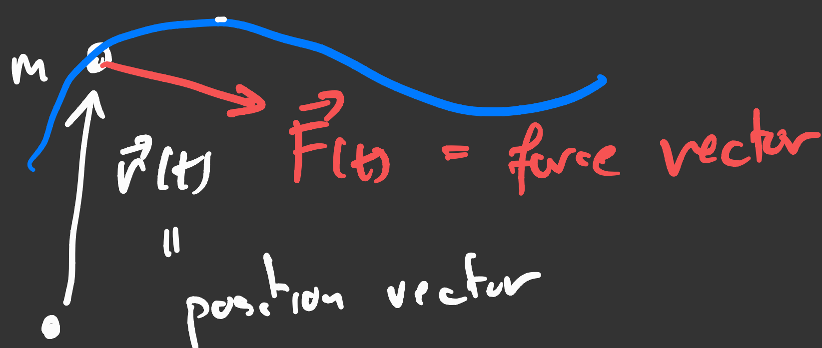
Some dynamical problems.

① 2d Newton's Law : mass m



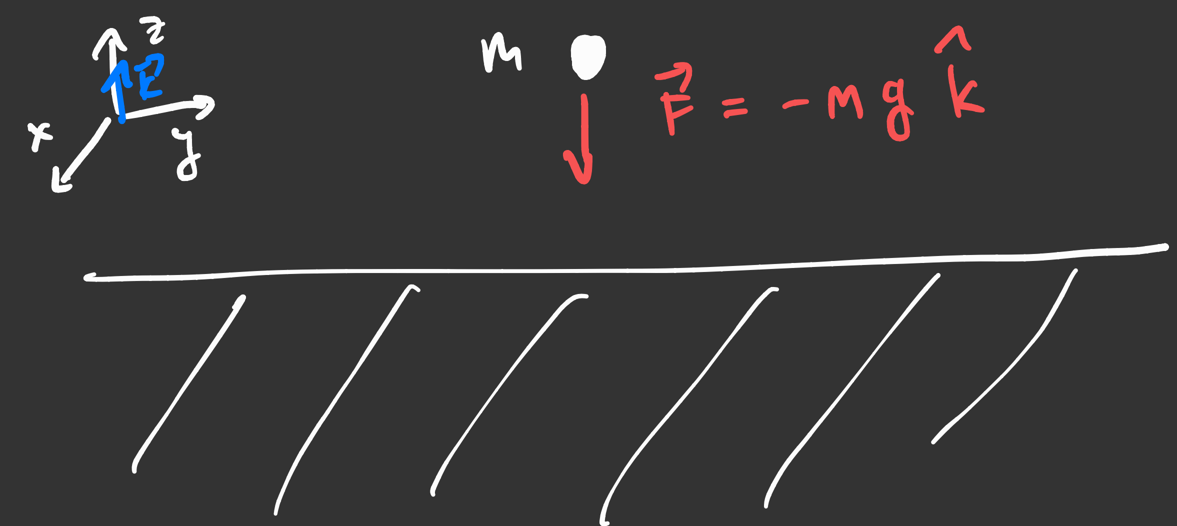
$$\boxed{\vec{F}(t) = m\,\vec{a}(t) = m\,\vec{r}\,''(t)}$$

acceleration.

Vectorial formulation of Newton's Law

Examples: Motion in gravitational field

Assume the earth is horizontal and flat, and the force of gravity acts vertically:



$$\vec{a} = \frac{\vec{F}}{m} = -g\hat{k} = (0, 0, -g)$$

$$\vec{a} = \frac{d}{dt}\vec{v}(t) \Rightarrow \quad \vec{v}(t) = \int \vec{a}(t)\, dt + \vec{v}_0$$

$$= -\int g\vec{k}\, dt + \vec{v}_0$$

$$= -gt\vec{k} + \vec{v}_0$$

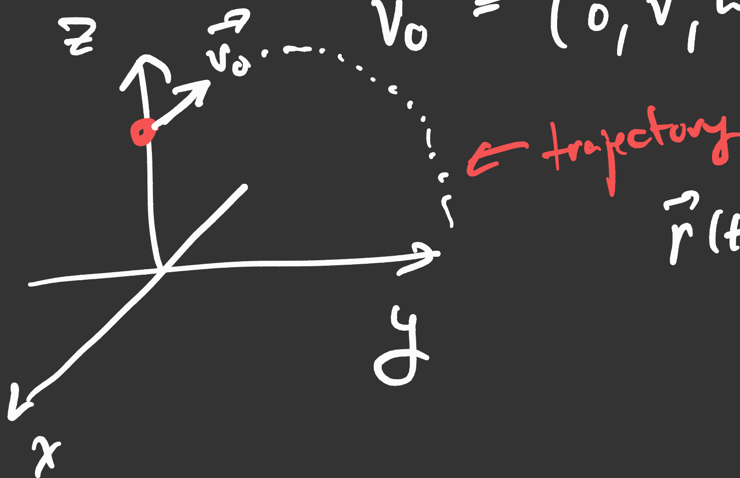
$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) \qquad \vec{r}(t) = \int \vec{v}(t)\, dt + \vec{C}$$

$$= -g\frac{t^2}{2}\vec{k} + t\vec{v}_0 + \vec{r}_0$$

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}_0 - g\frac{t^2}{2}\vec{k}}$$

Example: $\vec{r}_0 = (0, 0, H)$ height $= H$

$\vec{v}_0 = (0, v, w)$ ← motion in y-z plane



$$\vec{r}(t) = \left(0, 0, -g\frac{t^2}{2}\right) + (0, tv, tw)$$
$$+ (0, 0, h)$$
$$= \left(0, tv, h + tw - g\frac{t^2}{2}\right)$$

Lets find equation for the trajectory in the y-z plane, e.g. $z = z(y)$

Note $y = tv \Rightarrow t = \frac{y}{v}$

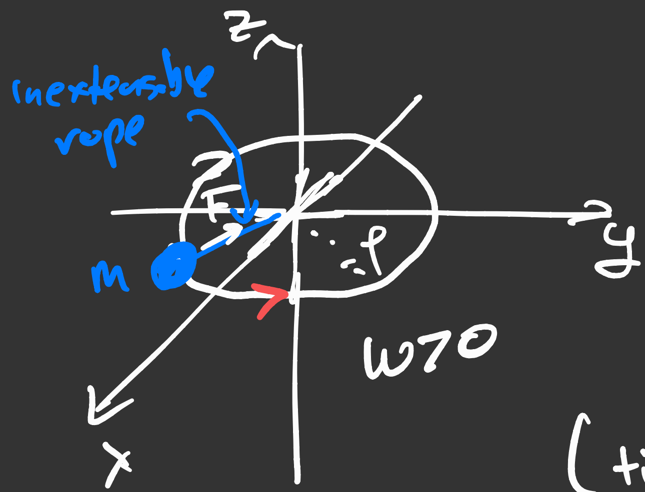
$$z = H + tw - g\frac{t^2}{2}$$
$$= H + \frac{w}{v}y - \frac{g}{2v^2}y^2$$

<u>upside-down parabola.</u>

2) Circular motion

$\rho > 0 \leftarrow$ radius

$w =$ frequency



$$\vec{r}(t) = (\rho \cos wt, \rho \sin wt, 0)$$

period $T = \frac{2\pi}{w}$

(time it takes to make one turn)

Rope pulls body to the center, so force $\vec{F}$ is directed in. Let's find $\vec{F}$:

$$\vec{F} = m\vec{a} = m\vec{r}''(t)$$

while

$$\vec{r}'(t) = w(-\rho \sin(wt), \rho \cos(wt), 0)$$

$$\vec{r}''(t) = -w^2 (\rho \sin(wt), \rho \cos(wt), 0)$$

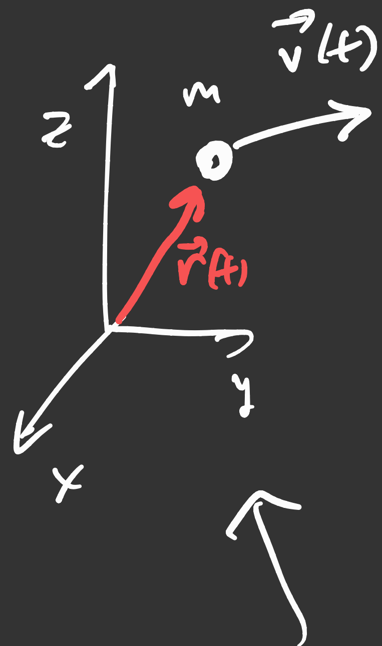
$$= -w^2 \vec{r}(t).$$

Thus

$$\boxed{\vec{F}(t) = -mw^2 \vec{r}(t)}$$

centrifugal force

3) Angular momentum and torque.



$$\boxed{\begin{array}{l}\text{Angular Momentum}\\[1em] \vec{M} = m\,\vec{r} \times \vec{v}\end{array}}$$

perpendicular to $\vec{r}, \vec{v}$
and directed so that
$(\vec{r}, \vec{v}, \vec{M})$ form a right triple

In this case, $\vec{M}$ points inside the page.

In previous example

$$\vec{r} = (\rho\cos(\omega t), \rho\sin(\omega t), 0)$$

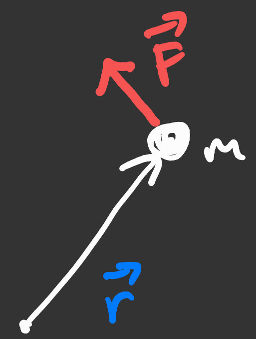
$$\vec{v} = (-\rho\sin(\omega t), \rho\cos(\omega t), 0)$$

$$\vec{M} = m\,\vec{r} \times \vec{v} = m \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \rho\cos\omega t & \rho\sin\omega t & 0 \\ -\rho\omega\sin\omega t & \rho\omega\cos\omega t & 0 \end{vmatrix}$$

$$= m \begin{vmatrix} \rho\cos(\omega t) & \rho\sin(\omega t) \\ -\rho\omega\sin(\omega t) & \rho\omega\cos(\omega t) \end{vmatrix} \vec{k}$$

$$= m\rho^2\omega\left(\cos^2(\omega t) + \sin^2(\omega t)\right)\vec{k} = m\rho^2\omega\vec{k}$$

Torque



Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$

If a force is acting on a body, it changes it's angular momentum. How does it change? Analogue of 2nd Newton's Law.

Theorem: If the mass is moving along a trajectory $\vec{r}(t)$ under the influence of force $\vec{F}$, then

$$\frac{d}{dt} \vec{M}(t) = \vec{\tau}(t)$$

where $\vec{M} = m \vec{r} \times \vec{v}$ is the angular momentum.

(Mass) x velocity = linear momentum.

$\vec{M}$ is angular momentum

Proof: $\vec{M} = \vec{r} \times (m\vec{v})$

$$\frac{d}{dt}\vec{M}(t) = m\vec{r}\,' \times \vec{v} + m\vec{r} \times \vec{v}\,' \qquad \text{(Leibnitz)}$$

$$= \cancel{m\vec{v} \times \vec{v}}\ 0 + m\vec{r} \times \vec{r}\,''$$

(since $\vec{a} \times \vec{a} = 0$ for all $\vec{a}$)

Pf: change places $\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$

$$= \vec{r} \times (m\vec{r}\,'')$$

$$= \vec{r} \times \vec{F}.$$

$$= \vec{\tau}.$$

∎

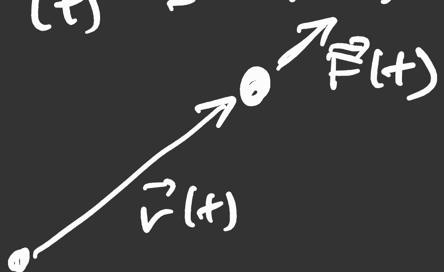
Remarkable particular case:

If $\vec{\tau}(t) = 0$, then $\vec{M}(t) = \text{const}$!

$\left(\frac{d}{dt}\vec{M}(t) = \vec{0} \quad \text{so} \quad \vec{M}(t) = \vec{M}_0 \right)$

What's an example of this? <u>Central force</u>

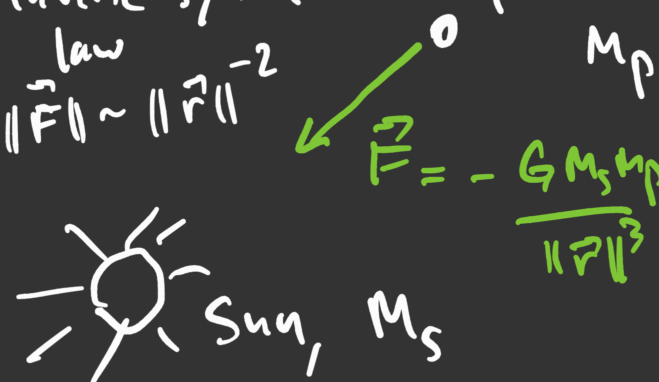
$$\vec{F}(t) = k(t)\,\vec{r}(t)$$



$\tau = \vec{r} \times \vec{F}$
$= k\,\vec{r} \times \vec{r} = 0.$

<u>Example:</u>

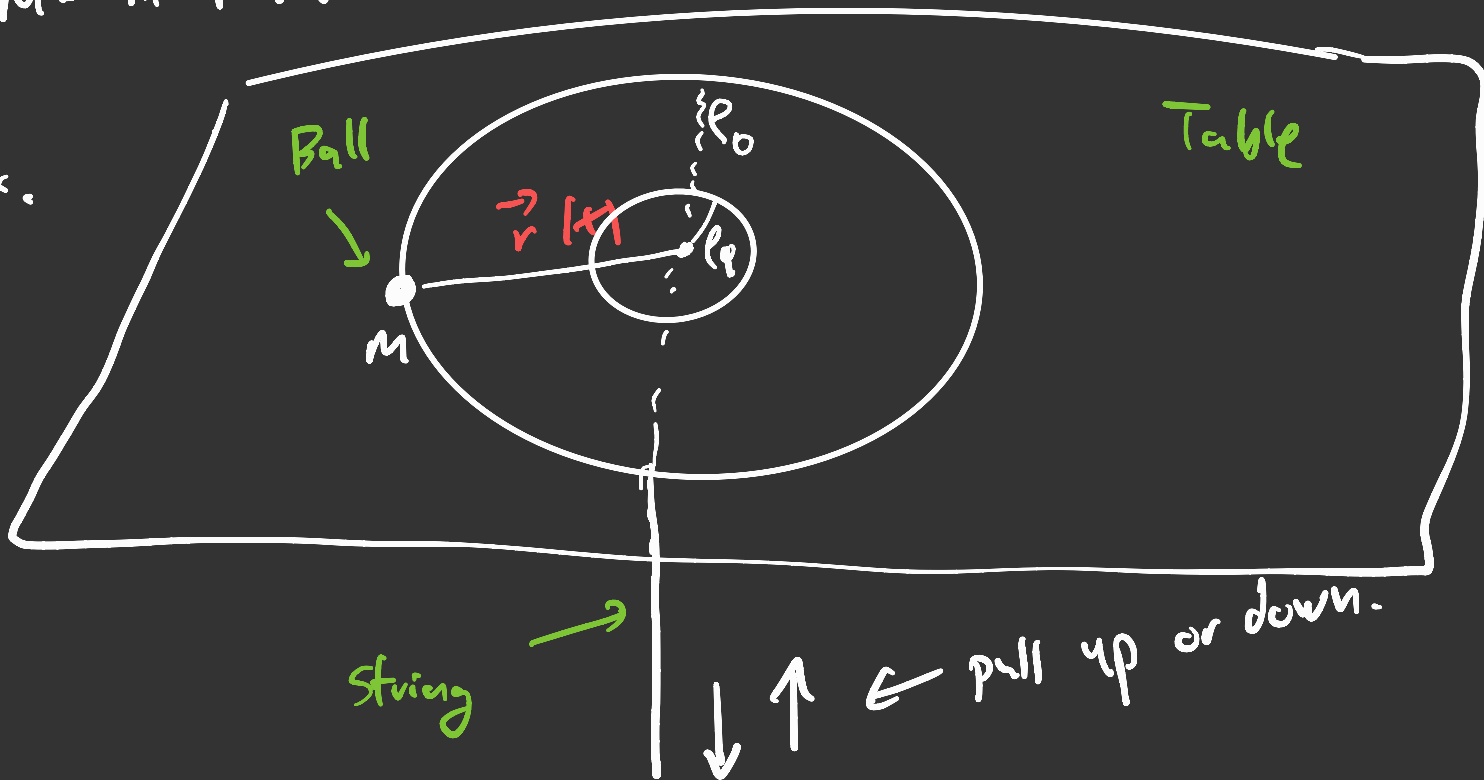
Inverse square law
$\|\vec{F}\| \sim \|\vec{r}\|^{-2}$



Thus, if $\vec{F}$ is a central force, then

$\vec{M} = m\,\vec{r}\times\vec{v}$ is constant.

Ex.



Thus we can prescribe $\|\vec{r}(t)\|$ arbitrarily.

$$\vec{M} = m\,\vec{r}(t)\times\vec{v}(t) = \text{constant}$$

$$\vec{M} = m\rho^2\omega\,\vec{k} \qquad \text{angular momentum.}$$

$$\vec{M}_0 = m\rho_0^2\omega_0\,\vec{k} \qquad \vec{M}_1 = m\rho_1\,\omega_1^2\,\vec{k}$$

$$\Rightarrow \quad \frac{\omega_1}{\omega_0} = \frac{\rho_0^2}{\rho_1^2} \iff \boxed{\omega_1 = \frac{\rho_0^2}{\rho_1^2}\,\omega_0}$$

If you reduce length by half, $\rho_1 = \frac{1}{2}\rho_0$

then $\omega_1 = 4\omega_0$ and $\|v_1\| = 2\|v_0\|$.