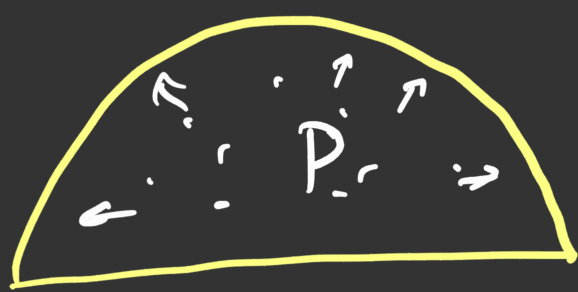
Further Applications of Divergence Theorem

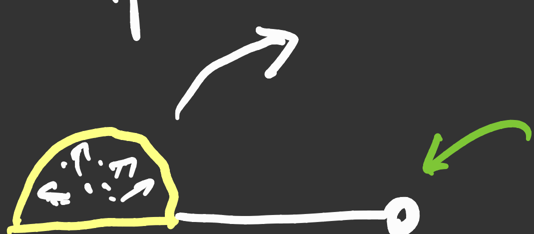
"Perpetuum mobile": device which violates conservation of energy

Consider a thin, hemispherical shell



Pump in air, inside there is air pressure P.

The flat part has less surface area than the round part. Thus the force up should be more than the force down. As a result which should push the device up — this is perpetuum mobile:



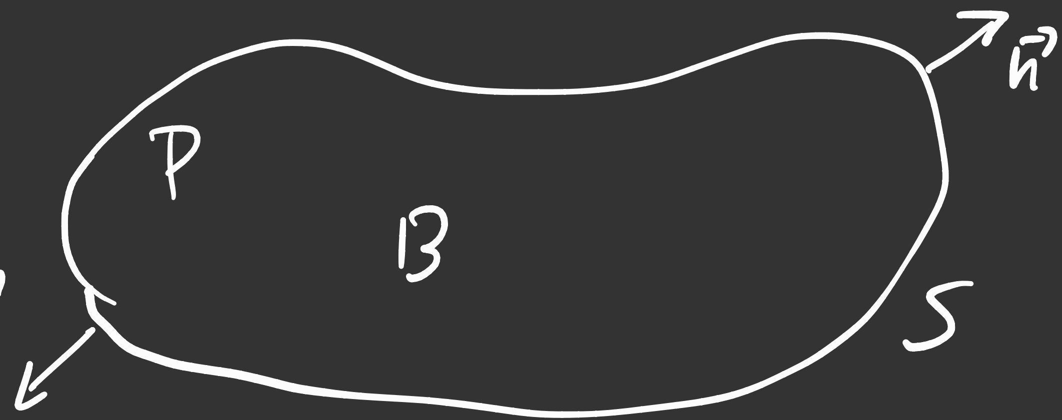
fix it here.

by rotating it can produce mechanical work

By law of conservation of energy, this should not work.

What is wrong?

Consider any shell S bounding domain B with internal pressure P.



What is the total force $\vec{F}$ on the shell by the pressure P.

The force $\vec{F}$ is

$$\vec{F} = \iint_S P \vec{n} \, dS = P \iint_S \vec{n} \, dS$$

Since P is constant.

Note

$$\vec{F} = P \iint_S \vec{n}\, dS$$

$$\vec{i} \cdot \vec{F} = P \iint_S \vec{n} \cdot \vec{i}\, dS \qquad \vec{i} = (1,0,0)$$

$$= P \iiint_B \operatorname{div} \vec{i}\, dV = 0 \qquad \text{as } \operatorname{div} \vec{i} = 0$$

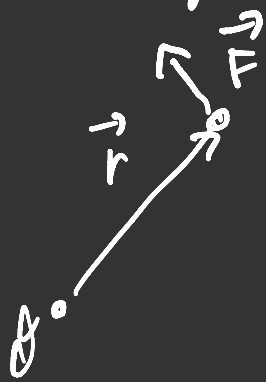
Also

$$\vec{j} \cdot \vec{F} = 0 \quad \text{and} \quad \vec{k} \cdot \vec{F} = 0.$$

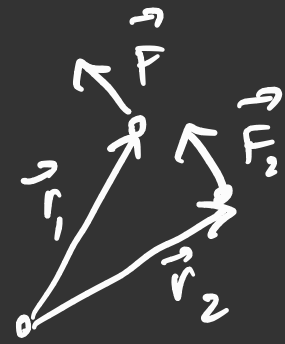
Thus $\vec{F} = 0$, there is no force!

Maybe there exists a total torque, so that the construction will rotate?

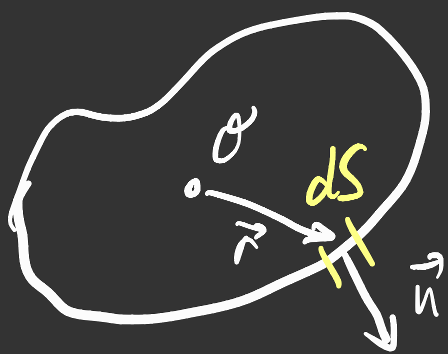
The torque is



$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\vec{\tau}_{tot} = \sum_i \vec{r}_i \times \vec{F}_i$$



$$\vec{F} = P\,\vec{n}\,dS$$

$$\vec{\tau} = P\,\vec{r} \times \vec{n}\,dS$$

Total torque
$$\vec{\tau} = P \iint_S \vec{r} \times \vec{n}\,dS$$

$$\vec{r} = (x, y, z)$$
$$\vec{n} = (n_x, n_y, n_z)$$

$$\vec{r} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ n_x & n_y & n_z \end{vmatrix}$$
$$= (yn_z - zn_y)\,\vec{i} + (zn_x - xn_z)\,\vec{j} + (xn_y - yn_x)\,\vec{k}$$

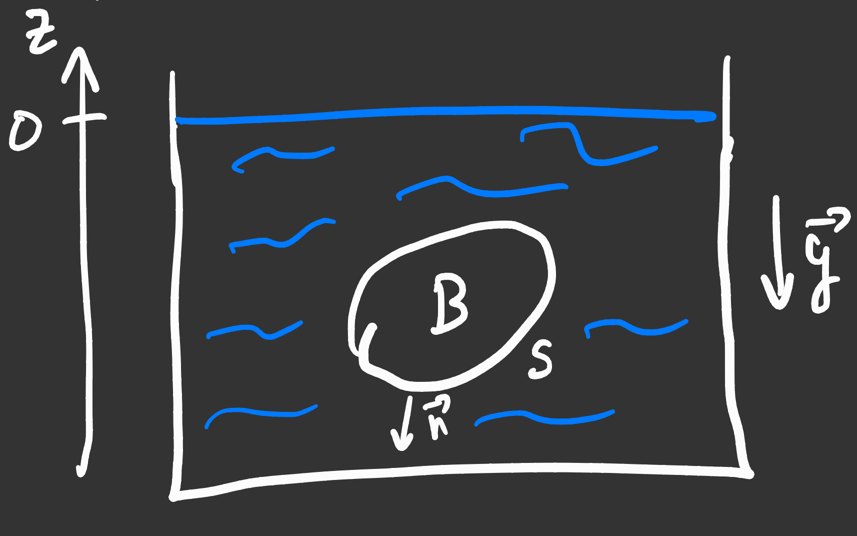
$$\vec{\tau} \cdot \vec{i} = P \iint_S (\vec{r} \times \vec{n}) \cdot \vec{i}\,dS$$
$$= P \iint_S (yn_z - zn_y)\,dS$$
$$= P \iint_S \underbrace{(0, -z, y)}_{\vec{F}_1} \cdot (n_x, n_y, n_z)\,dS$$
$$= P \iiint_B \operatorname{div} \vec{F}_1\,dV = 0.$$

$$\operatorname{div} \vec{F}_1 = 0$$

Similarly $\vec{\tau} \cdot \vec{j} = 0$ and $\vec{\tau} \cdot \vec{k} = 0$. Thus $\vec{\tau} = 0$!

Not a perpetual motion machine!

Archimedes Law



Body B submerged in fluid.

ρ = density of fluid

g = gravity constant

We know the fluid pushes the body up with force equal to the weight of the liquid occupying the same volume as B. We take this as a fact, law of nature. Archimedes himself had a good deduction of it, nearly a proof.

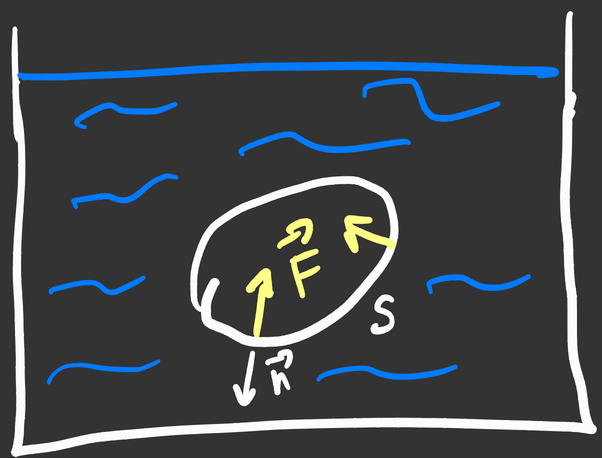
Here we derive this law using Calculus.

Pascal's Law:

$$P = -g\rho z$$

pressure grows linearly when you go down into the fluid.

The force the fluid exerts on the body is inward and



$$d\vec{F} = -P\,\vec{n}\,dS$$

$\downarrow \vec{g}$
$$= -(-g\rho z)\,\vec{n}\,ds$$
$$= g\rho z\,\vec{n}\,dS$$

Since z is negative, force decreases as you submerge the body.

Total force

$$\vec{F} = \iint_S g\rho z\vec{n}\,dS$$

$\vec{F}_1 = (z,0,0)$
$\operatorname{div}\vec{F}_1 = 0$

$\vec{F}_1$

$$\vec{F}\cdot\vec{i} = \iint_S g\rho(z\vec{i})\cdot\vec{n}\,dS = \iiint_B g\rho \operatorname{div}\vec{F}_1\,dV = 0$$

$$\vec{F}\cdot\vec{j} = \iint_S g\rho(z\vec{j})\cdot\vec{n}\,dS = \iiint_B g\rho \operatorname{div}(z\vec{j})\,dV = 0$$

$$\vec{F}\cdot\vec{k} = \iint_S g\rho(z\vec{k})\cdot\vec{n}\,dS = \iiint_B g\rho \operatorname{div}(z\vec{k})\,dV = g\rho\operatorname{Vol}(B)$$

Thus $\vec{F} = (0, 0, g\rho\operatorname{Vol}(B))$

This is Archimedes' Law

$$\vec{F} = (0, 0, g\rho \, \mathrm{Vol}(B))$$

The total force is directed vertically, and it is proportional to the mass of liquid occupying the volume of the body.

Archemedes did not derive it in this way, of course. But he came thousands of years before! This is the "Eureka" moment.

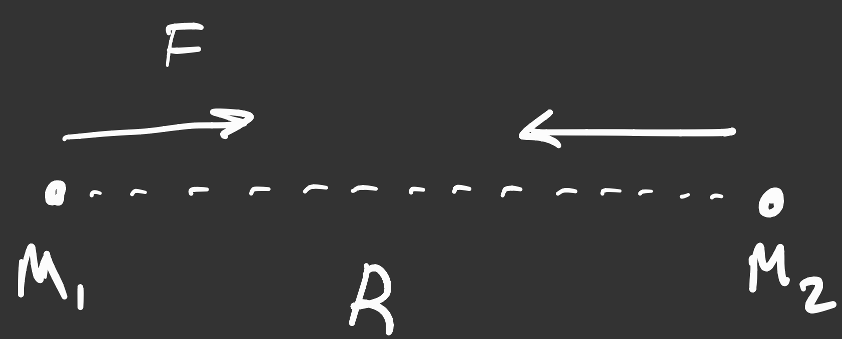
Using this law, Archemedes could tell if a crown was made of pure gold: balance the crown and pure gold on a scale in air, and then submerge it under water. If the density differs, the scale will be out of balance since the volumes will differ.

Attraction of two balls

Newton tried to build a theory of motion of the moon around the earth.

His idea was that the force that keeps the moon near the earth is the same force which attracts bodies on the surface to the center, namely gravity.

Newton's gravitational law (axiom):
Given two point masses, M_1 and M_2



distance between bodies

attractive force:

$$F = G \frac{M_1 M_2}{R^2}$$

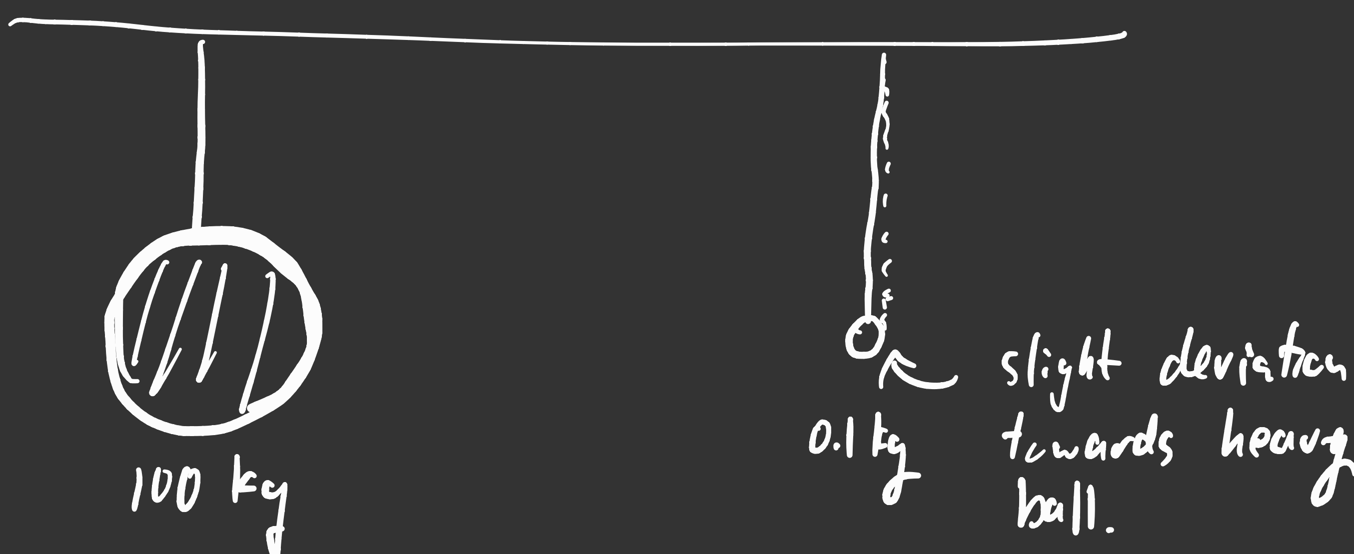
G is a constant

"Inverse-Square law"

This law cannot be varified experimentally since there are no point masses.

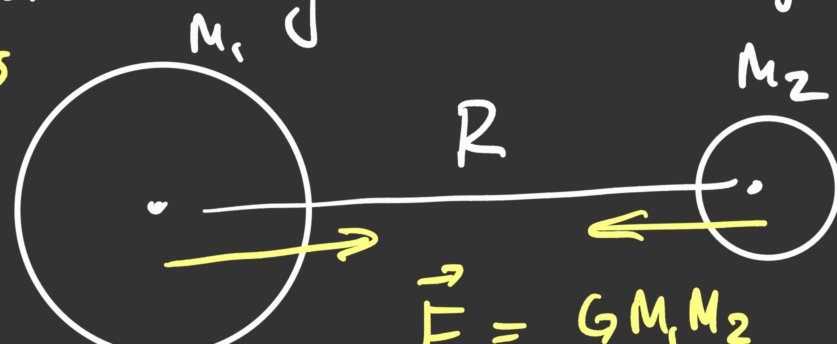
One has to use instead small masses, but then the attraction is very weak and hence hard to measure.

Intead, we can measure slight attraction of masses



Better yet, planets. But this requires work since they are no longer point masses

Nevertheless

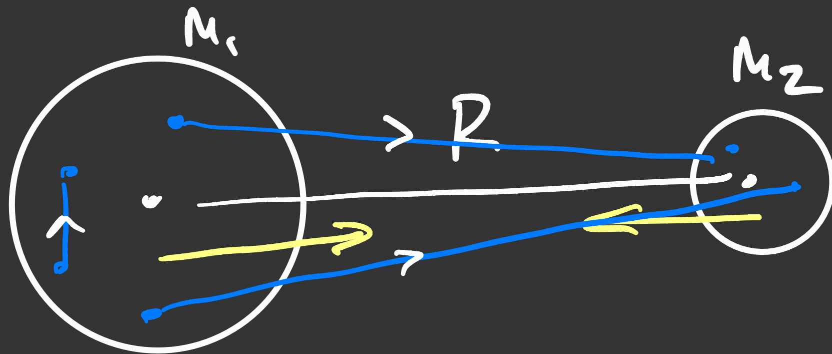


$$\vec{F} = \frac{G M_1 M_2}{R^2} \; !$$

Same formula!

Why??

Namely, if you assume Newton's axiom of attraction of point masses, how can you deduce the same holds for large balls?



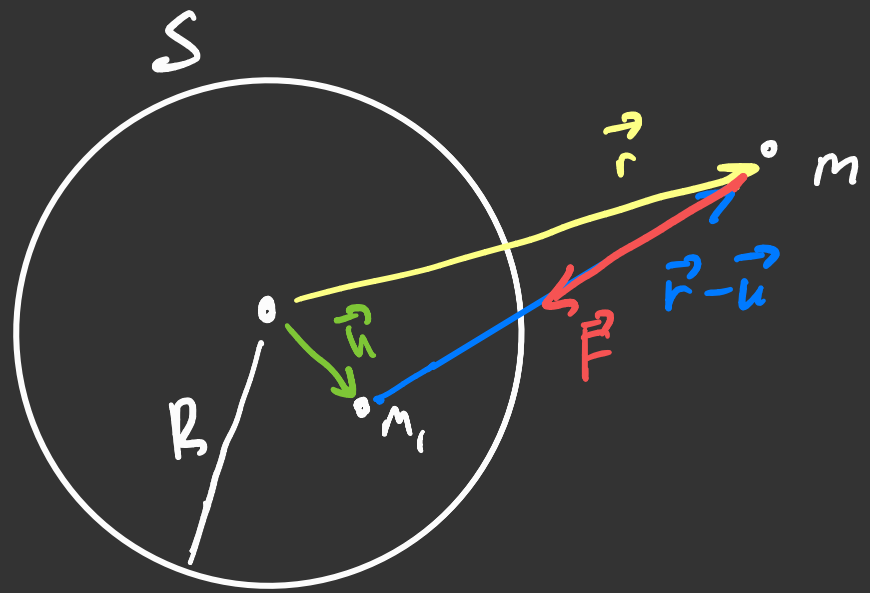
All pairs of particles attract eachother according to Newton's law. We must add up all these contributions.
The result is the attraction of the masses.

This looks to be a terribly hard problem since you have to integrate over both 3D balls, making a 6-dimensional integral.

This took Newton about 20 years to do...
When he started, calculus did not exist so he had to invent it!

Even with calculus, this is a difficult problem if you approach it in the straightforward way. But we shall use Divergence theorem, which had to wait for Gauss to come around.

Consider a sphere of radius R



To find the vector equation for the force that mass m_1 attracts mass m by

$$\vec{F} = -G m_1 m \frac{\vec{r}-\vec{u}}{\|\vec{r}-\vec{u}\|^3}$$

Force is directed opposite to $\vec{r}-\vec{u}$, thus from m to m_1.

So, the force vector field

$$\vec{F} = \vec{F}(\vec{r})$$

has divergence zero

$$\operatorname{div} \vec{F}(\vec{r}) = 0$$

as

$$\operatorname{div}(\vec{r}-u) = \operatorname{div} \vec{r} = 3$$

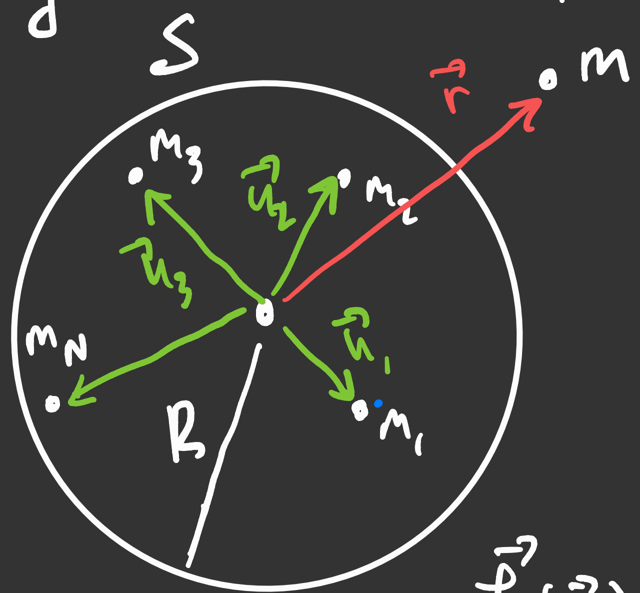
$$\operatorname{div} \vec{F}(\vec{r}) = -Gm_1 m \left[\frac{\operatorname{div}(\vec{r}-u)}{\|\vec{r}-\vec{u}\|^3} - 3 \frac{1}{\|\vec{r}-\vec{u}\|^3} \right] = 0$$

Thus the flux of force generated by m_1 over the sphere S is

$$\Phi = \iint_S \vec{F} \cdot \vec{n} \, dS = -4\pi G m_1 m$$

This flux does not depend on where m_1 is located inside the sphere.

Now suppose that instead of one point being inside the sphere, you have many $m_1, m_2, \ldots, m_N$



The force $\vec{F}(\vec{r})$ is then the sum of the forces that act from each of these points.

$$\vec{F}(\vec{r}) = -\sum_{i=1}^{N} G m_i m \frac{\vec{r}-\vec{u}_i}{\|\vec{r}-\vec{u}_i\|^3}$$

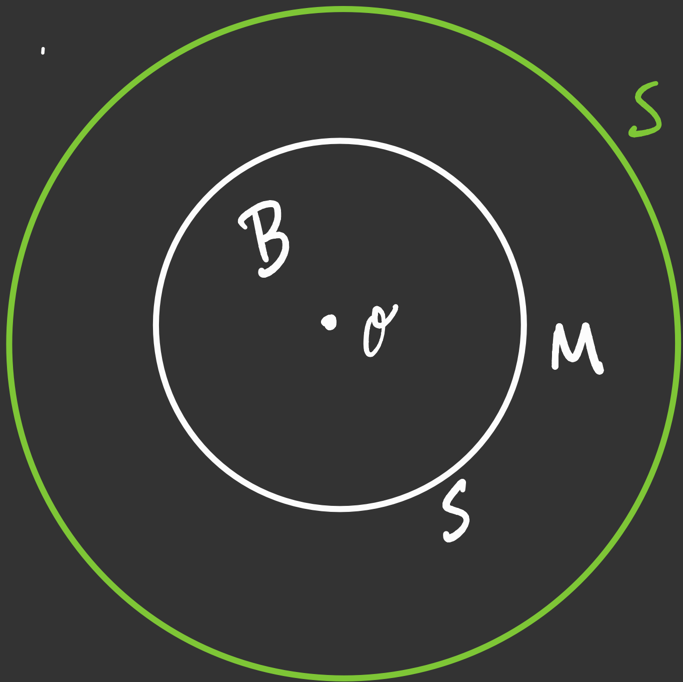
Suppose we have not just point masses, but instead a continuum with density ρ inside S. Then

$$\vec{F}(\vec{r}) = -\iiint_B G\, \rho(\vec{u}) m \frac{\vec{r}-\vec{u}}{\|\vec{r}-\vec{u}\|^3}\, dV_u$$

The flux of this vector field is a sum of the flux of the elementary fields (from one particle)

$$\Phi = \iint_S \vec{F}\cdot\vec{n}\, dS = -4\pi G \iiint_B \rho(\vec{u}) m\, dV_{\vec{u}}$$

$$= -4\pi G \underbrace{M}_{\text{mass inside the ball}} m$$



The flux through any sphere containing the ball B is the same as the flux through S.

We now assume the following thing:

the density $\rho(\vec{u})$ depends only on $\|\vec{u}\|$, e.g. ρ depends only on distance from center.

Thus, if you rotate the ball, the density does not change. Thus the force field $\vec{F}$ also does not change:

$\vec{F}$ is spherically symmetric!

It must be directed towards the center.
The strength depends only on distance to origin

Thus, the general form of such a force is

$$\vec{f}(\vec{r}) = \varphi(\|\vec{r}\|)\,\vec{r}$$

We must find the form of φ. This can be done using the flux:

$$\Phi = \iint_S \vec{f}(\vec{r})\cdot\vec{n}\,dS = 4\pi R^2\,\varphi(R)\cdot R$$

On the other hand, from before we found.

$$\Phi = -4\pi G M m.$$

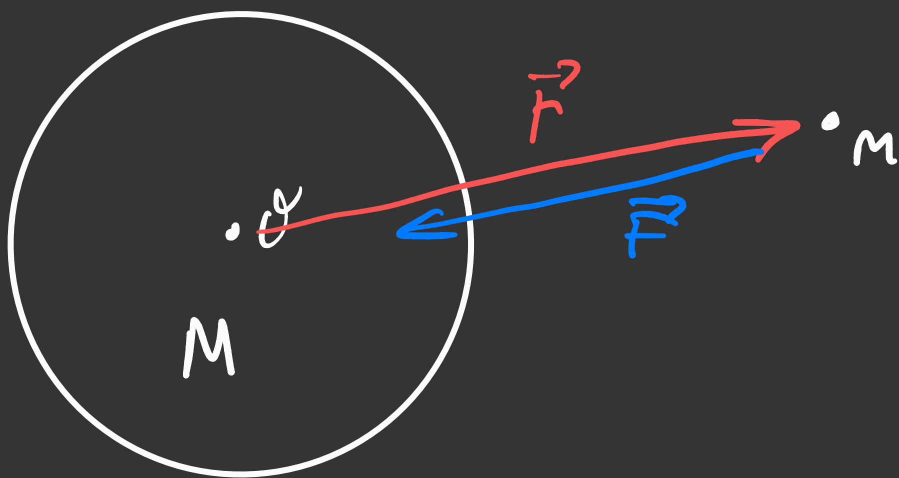
Equating the two fluxes we find

$$\varphi(R) = -\frac{GMm}{R^3}$$

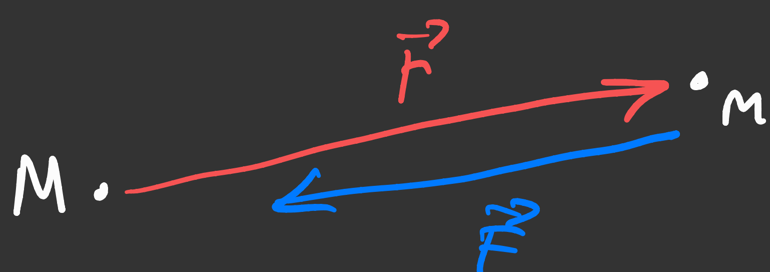
$$F(\vec{r}) = -\frac{GMm}{\|\vec{r}\|^3}\,\vec{r}$$

We found the following remarkable fact.

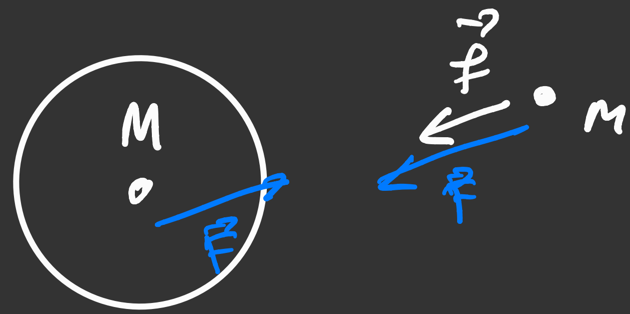
The force that a point mass m is attracted to a ball of mass M



is equal to the corresponding force replacing the ball by a point of equal mass at its center.

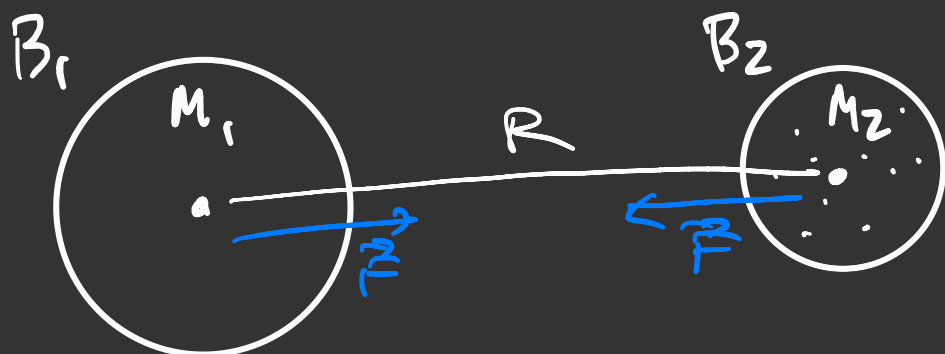


We started from a ball and a particle



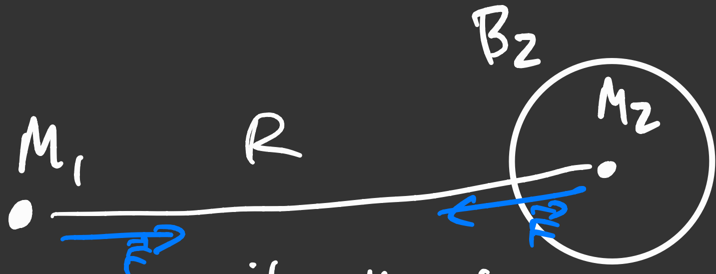
The force which particle is attracted to ball is same if you replace ball by point mass of M at center.

Now suppose you have two balls



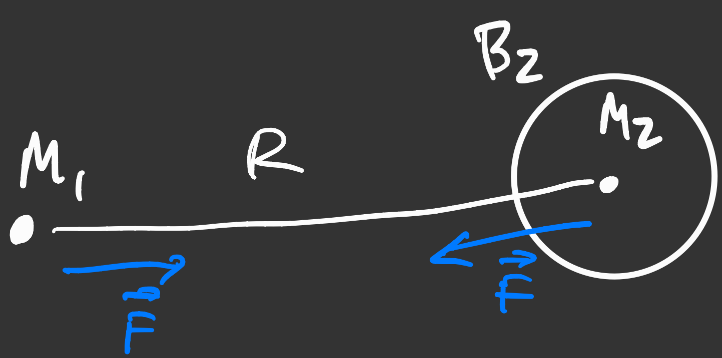
Ball B_1 attracts all particles in B_2.

But if you replace B_1 by a point mass

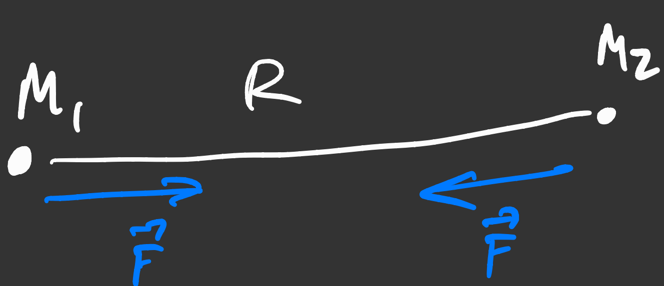


The force that it attracts all particles in B_2 will remain the same (we just proved it)

Thus the total attraction force does not change if you replace mass M_1 by a point mass at the center

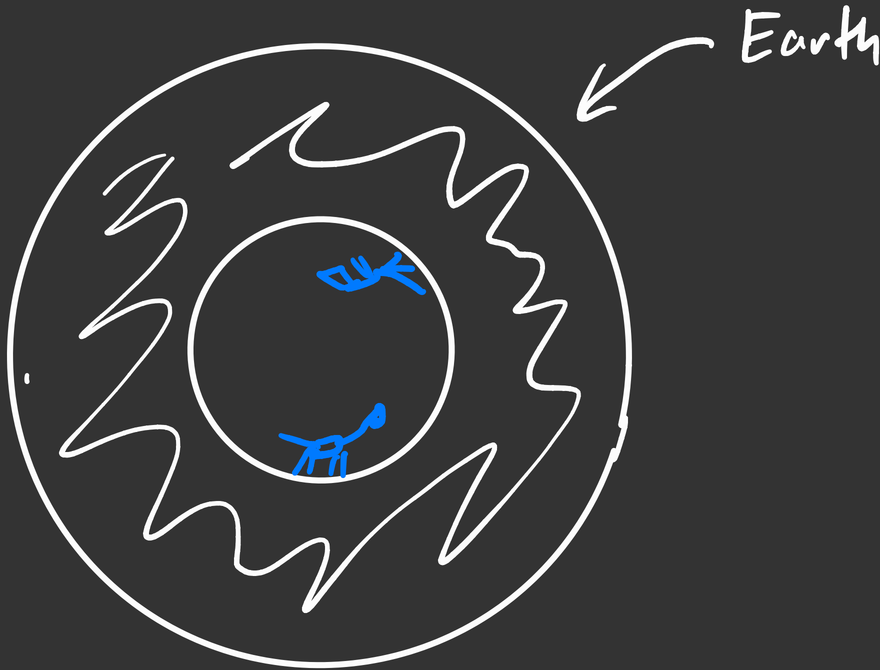


Now we repeat this reasoning.
The attraction force between m_1 and M_2 is same as if B_2 were a point mass at the center



This is what we aimed to prove!

Journey to the center of the earth?



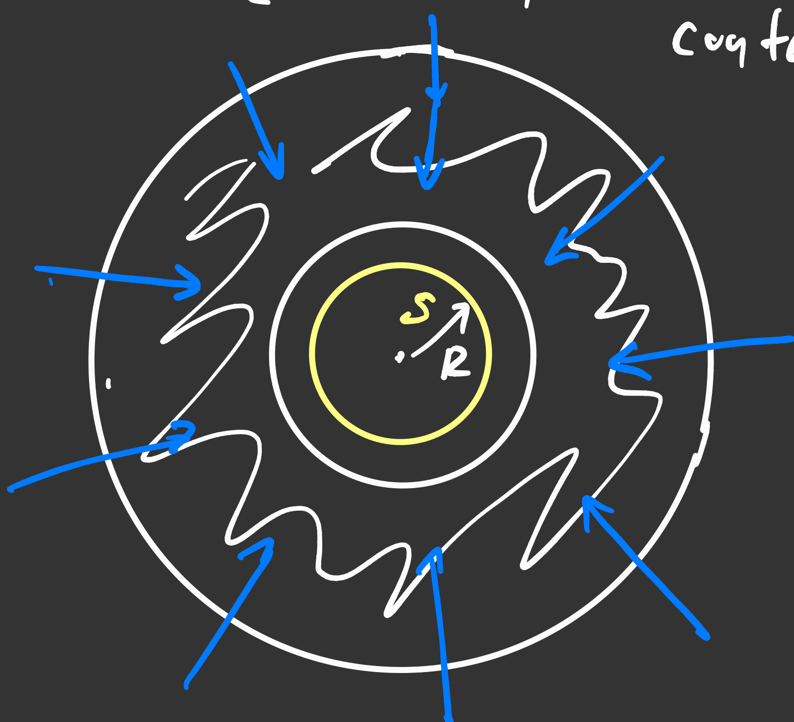
Suppose the earth is hollow, with an empty spherical cavity inside.
Can there be dinosaurs walking around there, as claimed by Jules Verne??

NO!

Indeed for the same reason as for Newton's problem, the gravity force inside the cavity is identically zero

Why? On one hand, the hollowed earth is symmetric, so if you turn it the distribution of masses does not change.

On other hand, consider any sphere S contained in the cavity.



$$\vec{F} = \varphi(\|r\|)\,\vec{r}$$

Since there is no mass inside, the flux Φ of the gravitational field is zero (by divergence Thm).

But since $\vec{F}$ is directed towards the center.

Only way for $\Phi = 0$ is for $\vec{F} = 0$.

The dinosaurs cannot walk around inside

At best, they are flying around.

The End