

To compute it, we could divide the surface into pieces such that, within each piece, the surface is the graph of some function. Either Z=f(x,y), x=g(y,Z), y=h(z,x). In this case, we can write the integral as the corresponding integral over a 2d domain Then we compute the integral as we have been (if we can). Ideally, it reduces to some iterated integral.

Our goal here is to avoid this way of computing, and to do it in a much simpler way.

Theorem: Let S be a regular surface,
Which is the boundary of a 3d domain
B. Let
$$\vec{F}(x,y,z)$$
 be a regular vector
Field in B. Then
 $\iint \vec{F} \cdot \vec{n} \, dS = \iint div \vec{F} \, dV$
S B
Example: B: $x^2 + y^2 + z^2 \leq P^2$ (Ball of reduces P)
 $\int x^2 + y^2 + z^2 = P^2$ (Sphen)
 $f(x,y,z) = (f(x+1) + f(x)y)^2$, $f(x+1) + f(x)y^2$)
 $\iint \vec{F} \cdot \vec{n} \, dS$ looks complicated
S
hut...
 $div \vec{F} = 0$ [$\int_0^{\infty} \int \vec{F} \cdot \vec{n}^2 \, dS = 0$.
This situation is typical in applications and very insetul.
(3)

 $\frac{Proof of Divergence (hronem;}{\vec{F}(Y,Y,Z) = P(Y,Y,Z)\vec{i} + O(Y,Y,Z)\vec{j} + F(Y,Y,Z)\vec{k}}$ $= \vec{F}_{1}(\vec{r}) + \vec{F}_{2}(\vec{r}) + \vec{F}_{3}(\vec{r})$ (when $\vec{F}_{1}(\vec{r}) = P(\vec{r})\vec{i}$, etc.)

We shall prove the theorem for each Fireperfly the general case will follow by additivity. It is enough to prove it for one, say Fz, as others follow similar argument.

Must Show

$$\int \overline{F_3} \cdot \overline{n} \, dS = \int \int div \overline{F_3} \, dV$$

S B

We start proving this for simple domains B...

Suppose B is of type I.
e.g. B:
$$(x,y) \in D$$
 and $g(x,y) \leq 2 \leq h(x,y)$
div $\overline{F_3}(\overline{r}) = \frac{\partial P}{\partial 2}$ (since $\overline{F_3} = (0,0,P)$)
Thus, the div integral is
 $\iiint d_1 \sqrt{F_3} dV = \iint \int_{D} \int_{D}^{h(x,y)} \frac{\partial P}{\partial 2} d2 dy dx$
 $B = \iint (P(x,y,h(x,y)) - P(x,y,g(x,y))) dy dx$

Now the Flax...



Thus, for $B_{ij}B_{2j}B_{3}$ are domains of type I and Theorem is proved $\iint \vec{F}_{3} \cdot \vec{n} \, ds = \iint div \vec{F}_{3} \, dv$ $S_{i} \quad R_{i}$ $S_{i} \quad R_{i}$ $S_{ince} \quad \vec{F}_{3} \quad s \quad tangent \quad to \quad vertical \\boundaries$



This domain is not of type I, but



We can cut it with a cylinder. Then B, B₂, B₃ are all type I domains.

Full mathematical proof applying to any negular domain is subtle, but this is main idea.

Not fully general ...

namely Sz doesn't Contain Si.

Introduce Sz ...

(IV

but also $\iint \vec{F} \cdot \vec{n} \, dS = \iint \vec{F} \cdot \vec{n} \, dS.$ Sz

Thus

 $\iint \vec{F} \cdot \vec{n} \, dS = \iint \vec{F} \cdot \vec{n} \, dS$ 5, S2

$$F(x,y,z) = \frac{(x,y,z)}{(x^{2}+y^{2}+z^{2})^{3}/2}$$

$$\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|^3}$$

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DV

This field enters in
a) Newton's gravitational law
b) Electro-static field (Coulumb)
c) arises also in fluid dynamics.

$$\vec{F}$$
 is resulter for all points except $\vec{P}=0$.
Moneuer $\prod_{i} \vec{F} \parallel = \frac{N\vec{r} \parallel}{N\vec{r} \parallel^{2}} = \frac{1}{N\vec{r} \parallel^{2}}$
Rul for $\vec{r} \neq 0$,
 $div \vec{F}(\vec{r}) = 0$
Check thic 1

Flow of incompressible fluid incompressible means every clement of fluid keeps its volume over time Water is close to incompnessible. $v(\vec{r}) = velocity of in compressible fluid$ $div \vec{v}(\vec{r}) = 0$ implies fluid is incompressible. Consider (Sansider $\vec{u}(\vec{r}) = \frac{q}{4\pi} \frac{\vec{r}}{\|\vec{r}\|^3}$ Q is a constant flow rate point source of water. $\overline{\phi} = \iint \overline{h} \cdot \overline{n} \, ds = Q$ 1 S flux of fluid through surface S, for any surface Amongt of water through Surgace is a lady equal to what is put in at 2000 (JS)

now the situation Consider F(7) diu 7 >0. with SF.rds. This can be hard (Find Byt consider also another surface. e.g. z = h(x,y) $Q \int S' = g(x, y)$ normal Swit Claim: $\iint \vec{F} \cdot \vec{n} \, dS = \iint \vec{F} \cdot \vec{n} \, dS.$ orintation Indeed, consider B: g(x,y) SZS h(x,y) V $O = \int (div f dV) = \int F \cdot \vec{n} dS = \int F \cdot \vec{n} dS - \int F \cdot \vec{n} dS$ B bonadary of B 51 ß

This can aid in compating integrals.
Consider
S: $2 = e^{\sqrt{\lambda^2 + y^2}} + e^{-\sqrt{\lambda^2 + y^2}}$
$0 \leq \chi^2 + \chi^2 \leq 1$
$\vec{F} = (x_1 y_1 - 2z)$ $div \vec{F} = 0$
what is contour on the boundary C
$2 = \frac{e+e}{2}$ $x^{2}+y^{2} = 1$.
Let us replace S by S':
$z = \frac{e+e'}{x^2+y^2} = 1$
Now $\vec{h} = (0,0,1)$. Thus $\vec{F} \cdot \vec{n} = -22 = e + e^{-1}$.
Thus $((\vec{E},\vec{h},ds) = ((\vec{F},\vec{h},ds) = (e+\vec{e})) Ana(dist)$
$S = (e+e^{-1}) \pi$