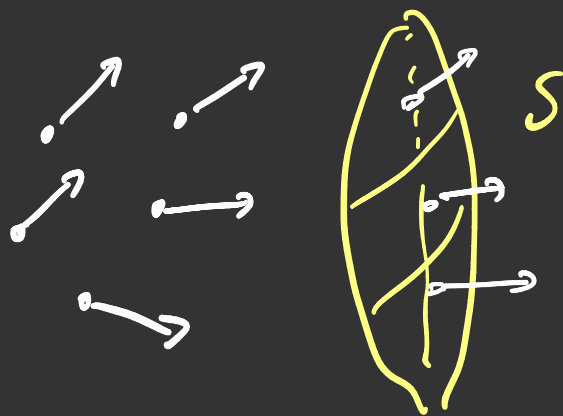
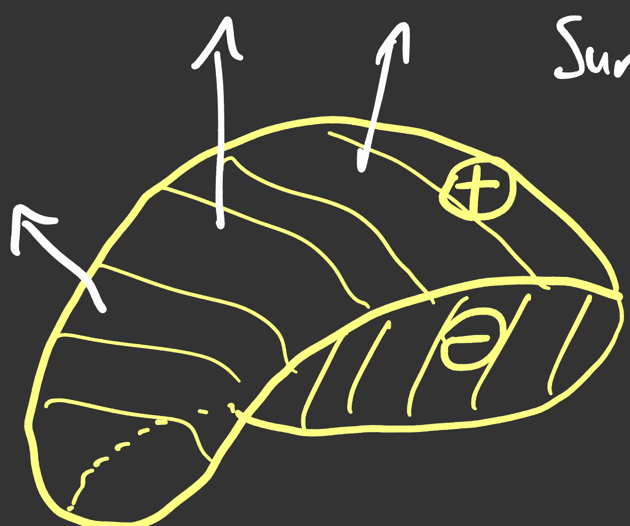
Flux of a vector field through a surface

$\vec{u}(x,y,z)$ = field of velocity of some fluid.



Flux of $\vec{u}$ through S tells how much fluid passes through S.

(Orientable)

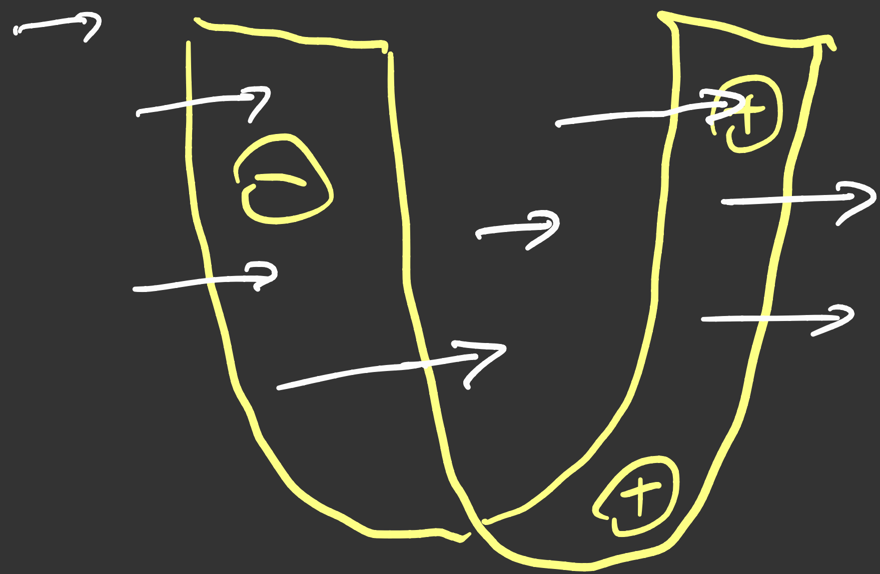


Surface has "positive side" and "negative side".

To get from one to the other, we need to cross an edge

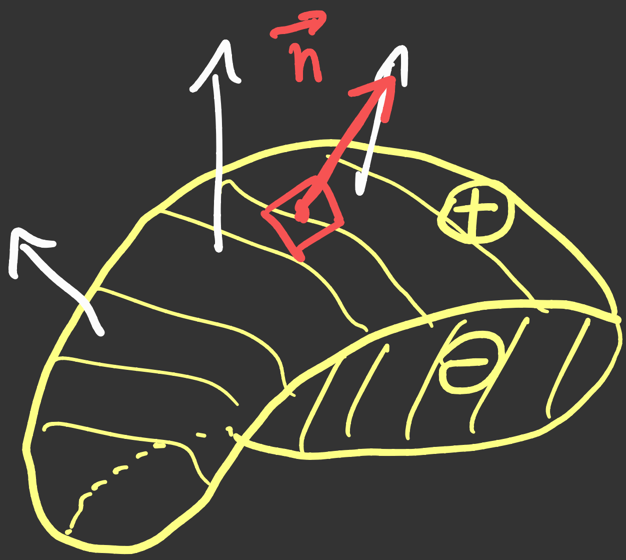
How much fluid flows through surface, from ⊖ → ⊕ side?

Flux is positive if fluid crosses from ⊖ to ⊕. Neg otherwise



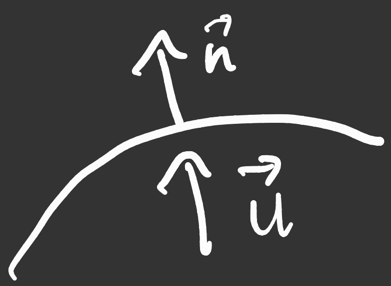
Here the total amount of fluid passing through surface may be zero, since equal amounts cross from ⊖ → ⊕ and ⊕ → ⊖.

Sometimes the flux is not zero:

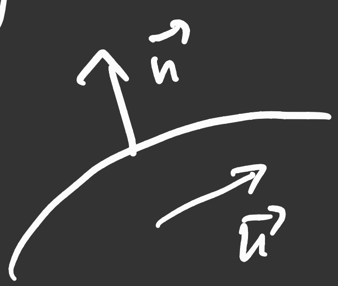


unit normal vector field $\hat{n}(x,y,z)$.

If flow goes in direction of normal, then the flux is maximal.



If, on the other hand, $\vec{u}$ is tangent then it gives no contribution.



Formula for flux:

$$\Phi = \iint_S \vec{u} \cdot \vec{n} \, dS$$

Greek letter "Phi"

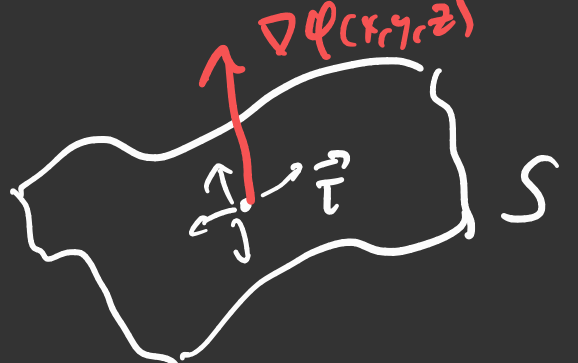
Suppose : $\vec{u} = (P(x,y,z), Q(x,y,z), R(x,y,z))$

Surface : $S: \ z = f(x,y) \qquad (x,y) \in D \subset \mathbb{R}^2$

or

$\varphi(x,y,z) = 0$ with $\varphi(x,y,z) := z - f(x,y)$

Normal vector : Normal vector is in direction of gradient φ;



since φ does not change when moving along S, infinitesimally, thus

$$\vec{t} \cdot \nabla\varphi = 0$$

for all tangential directions $\vec{t}$.

We compute: $\vec{n} = \dfrac{\nabla \varphi}{|\nabla \varphi|}$ where

$\varphi = z - f(x,y)$

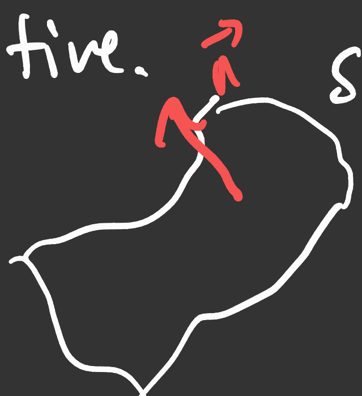
$$\nabla \varphi = \left(-f_x, -f_y, 1\right)$$

$$|\nabla \varphi| = \sqrt{f_x^2 + f_y^2 + 1}$$

Thus

$$\boxed{\vec{n}(x,y,f(x,y)) = \frac{\left(-f_x, -f_y, 1\right)}{\sqrt{f_x^2 + f_y^2 + 1}}}.$$

This vector looks "up", as third component is positive.



surface with upperside positive

$\vec{n}$ points from $\ominus$ to $\oplus$ side.

$$\Phi = \iint_S \vec{u} \cdot \vec{n} \, dS$$

$$= \iint_S (P, Q, R) \cdot \left(\frac{-f_x}{\sqrt{f_x^2 + f_y^2 + 1}}, \frac{-f_y}{\sqrt{f_x^2 + f_y^2 + 1}}, \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} \right) dS$$

$dS = \sqrt{f_x^2 + f_y^2 + 1} \, dA$

$$= \iint_D (P, Q, R) \cdot (-f_x, -f_y, 1) \, dA$$

$$= \iint_D \Big(P(x, y, f(x,y)) (-f_x) + Q(x, y, f(x,y)) (-f_y) + R(x, y, f(x,y)) \Big) \, dA$$

Luckily, nasty square roots disappear!

$$\phi = \iint_D (-Pf_x - Qf_y + R)\, dA$$

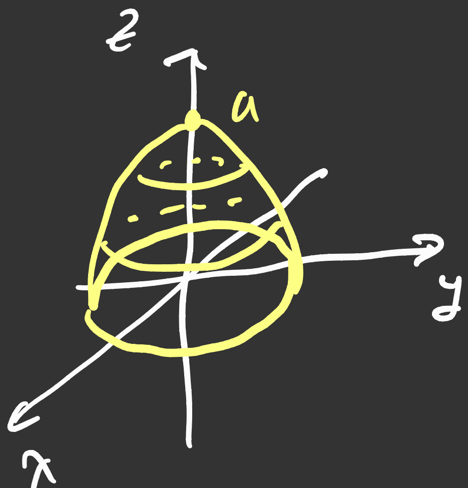
Example $\vec{u}(x,y,z) = (x, y, -2z)$

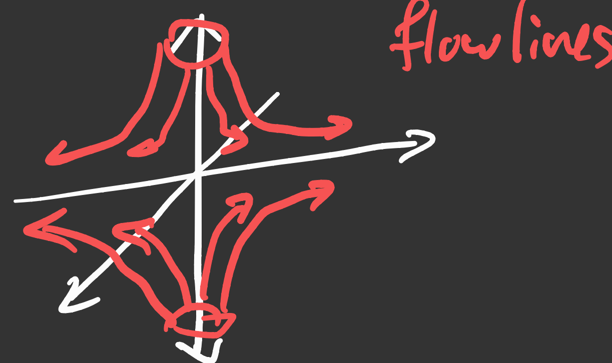
$S: \quad z = a(1-x^2-y^2) \qquad a \in \mathbb{R}.$

$D: \quad x^2+y^2 \le 1$

Surface is a piece of paraboloid!

$P = x, \quad Q = y, \quad R = -2z$





$f = a(1-x^2-y^2) \qquad f_x = -2ax \qquad f_y = -2ay$

$$\phi = \iint_D \left(2ax^2 + 2ay^2 - 2a(1-x^2-y^2)\right) dA$$

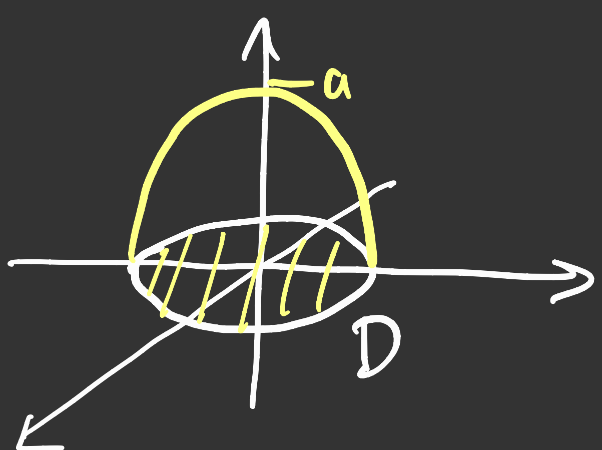
$$= \iint_D \left(4a(x^2+y^2) - 2a\right) dA$$

$$\Phi = \iint_D \left(4a(x^2+y^2) - 2a\right) dA$$

$$= \int_0^{2\pi}\int_0^1 (4ar^2 - 2a)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^1 (4ar^3 - 2ar)\, dr$$

$$= 2\pi\left(4a\cdot\frac{1}{4} - 2a\cdot\frac{1}{2}\right) = 0 \quad !$$



Look! we had a parameter "a". If $a=0$, our surface was the disk D. But $\vec{u}$ is tangent to D, so there is no flux.

For positive "a", the flux is still zero since fluid enters/exists in the same proportion. This will be a special case of a very general result...