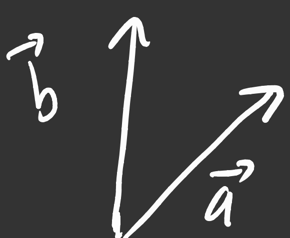
MAT 203: Multivariable Calculus — Lecture 2

The next operation, <u>cross product</u>, is more interesting. It is defined for vectors in 3 dimensional space, $\vec{a}$ and $\vec{b}$, and their cross product is a third vector $\vec{a} \times \vec{b}$.

To explain this operation, we require several preliminary concepts.

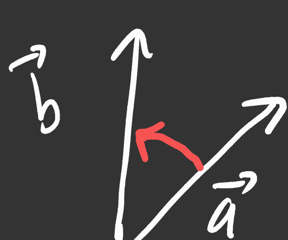
<u>Left and Right pairs of vectors the plane</u>



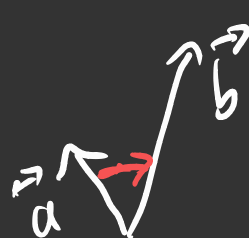
← pair of vectors

two different types of pairs. $(\vec{a}, \vec{b})$

Consider the angle between vector $\vec{a}$ and $\vec{b}$ which is smaller than π.
consider which direction should we rotate $\vec{a}$ so that its direction eventually coincides with $\vec{b}$.



$(\vec{a}, \vec{b})$
counterclockwise
<u>right pair</u>



clockwise
<u>left pair</u>

If $(\vec{a}, \vec{b})$ forms a right pair then
$(\vec{b}, \vec{a})$ is a left pair,

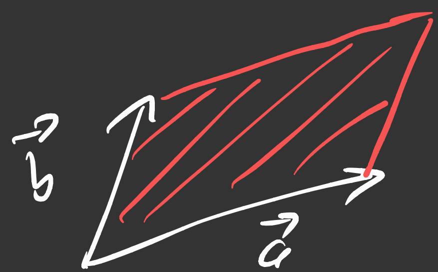
and conversely.

Different pairs of vectors are either right or left, unless the angle is 0 ($\vec{a}$ and $\vec{b}$ have same direction) or angle is 2π ($\vec{a}$ and $\vec{b}$ are antiparallel).

These are degenerate cases.

Signed area of a parallelogram

Say a parallelogram is defined by vectors $\vec{a}$ and $\vec{b}$.



Consider the signed area defined by

$$A(\vec{a},\vec{b}) = \begin{cases} \text{usual area} & \text{if } (\vec{a},\vec{b}) \text{ is right} \\ -\text{usual area} & \text{if } (\vec{a},\vec{b}) \text{ is left} \end{cases}$$

positive area is just area of piece of paper.

negative area is if you make a hole in shape of a parallelogram, you can count it as negative area (it is what is deficient in the whole sheet of paper).

Signed area behaves much more regularly than usual area.

How to find it?

Let us look for the formula.

We will do so step by step by examining the properties of this function.

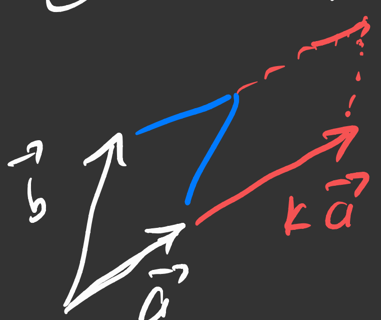
Properties of $A(\vec{a}, \vec{b})$

unlike dot product!

① $A(\vec{a}, \vec{b}) = -A(\vec{b}, \vec{a})$ (anti-symmetric)

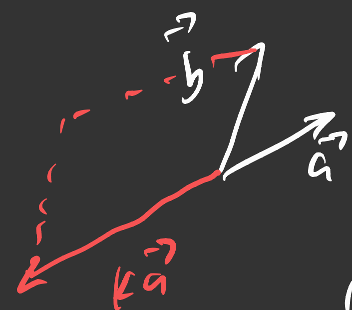
because if you take a right pair and change places, then $(\vec{b}, \vec{a})$ is left.

② $A(k\vec{a}, \vec{b}) = k\,A(\vec{a}, \vec{b})$ for $k \in \mathbb{R}$



if k positive

Since area of parallelogram is proportional to $\|\text{side}\|$, if follows



If pair $(\vec{a}, \vec{b})$ is right then $(k\vec{a}, \vec{b})$ is left if k negative

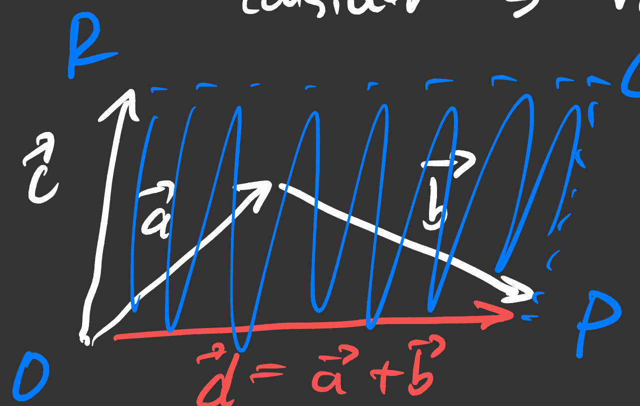
2) $A(\vec{a}, k\vec{b}) = k\, A(\vec{a}, \vec{b})$

3) $A(\vec{a}+\vec{b}, \vec{c}) = A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

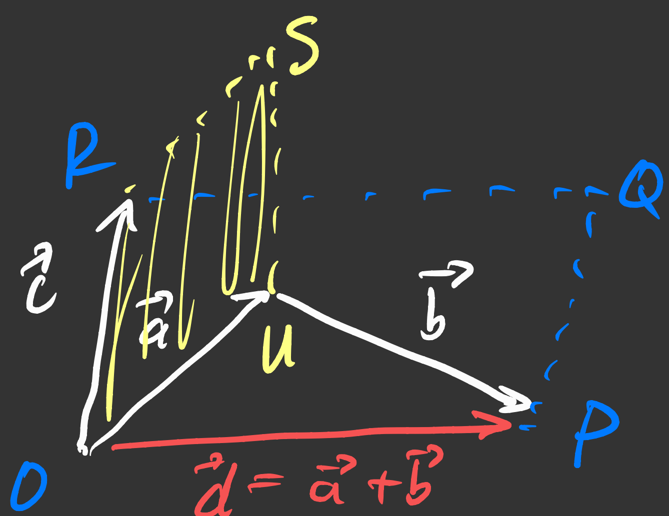
$A(\vec{a}, \vec{b}+\vec{c}) = A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

distributive property.

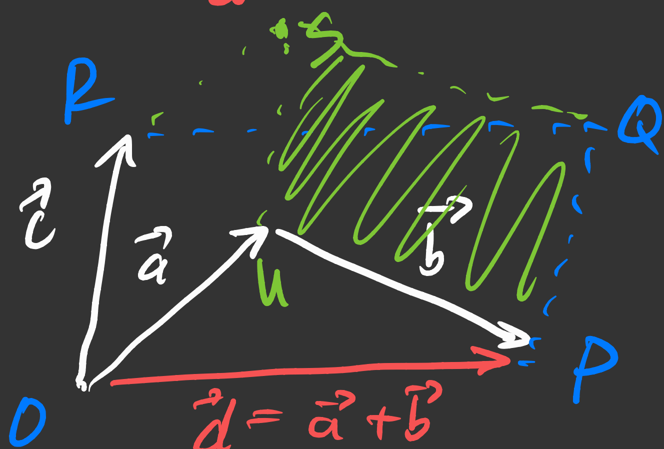
Pf: consider 3 vectors, $\vec{a}, \vec{b}, \vec{c}$



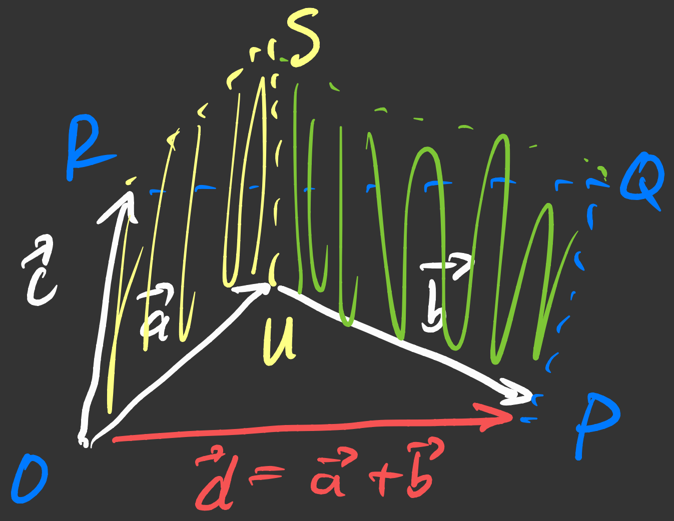
$A(\vec{a}+\vec{b}, \vec{c}) =$ signed area of OPQR



$A(\vec{a}, \vec{c}) =$ signed area of OUSR



$A(\vec{b}, \vec{c}) =$ signed area of UPQS



$A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

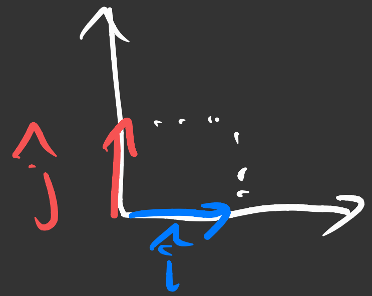
The difference between this area and that of OPQR is that we added $\triangle$ RSQ and subtracted $\triangle$ OUP.

These two triangle are shifts of one another, thus have same signed area.

Here we considered simplest case when all pairs of vectors are right, and so all signed areas are simple areas.

One needs to consider more general configurations, but it is the same (after more work).

4)



$A(\hat{i},\hat{j}) = 1$

since $(\hat{i},\hat{j})$ is a right pair

$A(\hat{j},\hat{i}) = -1$

$A(\hat{i},\hat{i}) = 0$ (degenerate parallelogram has zero width)

$A(\hat{j},\hat{j}) = 0$

Now we may derive the formula.

Suppose $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$

$$
\begin{aligned}
A(\vec{a},\vec{b}) &= A(a_1\hat{i} + a_2\hat{j},\ b_1\hat{i} + b_2\hat{j}) \\
&= A(a_1\hat{i}, b_1\hat{i}) + A(a_1\hat{i}, b_2\hat{j}) \\
&\quad + A(a_2\hat{j}, b_2\hat{j}) + A(a_2\hat{j}, b_1\hat{i}) \\
&= \cancelto{0}{a_1b_1 A(\hat{i},\hat{i})} + a_1b_2 \cancelto{1}{A(\hat{i},\hat{j})} \\
&\quad + \cancelto{0}{a_2b_2 A(\hat{j},\hat{j})} + a_2b_1 \cancelto{-1}{A(\hat{j},\hat{i})} \\
&= a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}
$$

← determinant

We have our formula:

$$A(\vec{a}, \vec{b}) = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

We can see the origin of <u>determinants</u>, at least in 2d. It is simply the signed area of parallelogram.

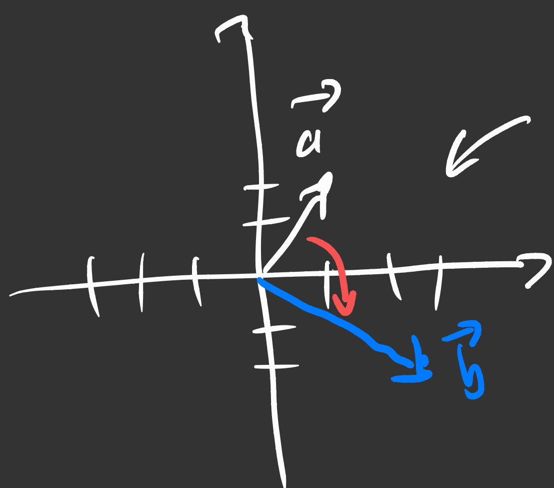
The usual area of parallelogram is

$$\left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$$ absolute value.

<u>Example</u> $\vec{a} = (1,2)$ $\vec{b} = (3,-2)$

$$A(\vec{a},\vec{b}) = 1\cdot(-2) - 2\cdot 3$$
$$= -8.$$

Since $A(\vec{a},\vec{b})$ is negative, it means that $\vec{a}$ and $\vec{b}$ form a left pair.



rotate clockwise, forms a left pair.

This is the concept leading to the cross product.