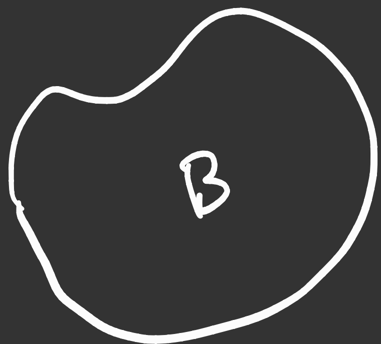
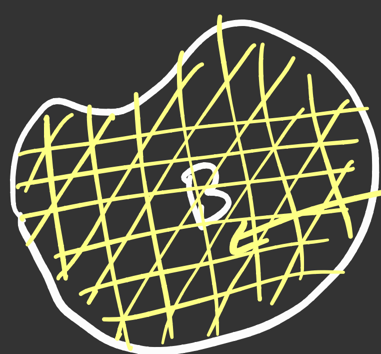
Triple integral: integral over domain in 3d space



$B \subset \mathbb{R}^3$

partition B into small pieces



B_i has center (x_i, y_i, z_i)

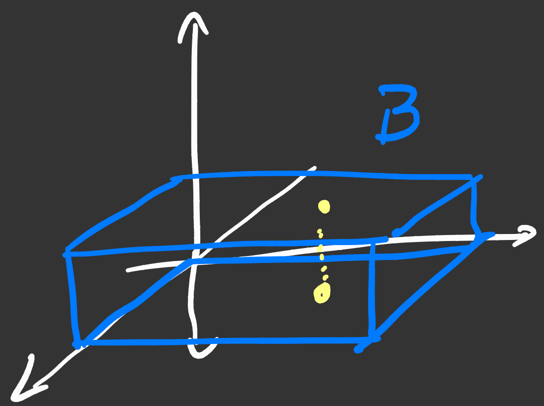
say $f(x,y,z)$ is a continuous function in B.

$$S_n := \sum_{i=1}^{n} f(x_i, y_i, z_i)\, V(B_i) \xrightarrow[\text{size}(B_i) \to 0]{n \to \infty} \iiint_B f(x,y,z)\, dV$$

for instance, if B_i is a small parallelpiped, volume is product of lengths.

Iterated integral

Ex: $B : \left\{ \begin{array}{l} a_1 \le x \le a_2 \\ b_1 \le y \le b_2 \\ c_1 \le z \le c_2 \end{array} \right\}$



$$\iiint_B f \, dV = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \left(\int_{c_1}^{c_2} f(x,y,z) \, dz \right) dy \right) dx$$

first fix x & y and integrate along z.

ex B : $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

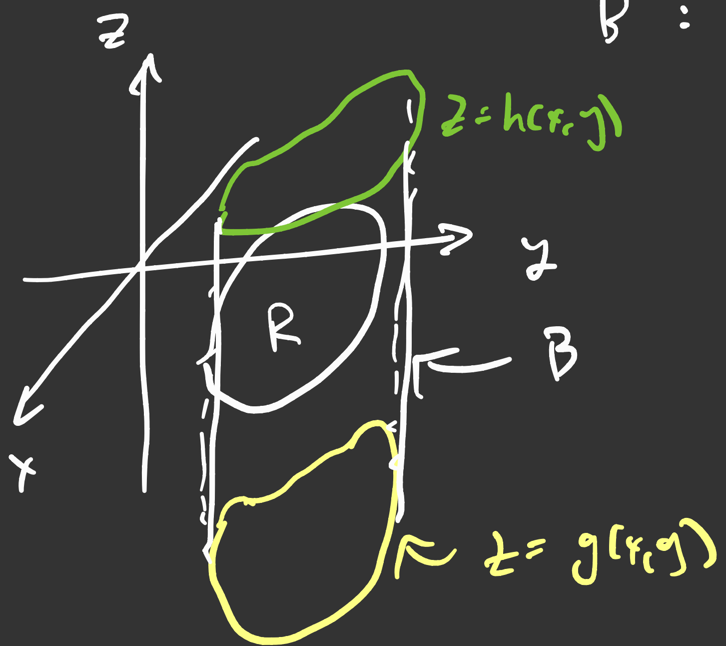
$f(x,y,z) = xy - z^4$

$$\int_0^1 \int_0^1 \int_0^1 (xy - z^4) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(xy - \frac{1}{5}\right) dy \, dx$$

$$= \int_0^1 \left(\frac{x}{2} - \frac{1}{5}\right) dx = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

More general domain

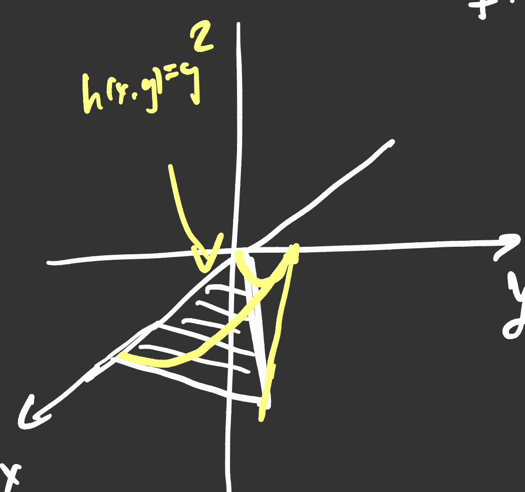
$B: \quad (x,y) \in R, \quad g(x,y) \le z \le h(x,y)$



$$\iiint_B f \, dV = \iint_R \int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \, dA_{xy}$$

Ex: $B: \quad 0 \le x \le 1, \quad 0 \le y \le x, \quad 0 \le z \le y^2$

$f = x + y + z$



$$\iiint_B f \, dV = \int_0^1 \int_0^x \int_0^{y^2} (x+y+z) \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^x \left(xy^2 + y^3 + \frac{y^4}{2} \right) dy \, dx$$

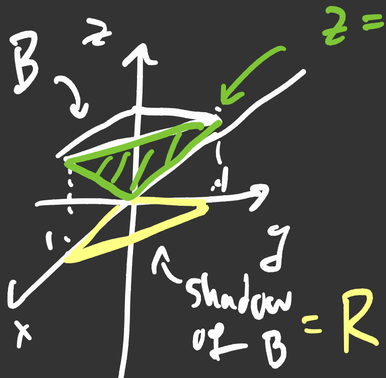
$$= \int_0^1 \left(\frac{x^4}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right) = \frac{1}{15} + \frac{1}{20} + \frac{1}{60}$$

$$= \frac{2}{15}$$

Here we distinguished the z-direction, with our body extruded over a domain in xy place. But, we can change which axis is distinguished, having x or y distinguished, for example.

Sometimes changing the order of integration can make an integral doable.

Ex: B: $x \geq 0$, $y \geq 0$ $x+y \leq z$, $0 \leq z \leq 1$.



$f(x,y,z) = \sin(z^3)$

$$\iiint_B f \, dV = \iint_R \int_{x+y}^{1} f \, dz \, dA$$

$$= \int_0^1 \int_0^{1-x} \left(\int_{x+y}^{1} \sin(z^3) \, dz \right) dy \, dx.$$

cannot integrate in terms of simple functions

Instead, look from side:

$B = \; 0 \leq z \leq 1, \; 0 \leq x \leq z, \; 0 \leq y \leq z-x$

$$\iiint_B f \, dV = \int_0^1 \int_0^z \int_0^{z-x} \sin(z^3) \, dy \, dx \, dz$$

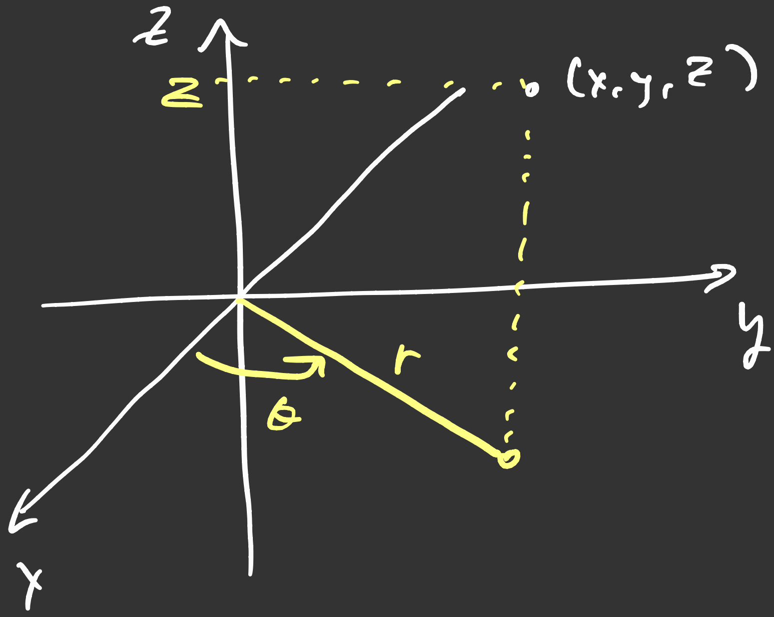
$$= \int_0^1 \int_0^z \sin(z^3)(z-x) \, dx \, dz = \int_0^1 \sin(z^3) \frac{z^2}{2} \, dz =$$

$$= \frac{1}{6} \int_0^1 \sin(t) \, dt$$

$$= \frac{1}{6} \left(-\cos t \right)_0^1$$

$$= \frac{1}{6} \left(1 - \cos(1) \right)$$

Cylindrical Coordinates



three parameters,

r: distance to z axis

θ: angle to x axis

z: height

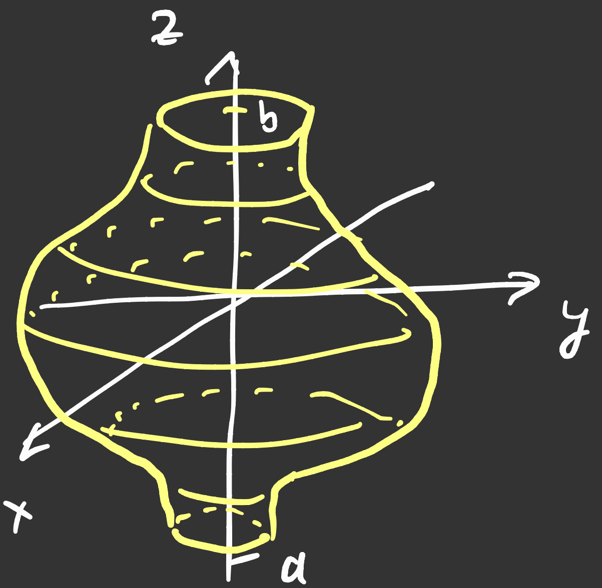
Characterise our point (x, y, z)

$x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad z = z$

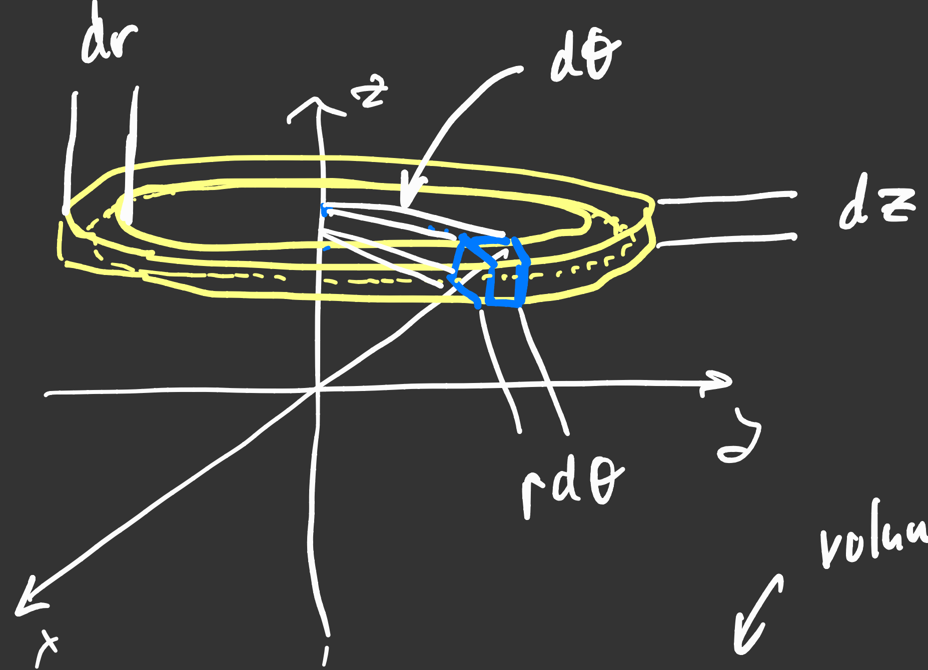
$r = \sqrt{x^2+y^2}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$ $\quad z = z$

Body of revolution

$a \le z \le b$, $\quad 0 \le r \le g(z) \quad 0 \le \theta \le 2\pi$



Volume element in cylindrical coordinates



Take small piece

$z_0 \le z \le z_0 + dz$

$r_0 \le r \le r_0 + dr$

$\theta_0 \le \theta \le \theta_0 + d\theta$

volume of small piece

$$dV = r\,dr\,dz\,d\theta$$

$B: \quad a \le z \le b \qquad 0 \le \theta \le 2\pi \qquad 0 \le r \le g(z)$

$$\iiint_B f \, dV = \int_0^{2\pi} \int_a^b \int_0^{g(z)} f(r, z, \theta) \, r \, dr \, dz \, d\theta$$

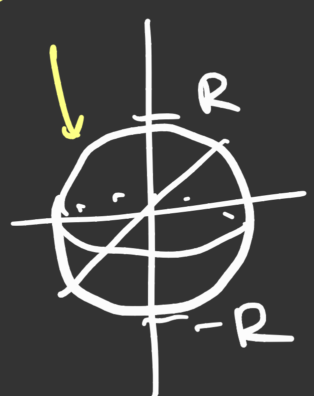
ex: Volume of B:

$$\iiint_B 1 \, dV = \int_0^{2\pi} \int_a^b \int_0^{g(z)} r \, dr \, dz \, d\theta$$

$$= 2\pi \int_a^b \frac{g(z)^2}{2} \, dz = \pi \int_a^b g(z)^2 \, dz.$$

For example, if B is a ball of radius R

$z^2 + r^2 = R^2$



$B: \quad -R \le z \le R$

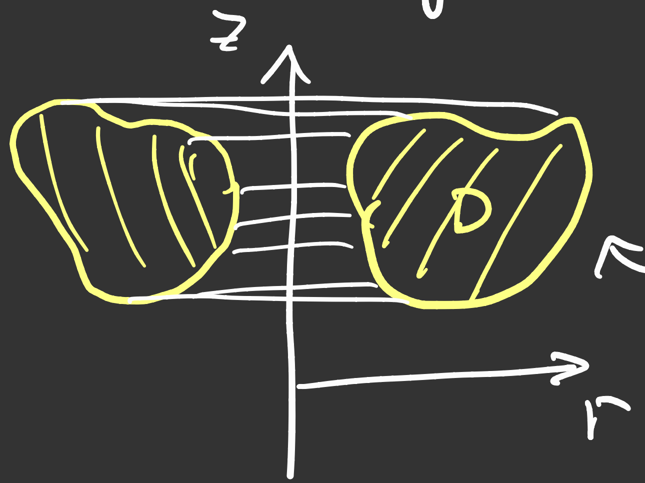
$\quad\;\; 0 \le r \le \sqrt{R^2 - z^2}$

Thus $g(z) = \sqrt{R^2 - z^2}$

$$\mathrm{Vol}(B) = \pi \int_{-R}^{R} (R^2 - z^2) \, dz = \pi \left(2R^3 - 2\frac{R^3}{3} \right)$$

$$= \frac{4\pi}{3} R^3.$$

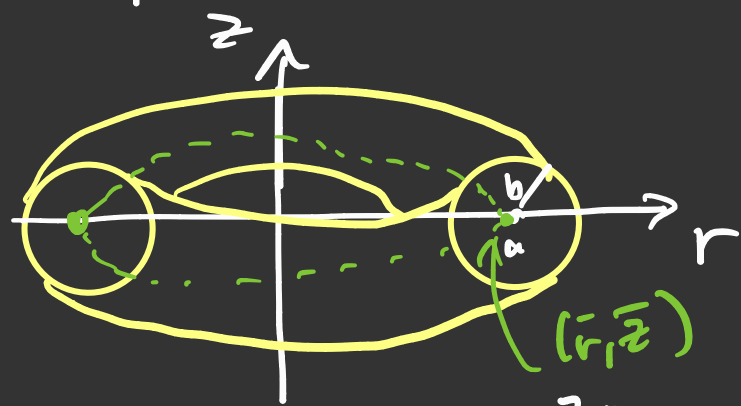
General body of revolution.



$B: \ (r, z) \in D \quad \theta \in [0, 2\pi).$

← sweep around z axis

Example: Volume inside torus (doughnut)



$D: (r-a)^2 + z^2 \le b^2$

$B: (r, z) \in D, \quad \theta \in [0, 2\pi)$

$(\bar{r}, \bar{z})$ center of mass

$$V(B) = \iiint_B dv = \int_0^{2\pi} \iint_D r\,dr\,dz\,d\theta = 2\pi \iint_D r\,dr\,dz$$

moment w.r.t. z

$$= 2\pi M_z(D) = 2\pi \bar{r} A(D)$$

Gulden Theorem

= (length of circle generated by rotating center of mass around z-axis)

× (area of cross-section in z-r plane).

$\bar{r} = \dfrac{M_z(D)}{M(D)}$

$M(D) = \text{area}(D)$

In the case of our doughnut,

length of circle $= 2\pi a$

$A(D) = \pi b^2$

Thus

$$\boxed{V(B) = 2\pi^2 a b^2.}$$

Note, we may compute directly

$$\iint_D r\,dr\,dz = \iint_D (r-a)\,dr\,dz + a\iint_D dr\,dz$$

$$\overset{\rho = r-a}{=} \iint_{\{\rho^2+z^2\le b^2\}} \rho\,d\rho\,dz + a(\pi b^2)$$

$$= \int_0^{2\pi}\int_0^b r\cos\theta\,dr\,d\theta + \pi a b^2$$

$$= \pi a b^2$$

Thus, again

$$V(B) = 2\pi \iint_D r\,dr\,dz = 2\pi^2 a b^2.$$