand...

$$\frac{d\vec{T}}{ds} = k\,\vec{N}.$$

Thus we found

$$-\frac{v'}{v^3}\vec{v} + \frac{1}{v^2}\vec{a} = \kappa \vec{N}$$

Isolating $\vec{a}$:

$$\frac{1}{v^2}\vec{a} = \frac{v'}{v^3}\vec{v} + \kappa \vec{N}$$

or

$$\vec{a} = \frac{v'}{v}\vec{v} + v^2 \kappa \vec{N}$$

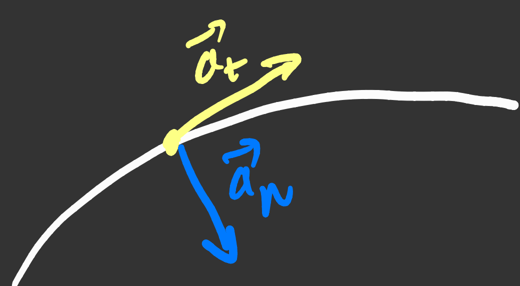
Note tha $\frac{\vec{v}}{v} = \frac{\vec{v}}{\|\vec{v}\|} = \vec{T}$. Thus

looks like centripetal force.

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{T} + \|\vec{v}\|^2 \kappa \vec{N}}$$

$(\text{speed})^2 \left(\begin{matrix}\text{inverse} \\ \text{radius}\end{matrix}\right)$. (unit normal)

↑ Scalar acceleration (comp of acceleration along the road



$$\vec{a} = \vec{a}_t + \vec{a}_N.$$

We must extract from here κ and $\vec{N}$.

$$\vec{a} = \frac{dv}{dt}\vec{T} + v^2 \kappa \vec{N}.$$

Crossing with $\vec{T}$, we find

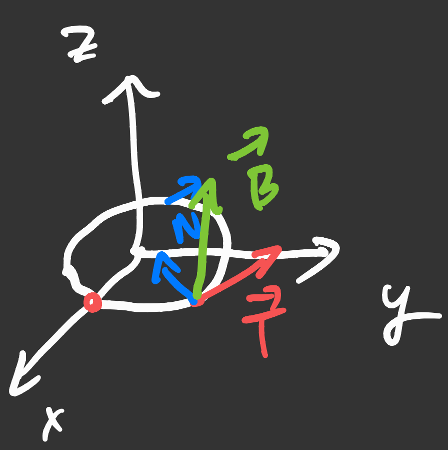
$$\vec{T} \times \vec{a} = \frac{dv}{dt} \cancel{\vec{T} \times \vec{T}}^{\,0} + v^2 \kappa \, \vec{T} \times \vec{N}$$

$$= v^2 \kappa \, \vec{T} \times \vec{N}$$

As $\vec{T}$ is a unit tangent, and $\vec{N}$ is unit normal, orthogonal to $\vec{T}$, the vector

$$\vec{B} = \vec{T} \times \vec{N} \qquad \text{(binormal)}$$

is orthogonal to both $\vec{T}$ and $\vec{N}$ and length 1.



$\vec{B} = (0,0,1) = \vec{k}$

for the unit circle

It is the same for all points on the circle.

Thus we found

$$\vec{T} \times \vec{a} = v^2 \kappa \vec{B}.$$

Now we find κ. Recall $\vec{V} = v\vec{T}$.

$$\vec{V} \times \vec{a} = v \vec{T} \times \vec{a} = v^3 \kappa \vec{B}.$$

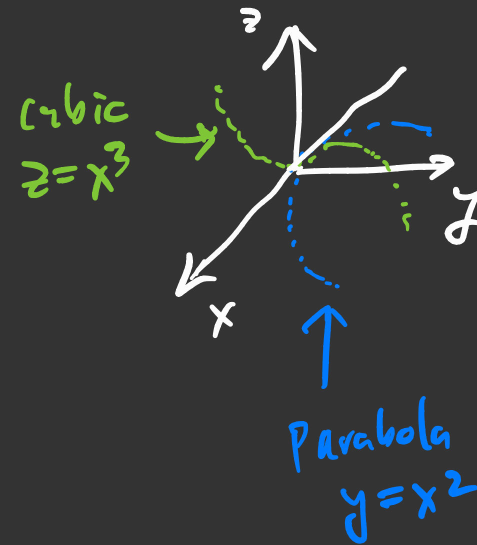
$$\|\vec{v} \times \vec{a}\| = v^3 \kappa \|\vec{B}\| = v^3 \kappa.$$

Since $\|\vec{B}\| = 1$. Thus

$$\boxed{\kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}} = \frac{\|\vec{v} \times \vec{a}\|}{(\vec{v} \cdot \vec{v})^{3/2}}$$

Now we can take any parametric curve, $\vec{r}(t)$, find $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$ and thereby find the curvature.

Example:



$$\vec{r}(t) = (t, t^2, t^3).$$

$$\vec{v}(t) = \vec{r}\,'(t) = (1, 2t, 3t^2)$$

$$\vec{a}(t) = \vec{r}\,''(t) = (0, 2, 6t)$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = i\left(12t^2 - 6t^2\right) - j(6t) + k(2)$$

$$= (6t^2, -6t, 2)$$

$$\vec{v} \cdot \vec{v} = 1 + 4t^2 + 9t^4$$

$$\kappa(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

Note $\kappa(0) = 2.$

How to find the principal normal $\vec{N}$?

Return to our formula:

$$\vec{a} = \frac{dv}{dt}\vec{T} + v^2 k \vec{N}.$$

We know that

$$\vec{v} \times \vec{a} = v^3 k \vec{B} \qquad \left(\text{since } \vec{v} \times \vec{T} = 0\right)$$

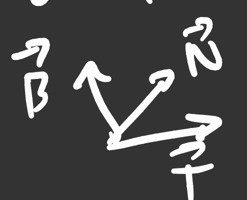
Now

$$(\vec{T}, \vec{N}, \vec{B})$$

form a right triple of orthogonal unit vectors

$$\|\vec{T}\| = \|\vec{N}\| = \|\vec{B}\| = 1$$

$$\vec{T}\cdot\vec{N} = 0 \quad \vec{T}\cdot\vec{B} = 0 \quad \vec{N}\cdot\vec{B} = 0$$



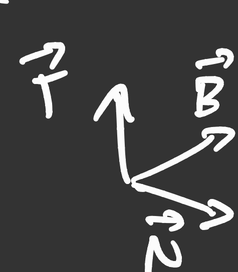
$$\vec{T} \times \vec{N} = \vec{B}$$

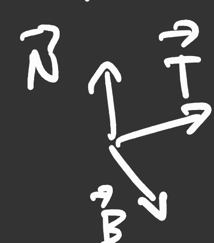
but also

$$\vec{B} \times \vec{T} = \vec{N}$$

$$\vec{N} \times \vec{B} = \vec{T}$$

since we took and turned them





Thus to find $\vec{N}$,

$$\vec{N} = \vec{B} \times \vec{T}$$

$$= \vec{B} \times \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \left(\frac{\vec{v} \times \vec{a}}{k\, v^3}\right) \times \frac{\vec{v}}{v} \qquad \left(\text{since } \vec{B} = \frac{1}{kv^3}\vec{v}\times\vec{a}\right)$$

$$= \frac{1}{kv^4}\left(\vec{v}\times\vec{a}\right)\times\vec{v}$$

$$= \frac{1}{v}\,\frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}\times\vec{v} \qquad \left(\text{using } k = \frac{\|\vec{v}\times\vec{a}\|}{v^3}\right)$$

Thus we found the formula

$$\boxed{\vec{N} = \frac{(\vec{v}\times\vec{a})\times\vec{v}}{\|(\vec{v}\times\vec{a})\times\vec{v}\|}}$$

$$\left(\begin{array}{l}\text{since } \vec{v} \perp \vec{v}\times\vec{a} \\ \text{so } \|\vec{v}\times(\vec{v}\times\vec{a})\| \\ \quad = \|\vec{v}\|\,\|\vec{v}\times\vec{a}\|\end{array}\right)$$