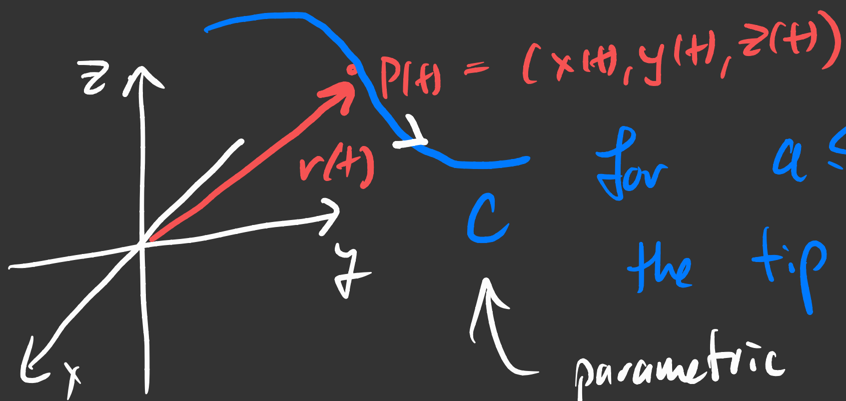
Vector functions: function whose values are vectors

$$\vec{r}(t) = (x(t), y(t), z(t))$$



for $a \le t \le b$, the tip sweeps out the curve C

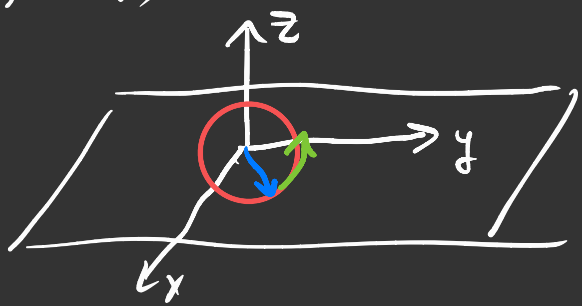
parametric curve oriented (directed) by the direction of motion of the tip.

Example: $r(t) = r_0 + t\vec{u}$. : Motion along straight line with constant speed.

Example: $r(t) = (\cos t, \sin t, 0)$

- planar motion
- note that $(x(t))^2 + (y(t))^2 = (\cos t)^2 + (\sin t)^2 = 1$

thus this motion remains on unit circle



motion is counter clockwise observed from tip of $\vec{K}$.

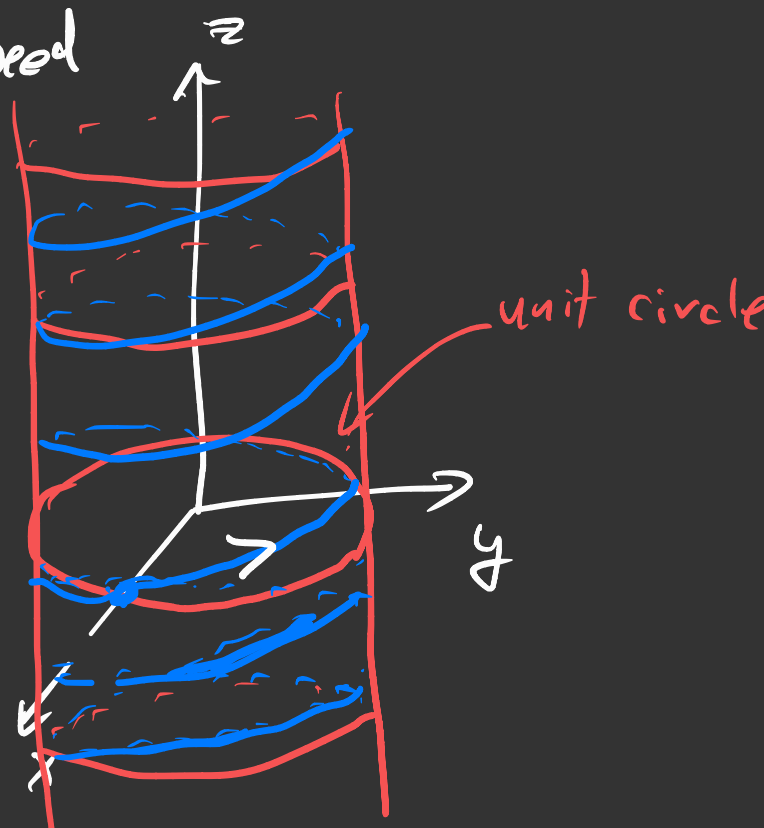
Example: $r(t) = (\cos(t), \sin(t), t)$

- $(x(t))^2 + (y(t))^2 = 1$

Moving up in z at const speed

$\boxed{\text{Helix}}$

ccw motion with respect to $\vec{k}$.



Differentiation

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$$

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Properties of derivative

1) Linearity:

$$(r_1(t) + r_2(t))' = r_1'(t) + r_2'(t)$$

2) $k \in \mathbb{R}$, $(k r(t))' = k\, r'(t)$.

<u>Proof</u>: Limit definition.

3) Leibnitz Rules

Scalar a) $(f(t)\, g(t))' = f'(t)\, g(t) + f(t)\, g'(t)$

Scalar multiple b) $(k(t)\, \vec{r}(t))' = k'(t)\, \vec{r}(t) + k(t)\, \vec{r}\,'(t)$

dot product c) $(\vec{r}_1(t) \cdot \vec{r}_2(t)) = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$

cross product d) $(\vec{r}_1(t) \times \vec{r}_2(t))' = \vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)$

all are proved in same way. Let's prove (d).

Write $(\vec{r}_1(t) \times \vec{r}_2(t))' = \lim\limits_{h \to 0} \dfrac{\vec{r}_1(t+h) \times \vec{r}_2(t+h) - \vec{r}_1(t) \times \vec{r}_2(t)}{h}$

$$= \lim_{h \to 0} \frac{(\vec{r}_1(t+h) - \vec{r}_1(t)) \times \vec{r}_2(t+h) - \vec{r}_1(t) \times (\vec{r}_2(t) - \vec{r}_2(t+h))}{h}$$

$$= \lim_{h \to 0} \left[\left(\frac{\vec{r}_1(t+h) - \vec{r}_1(t)}{h} \right) \times \vec{r}_2(t+h) + \vec{r}_1(t) \times \left(\frac{\vec{r}_2(t+h) - \vec{r}_2(t)}{h} \right) \right]$$

3) Chain Rule

$\vec{r}(t)$ and $t = t(s)$

$$\left(\vec{r}(t(s))\right)' = t'(s)\,\vec{r}\,'(t(s))$$

Example: $r(t) = (\cos(t), \sin(t), t)$

$t(s) = s^2$

$$r(t(s)) = (\cos(s^2), \sin(s^2), s^2)$$

$$\frac{d}{ds} r(t(s)) = \frac{d}{ds}\left(\cos(s^2), \sin(s^2), s^2\right)$$

$$= \left(-\sin(s^2)\cdot(2s), \cos(s^2)\cdot(2s), (2s)\right)$$

$$= 2s\,(-\sin(s^2), \cos(s^2), 1)$$

$$= \frac{dt}{ds} \cdot \frac{d\vec{r}}{dt}(t(s))$$

✓

$\frac{dt}{ds} = 2s.$

4) Higher derivatives

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$$

$$\vec{r}\,''(t) = (x''(t), y''(t), z''(t))$$

$$\vdots$$

$$\vec{r}^{\,(n)}(t) = (x^{(n)}(t), y^{(n)}(t), z^{(n)}(t))$$

<u>Example</u>: $(\vec{r}_1(t) \cdot \vec{r}_2(t))' = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$

$$(\vec{r}_1(t) \cdot \vec{r}_2(t))'' = \vec{r}_1\,''(t) \cdot \vec{r}_2(t) + 2\,\vec{r}_1\,'(t) \cdot \vec{r}_2\,'(t) + \vec{r}_1(t) \cdot \vec{r}_2\,''(t)$$

Rate of change of volume of a parallelepiped.

Consider

$$\vec{a}(t),\ \vec{b}(t) \text{ and } \vec{c}(t).$$

$$V(\vec{a}(t), \vec{b}(t), \vec{c}(t)) = (\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}(t)$$

$$\frac{d}{dt} V(\vec{a}(t), \vec{b}(t), \vec{c}(t)) = \frac{d}{dt}\Big((\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}(t) \Big)$$

$$= (\vec{a}\,'(t) \times \vec{b}(t)) \cdot \vec{c}(t)$$

$$+ (\vec{a}(t) \times \vec{b}\,'(t)) \cdot \vec{c}(t)$$

$$+ (\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}\,'(t)$$

$$= V(\vec{a}\,'(t), \vec{b}(t), \vec{c}(t))$$

$$+ V(\vec{a}(t), \vec{b}\,'(t), \vec{c}(t))$$

$$+ V(\vec{a}(t), \vec{b}(t), \vec{c}\,'(t))$$

Integral of vector function

Primative:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Def: $\vec{R}(t)$ is called the primative if $\vec{R}'(t) = \vec{r}(t)$.

$$\vec{R}(t) = (\vec{u}(t), \vec{v}(t), \vec{w}(t))$$

$$\vec{R}(t) = (\vec{u}'(t), \vec{v}'(t), \vec{w}'(t))$$

Thus

$$\vec{u}'(t) = \vec{x}(t)$$
$$\vec{v}'(t) = \vec{y}(t)$$
$$\vec{w}'(t) = \vec{z}(t)$$

Then

$$\vec{u}(t) = \int \vec{x}(t)\,dt + C_1$$
$$\vec{v}(t) = \int \vec{y}(t)\,dt + C_2$$
$$\vec{w}(t) = \int \vec{w}(t)\,dt + C_2$$

or

$$\vec{R}(t) = \int \vec{r}(t)\,dt + \vec{C}.$$

Ex: $\vec{r}(t) = (\cos t, \sin t, t)$

$$\vec{R}(t) = \left(\int \cos t \, dt + C_1, \int \sin t \, dt + C_2, \int t \, dt + C_3 \right)$$

$$= \left(\sin t, -\cos t, \frac{t^2}{2} \right) + (C_1, C_2, C_3)$$

Definite integral:

$$\int_a^b \vec{r}(t) \, dt = \vec{R}(b) - \vec{R}(a)$$

$$= \left(\int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right)$$

Ex: $\displaystyle\int_0^{\pi/4} (\cos t, \sin t, t) \, dt$

$$= \left(\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, \frac{1}{2}\left(\frac{\pi}{4}\right)^2 \right)$$

since $\displaystyle\int_0^{\pi/4} \cos t \, dt = \sin\left(\frac{\pi}{4}\right) - \sin(0) = \frac{1}{\sqrt{2}}$

$$\int_0^{\pi/4} \sin t \, dt = -\cos\left(\frac{\pi}{4}\right) + \cos(0) = 1 - \frac{1}{\sqrt{2}}$$