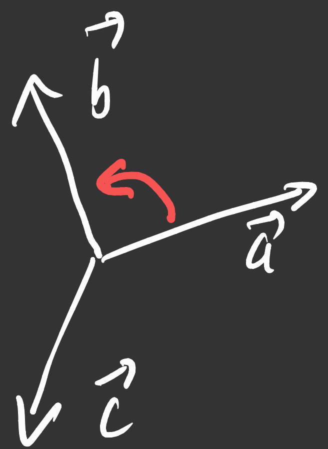
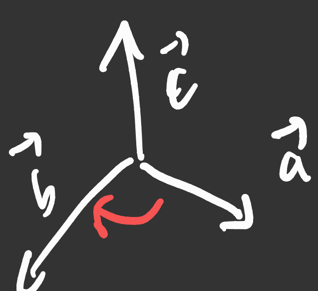
Signed volume of a parallelepiped

Consider three vectors $\vec{a}, \vec{b}, \vec{c}$ not lying in the same plane.



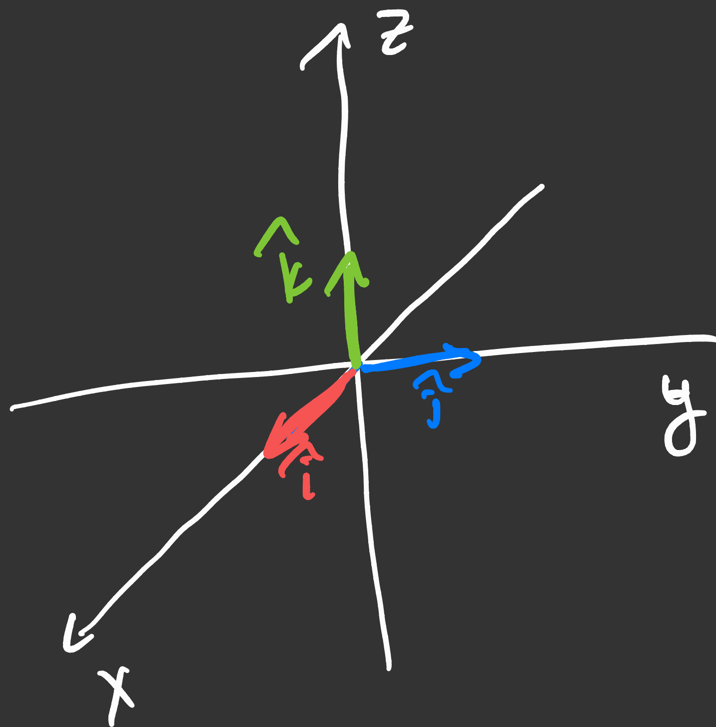
Def: These vectors form a right triple

If you look at $\vec{a}, \vec{b}$ from the view of vector $\vec{c}$, and you have to rotate $\vec{a}$ counterclockwise to align with $\vec{b}$, then $(\vec{a}, \vec{b}, \vec{c})$ is a right triple.

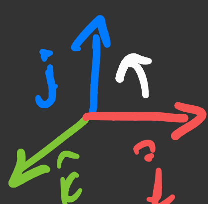


← if clockwise, then $(\vec{a}, \vec{b}, \vec{c})$ is a left triple.

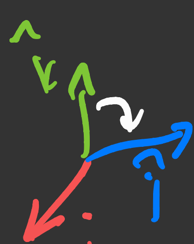
E.g.



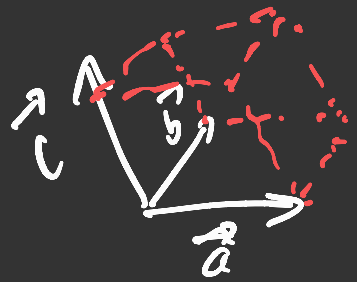
$(\hat{i}, \hat{j}, \hat{k})$ form a right triple



$(\hat{k}, \hat{j}, \hat{i})$



form a left triple



Π = parallelepiped

Def: The signed volume of Π defined by vectors $\vec{a}, \vec{b}, \vec{c}$ is

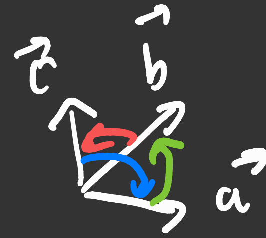
$$V(\vec{a},\vec{b},\vec{c}) = \begin{cases} \text{usual volume of } \Pi & \text{if } (\vec{a},\vec{b},\vec{c}) \text{ is right triple} \\ -\text{usual volume} & \text{if left triple} \end{cases}$$

What properties?

1) $$V(\vec{a},\vec{b},\vec{c}) = -V(\vec{b},\vec{a},\vec{c})$$
$$= -V(\vec{a},\vec{c},\vec{b})$$
$$= -V(\vec{c},\vec{b},\vec{a})$$

Since $(\vec{a},\vec{b},\vec{c})$ is right triple, then $(\vec{b},\vec{a},\vec{c})$, $(\vec{a},\vec{c},\vec{b})$ and $(\vec{c},\vec{b},\vec{a})$ are left triples.

On other hand,



$$V(\vec{a}, \vec{b}, \vec{c}) = V(\vec{c}, \vec{a}, \vec{b})$$
$$= V(\vec{b}, \vec{c}, \vec{a})$$

↑ all right (or left) triples.

There are 6 total permutations of $(\vec{a}, \vec{b}, \vec{c})$. 3 are right and 3 are left.

Volume is always the same.

3 cases signed volume is positive

3 cases " " is negative

2) $V(k\vec{a}, \vec{b}, \vec{c}) = k\, V(\vec{a}, \vec{b}, \vec{c})$

$V(\vec{a}, k\vec{b}, \vec{c}) = k\, V(\vec{a}, \vec{b}, \vec{c})$

$V(\vec{a}, \vec{b}, k\vec{c}) = k\, V(\vec{a}, \vec{b}, \vec{c})$.

3) $V(\vec{a}+\vec{d}, \vec{b}, \vec{c}) = V(\vec{a}, \vec{b}, \vec{c}) + V(\vec{d}, \vec{b}, \vec{c})$

Proved in the same way as in 2D.
draw the two parallelpipeds $\vec{a}, \vec{b}, \vec{c}$
and $\vec{d}, \vec{b}, \vec{c}$.

Their difference is some prism.

Same property for other slots:

$V(\vec{a}, \vec{b}+\vec{d}, \vec{c}) = V(\vec{a}, \vec{b}, \vec{c}) + V(\vec{a}, \vec{d}, \vec{c})$

$V(\vec{a}, \vec{b}, \vec{c}+\vec{d}) = V(\vec{a}, \vec{b}, \vec{c}) + V(\vec{a}, \vec{b}, \vec{d})$

4) $V(\hat{i},\hat{j},\hat{k}) = 1$ (unit cube)

$V(\hat{k},\hat{i},\hat{j}) = 1$ right triple

$V(\hat{j},\hat{k},\hat{i}) = 1$

$V(\hat{j},\hat{i},\hat{k}) = -1$ left triples

$V(\hat{k},\hat{j},\hat{i}) = -1$

$V(\hat{i},\hat{k},\hat{j}) = -1$

degenerate

If any two are same, e.g. $V(\hat{i},\hat{i},\hat{i}) = 0$
volume is zero

$\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ $= b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = (c_1, c_2, c_3) = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$V(\vec{a},\vec{b},\vec{c}) =$

$V(a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$$V(\vec{a}, \vec{b}, \vec{c}) =$$

$$V(a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= 27 \text{ terms} \quad (3 \times 3 \times 3)$$

$$= V(a_1\hat{i}, b_1\hat{i}, c_1\hat{i}) +$$

$$V(a_1\hat{i}, b_1\hat{j}, c_1\hat{i}) +$$

$$\vdots$$

$$V(a_3\hat{k}, b_3\hat{k}, c_3\hat{k})$$

$$= a_1b_1c_1 \cancel{V(\hat{i}, \hat{i}, \hat{i})}^{0} + \dots + a_3b_3c_3 \cancel{V(\hat{k}, \hat{k}, \hat{k})}^{0}$$

$$= a_1b_2c_3 \cancel{V(\hat{i}, \hat{j}, \hat{k})}^{1} + a_1b_3c_2 \cancel{V(\hat{i}, \hat{k}, \hat{j})}^{-1}$$

$$+ a_2b_1c_3 \cancel{V(\hat{j}, \hat{i}, \hat{k})}^{-1} + a_2b_3c_1 \cancel{V(\hat{j}, \hat{k}, \hat{i})}^{-1}$$

$$+ a_3b_1c_2 \cancel{V(\hat{k}, \hat{i}, \hat{j})}^{1} + a_3b_2c_1 \cancel{V(\hat{k}, \hat{j}, \hat{i})}$$

$$= a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$V(\vec{a}, \vec{b}, \vec{c})$

$$= a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

This funny formula turns out simply to be the determinant!

$$\boxed{V(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

Let us write

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

expand determinant relative to last row

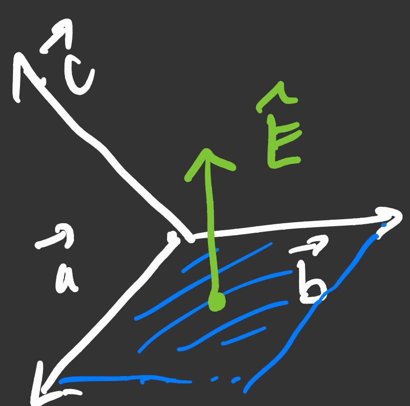
Let us define a vector

$$\vec{F}(\vec{a},\vec{b}) = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

With this, we have the formula

$$V(\vec{a},\vec{b},\vec{c}) = \vec{F}(\vec{a},\vec{b}) \cdot \vec{c}$$

But what is the volume geometrically?

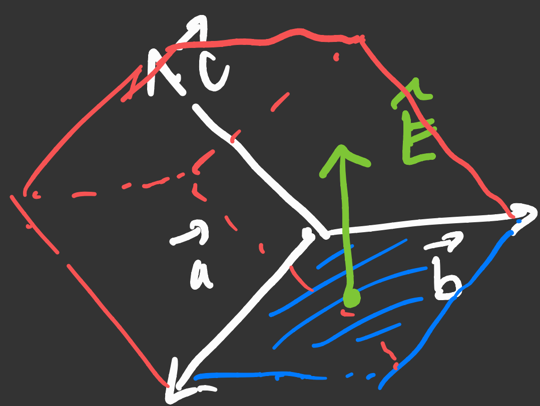


$\hat{E}$ is vector normal to the plane of parallelogram $(\vec{a},\vec{b})$, call it Π.

Moreover

- $\|\hat{E}\| = |\text{area}(\Pi)|$
- direction is so that $(\vec{a},\vec{b},\hat{E})$ form a right triple

Now, we see that

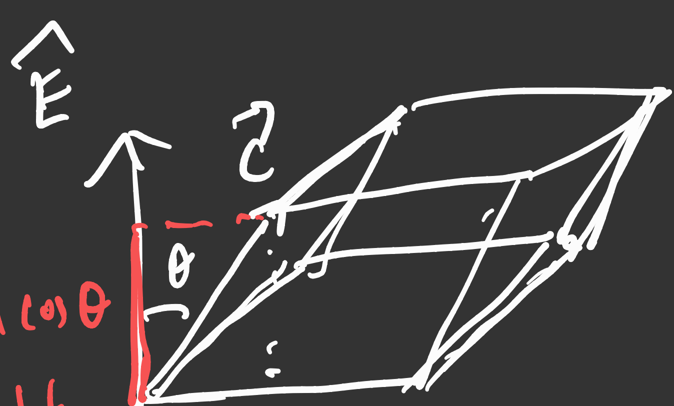


Parallelpiped defined by $(\vec{a}, \vec{b}, \vec{c})$

$$V(\vec{a}, \vec{b}, \vec{c}) = (\text{area of base}) \cdot (\text{height})$$

$$= \vec{E} \cdot \vec{c}$$

$$= \|\vec{E}\| \, \|\vec{c}\| \cos\theta$$



Thus

$$V(\vec{a},\vec{b},\vec{c}) = \vec{F}\cdot\vec{c} = \vec{E}\cdot\vec{c}$$

This holds for any $\vec{c}$. Thus

$$\boxed{\vec{E} = \vec{F}}$$

$\vec{E}$ is a vector orthogonal to plane spanned by $\vec{a},\vec{b}$ whose length is equal to the area of the parallelogram.

We shall name this quantity:

$$\vec{F}(\vec{a},\vec{b}) = \vec{a}\times\vec{b}$$

<u>Cross product</u>

The formula is

$$\vec{a}\times\vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

(symbolically)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$V(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Example: $\vec{a} = (1,2,3)$, $\vec{b} = (3,2,-1)$, $\vec{c} = (0,2,-3)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}\hat{k}$$

$$= (-2-6)\hat{i} - (-1-9)\hat{j} + (2-6)\hat{k}$$

$$= -8i + 10\hat{j} - 4k = (-8,10,-4)$$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (-8,10,-4)\cdot(0,2,-3) = 20+12 = 32$$

Since $V(\vec{a},\vec{b},\vec{c}) > 0$, $(\vec{a},\vec{b},\vec{c})$ is right triple.