Conditions of potentiality of a vector field

$\vec{F} = (P, Q, R)$

How to know if it is potential?

Two problems:

1. Which properties are necessary for potentiality? Namely, what are the properties common to all potential fields?

2. Which conditions are sufficient? Namely, if a field has this property, then it is potential.

In particular, properties that are simpler than "integral along every closed curve is zero"

Suppose $F(x,y) = (P(x,y), Q(x,y))$ is potential.
This means that there is an f so that

$$P(x,y) = \frac{\partial f}{\partial x}(x,y) \qquad Q(x,y) = \frac{\partial f}{\partial y}(x,y)$$

Consider $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

But partial derivatives commute (Clairot thm)
so that $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus, given a field in the plane,
you can look at $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$.
If equal, the field is potential.
If not, then the field is not potential.

Ex: $P = e^x \cos y \qquad Q = e^x \sin y$

$\vec{F} = (P, Q)$

$$\frac{\partial P}{\partial y} = -e^x \sin y \qquad \frac{\partial Q}{\partial x} = e^x \sin x$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not potential.

But

$$P = e^x \sin y \qquad Q = e^x \cos y$$

$$\frac{\partial P}{\partial y} = e^x \cos y \qquad \frac{\partial Q}{\partial x} = e^x \cos y$$

They are equal, so $\vec{F} = (P, Q)$ is potential. The potential function is

$P = f_x$ and $Q = f_y$

$f = e^x \sin y$ works!

What happens in 3D? What is condition for potentiality?

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}, \quad R = \frac{\partial f}{\partial z}$$

From $\frac{\partial P}{\partial y} = f_{yx}$ $\quad \frac{\partial Q}{\partial x} = f_{xy}$

$$\Rightarrow \boxed{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0}$$

$$\frac{\partial P}{\partial z} = f_{zx} \quad \frac{\partial R}{\partial x} = f_{xz}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0}$$

$$\frac{\partial Q}{\partial z} = f_{zy} \quad \frac{\partial R}{\partial y} = f_{yz}$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0}$$

These identities are familiar, they are the three components of $\operatorname{curl}\vec{F}$!

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= \left(R_y - Q_z ,\ P_z - R_x ,\ Q_x - P_y \right)$$

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

<u>Conclusion:</u> If $\vec{F}$ is potential, then $\operatorname{curl}\vec{F} = 0$.

Note:

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

in 2D defines a vector field in 3D

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + 0\,\vec{k}.$$

What is curl?

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y).$$

Thus 3D condition for potentiality reduces to the 2D condition:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

__Def:__ A vector field $\vec{F}(\vec{r})$ is called irrotational if curl $\vec{F} = 0$.

__Theorem:__ Potential fields are irrotational.

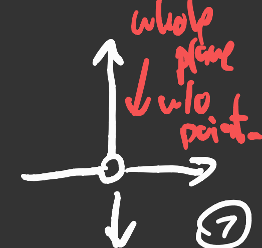
Is the opposite true?

To some extent, it is true (in general, not).
It depends on which domains we consider,
e.g. if the vector field is defined on all space, then yes. But in more complicated domains, it can be false.

__Def:__ Domain D in the plane is called __simply connected__ if it contains no holes.

e.g. D = whole plane, half plane, interior of circle.

NOT:



Simple but deep example

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Vector field is defined everywhere but at the origin. Thus, it's domain is not simply connected.

Is $\vec{F}$ irrotational? $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$

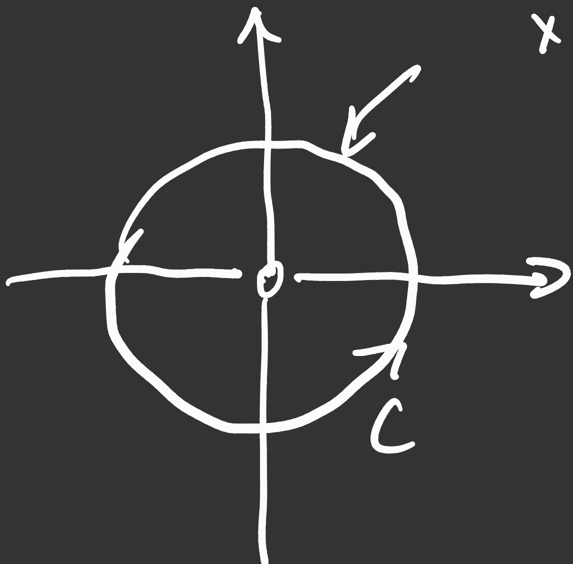
$$\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2} + \frac{y(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow$ irrotational.

Lets integrate



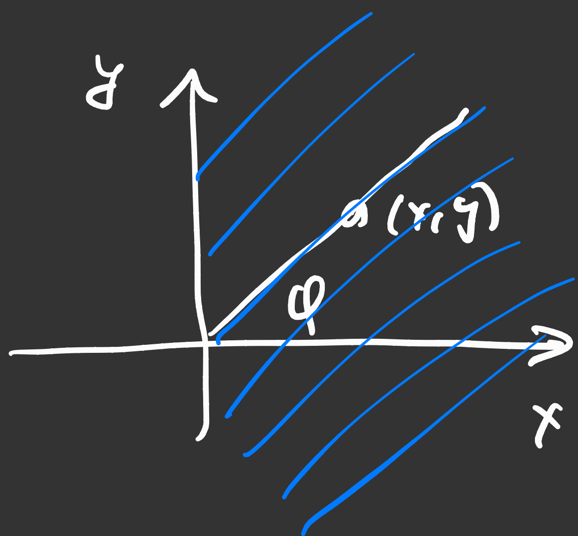
$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi.$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{(-\sin t)}{\sin^2 t + \cos^2 t}(-\sin t)\,dt$$

$$+ \int_0^{2\pi} \frac{(\cos t)\,(\cos t)}{\sin^2 t + \cos^2 t}\,dt = 2\pi \neq 0.$$

Something is wrong! If it were potential, this should be zero!

Consider a function that looks like it's potential:



$$\varphi(x,y) = \arctan\left(\frac{y}{x}\right) \qquad x > 0.$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = P$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = Q$$

Thus $\vec{F}(x,y) = \nabla \phi$ if $x > 0$.

If $y>0$

Now $\phi = \operatorname{arccot}\left(\frac{x}{y}\right)$ $y>0$.

One can check that

$$\vec{F}(x,y) = \nabla\phi \quad \text{for } y>0.$$

Thus, in right half plane, the potential is defined by

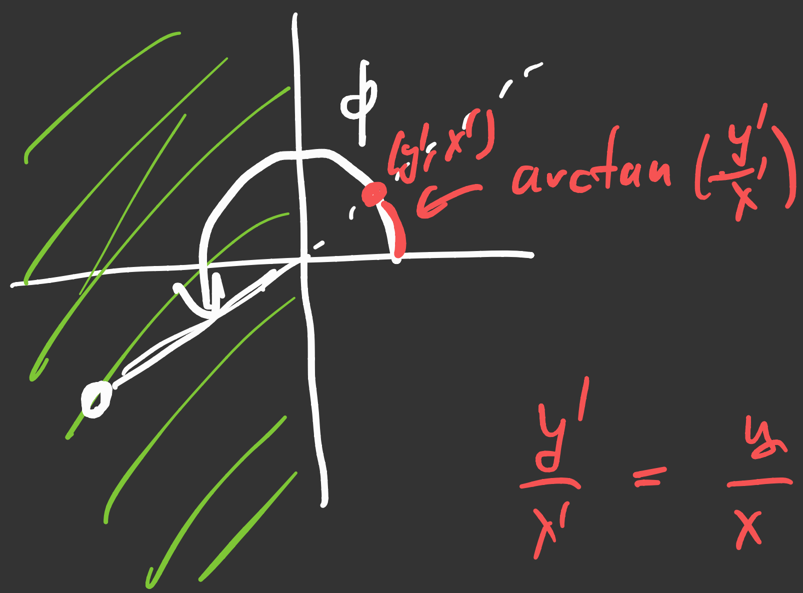
$$\phi = \arctan\left(\frac{y}{x}\right)$$

in upper half plane

$$\phi = \operatorname{arccot}\left(\frac{x}{y}\right)$$

In the overlapping region, both functions give same result.

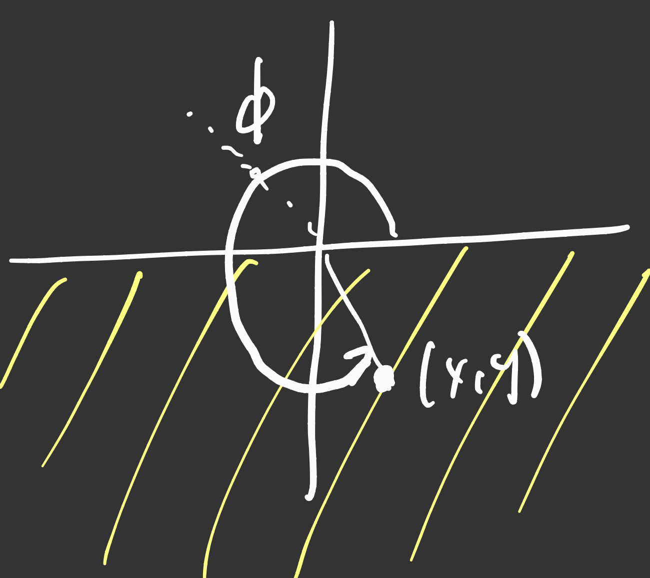
In the left half plane $(x<0)$



$$\frac{y'}{x'} = \frac{y}{x}$$

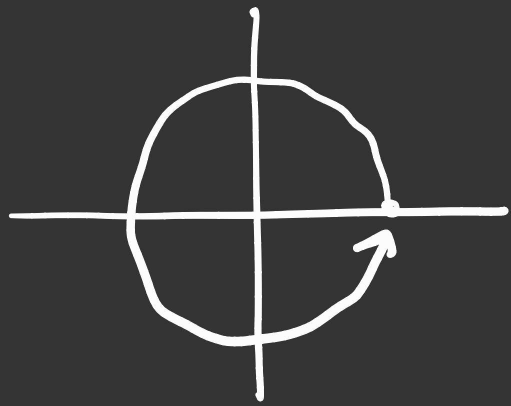
$$\phi = \arctan\left(\frac{y}{x}\right) + \pi. \qquad x < 0$$

In lower half plane $(y<0)$



$$\phi = \operatorname{arccot}\left(\frac{x}{y}\right) + \pi$$

Thus when we make a full turn



the angle becomes 2π (not 0).

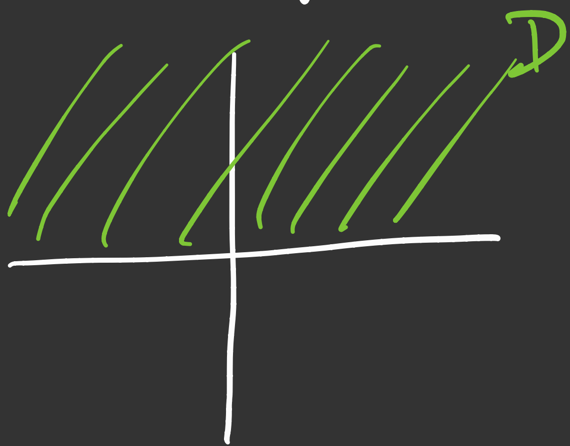
Thus, the "function" angle is not a single function. It grows from 0 to 2π in one rotation, and if you continue it keeps growing (if you want to keep it continuous).

Thus it is not a function, rather

$$\phi(x,y), \quad \phi(x,y) + 2n\pi \qquad n \in \mathbb{Z}$$

are all "branches" of the angle.

For it to be a well defined function, must restrict the domain so that φ is single valued, for instance



or

Domain should have no holes, so that φ is defined everywhere and would be continuously differentiable.

Theorem: If a field is irrotational in a simply connected domain, it is potential in this domain.

Problem Suppose $\vec{F}$ is irrotational field. How to find its potential $f(\vec{r})$?

Example $\vec{F}(x,y) = \left(4x^3+6x^2y,\ 2x^3+6y^2\right)$

$$P = 4x^3+6x^2y \qquad Q = 2x^3+6y^2$$

$$\partial_y P = 6x^2, \qquad \partial_x Q = 6x^2$$

$\Rightarrow$ X is irrotational defined on all the plane $\Rightarrow$ potential.

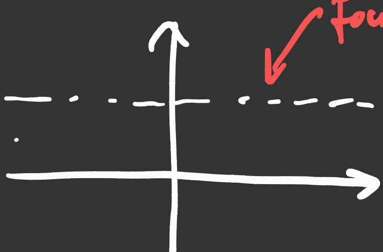
There exists f such that $P = \partial_x f$, $Q = \partial_y f$.

$$\frac{\partial f}{\partial x} = 4x^3 + 6x^2y$$

$$\Rightarrow \quad f(x,y) = \int (4x^3+6x^2y)\,dx + C$$

found f here

$$= x^4 + 2x^3y + C$$



So $C = g(y)$

$$f(x,y) = x^4 + 2x^3y + g(y)$$

Must find $g(y)$. Use...

$$\frac{\partial f}{\partial y} = Q$$

$$\frac{\partial f}{\partial y} = 2x^3 + g'(y) \quad \text{But} \quad Q = 2x^3 + 6y^2$$

here is where irrotationality appears. If done for an arbitrary vector field, will get

$g'(y) = h(x,y)$

inconsistent

Thus

$$2x^3 + g'(y) = 2x^3 + 6y^2$$

$$\Rightarrow \quad g'(y) = 6y^2$$

$$\Rightarrow \quad g(y) = \int 6y^2 + C = 2y^3 + C$$

Thus

$$f(x,y) = x^4 + 2x^3y + 2y^3 + C$$

arbitrary, since P, Q don't depend.

3D examples:

$$\vec{F}(x,y,z) = (3x^2y,\ x^3+6y^2z,\ 2y^3-3z^2)$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 3x^2y & x^3+6y^2z & 2y^3-3z^2 \end{vmatrix} = (6y^2-6y^2,\ 0,\ 3x^2-3x^2) = (0,0,0)$$

Thus $\vec{F}$ is irrotational. There exists f such that

$$f_x = 3x^2y \Rightarrow f = x^3y + g(y,z) \qquad ①$$

$$\Downarrow$$

$$f_y = x^3+6y^2z \qquad f_y = x^3 + g_y$$

$$\Downarrow$$

$$f_z = 2y^3-3z^2 \qquad g_y = 6y^2z$$

$$\Downarrow$$

$$g(y,z) = 2y^3z + h(z)$$

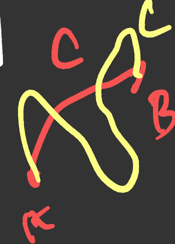
$$f = x^3y + 2y^3z + h(z)$$

$$f_z = 2y^3 + h'(z) \Rightarrow h'(z) = -3z^2$$

$$h = z^3 + C$$

$$\boxed{f(x,y,z) = x^3y + 2y^3z + z^3 + C}$$

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$



This comes up in ODE: exact equations.