**Potential fields** : vector fields which are gradients of functions

$$\vec{F}(\vec{r}) = \nabla f(\vec{r})$$

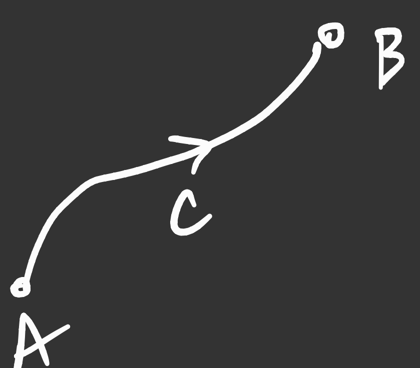
$$\vec{F}(x,y) = (f_x(x,y), f_y(x,y))$$

The function $f(x,y)$ is called the **potential**.

**Ex:** $\vec{F}(x,y) = (x,y)$ $\qquad f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\nabla f(x,y) = (x,y) = \vec{F}$$

**Observation:**




$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$

To see why this is, let's introduce a parametrization of

$$C: \quad \vec{r} = \vec{r}(t) \qquad a \leq t \leq b$$

$$\vec{r}(a) = A \qquad \vec{r}(b) = B$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\vec{r}(t) = (x(t), y(t))$$

$$= \int_a^b \left( x'(t) f_x(x(t), y(t)) + y'(t) f_y(x(t), y(t)) \right) dt$$

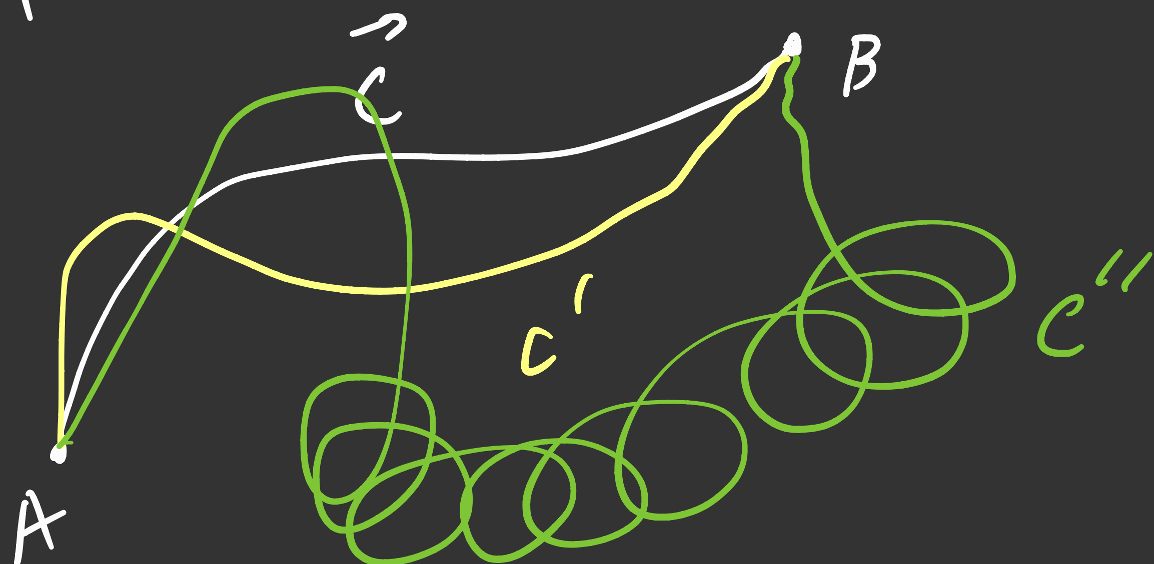
by chain rule

$$= \int_a^b \frac{d}{dt} f(x(t), y(t))\, dt$$

fundamental theorem of calculus : differentiation is inverse to integration

$$= f(B) - f(A)$$

Thus, as soon as we know the potential for a potential vector field, we immediatly know the work done by it along a curve by evaluating the potential at two end points.




$$\oint_C \vec{F}\cdot d\vec{r} = f(B) - f(A), \qquad \vec{F} = \nabla f$$

$$\parallel$$

$$\int_{C'} \vec{F}\cdot d\vec{r}$$

Thus $\oint_C \vec{F}\cdot d\vec{r}$ does not depend on $C$, provided $C$ connects A and B.

In 3d, the same holds.

$$\vec{F}(\vec{r}) = \nabla f(\vec{r}) = (f_x, f_y, f_z)$$

$\oint_C \vec{F} \cdot d\vec{r}$ depends only on A and B.

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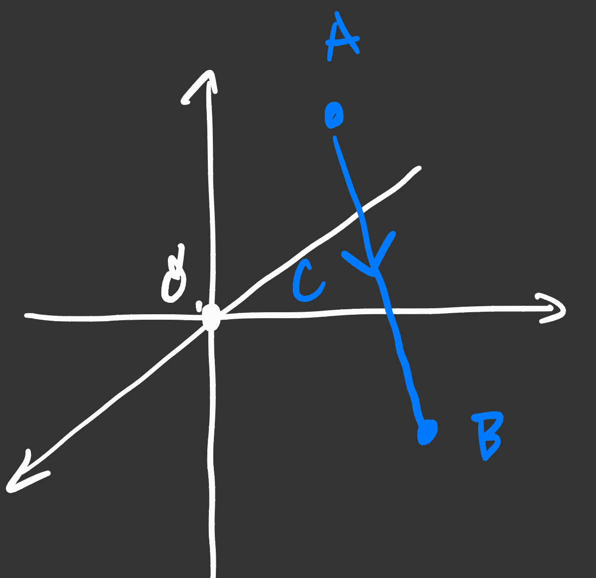
**Example:** $\vec{F}(\vec{r}) = -\dfrac{\vec{r}}{\|\vec{r}\|^3} = \left( \dfrac{-x}{\sqrt{x^2+y^2+z^2}}, \dfrac{-y}{\sqrt{x^2+y^2+z^2}}, \dfrac{-z}{\sqrt{x^2+y^2+z^2}} \right)$




$\oint_C \vec{F} \cdot d\vec{r} = ?$

↑ looks hard...

But: consider $f(r) = \dfrac{1}{\|\vec{r}\|} = (x^2+y^2+z^2)^{-1/2}$.

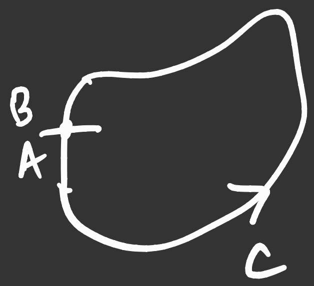
$$\nabla f = (f_x, f_y, f_z) = \left( \frac{-x}{\sqrt{x^2+y^2+z^2}}, \frac{-y}{\sqrt{x^2+y^2+z^2}}, \frac{-z}{\sqrt{x^2+y^2+z^2}} \right) = \vec{F} \;!$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = \frac{1}{|\vec{OB}|} - \frac{1}{|\vec{OA}|}$$

# Integral along a closed curve

$$\vec{F} = \nabla f$$

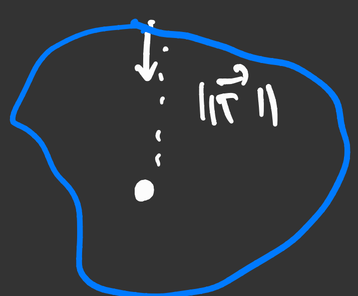



$$\oint_C \vec{F} \cdot dr = f(B) - f(A)$$

$$= 0 \quad \text{as } A = B.$$

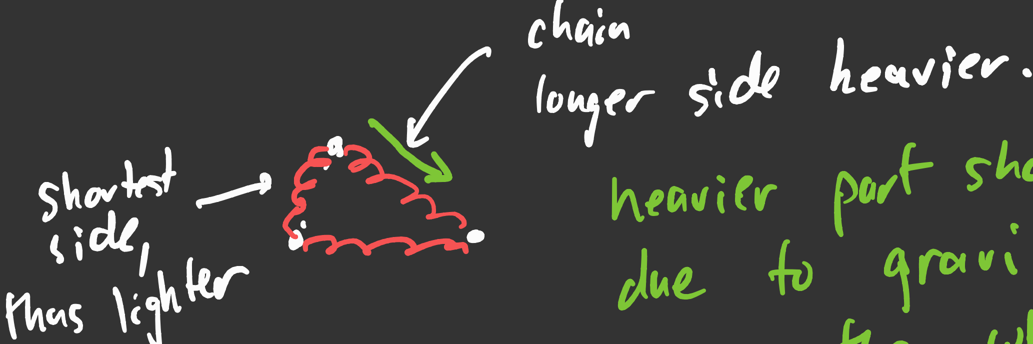
Hence $\oint_C \vec{F} \cdot dr = 0$, integral of a potential field over a closed curve is zero.

Note: $\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$




Work done moving a body along a closed curve in a gravitational field is zero.

Most "perpetual motion" machines are based on the idea of moving a body in a field on a closed curve and hope to get some work done. For example




heavier part should slide down due to gravity and cause the whole chain to turn and to produce some work (by gravitation).

But, according to our theorem, the gravitational work done here is zero!

Nothing will be done. Oh well...

Suppose the vector field $\vec{F}(\vec{r})$ is such that

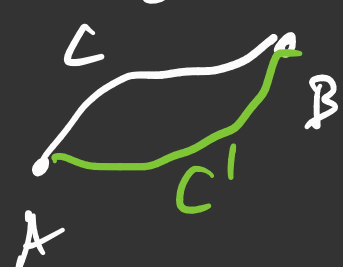
$$\oint_C \vec{F} \cdot d\vec{r} = 0 \quad \text{for any closed curve } C.$$

Then $\vec{F}$ is potential.

This is the reverse theorem to what we now know, that $\oint_C \nabla f \cdot d\vec{r} = 0$.

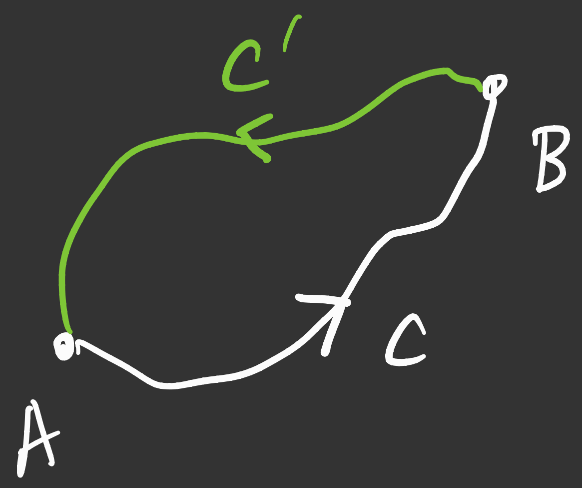
Proof: (1) The above field $\vec{F}$ is such that $\int_C \vec{F} \cdot dr$ is path independent

e.g.




$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$

Indeed, to see this consider the following curve




D = C and C' together

Call that closed curve from A → A, D.
Now, since D is a closed path, so

$$\oint_D \vec{F} \cdot d\vec{r} = 0$$

changes sign because orientation reversed

but

$$\oint_D \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} - \int_{C'} \vec{F} \cdot d\vec{r}$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}.$$

C and C' are arbitrary, thus integral does not depend. ✓

(2) We want to prove that $\vec{F}$ is potential. To do this, we must produce some potential function, $f$.

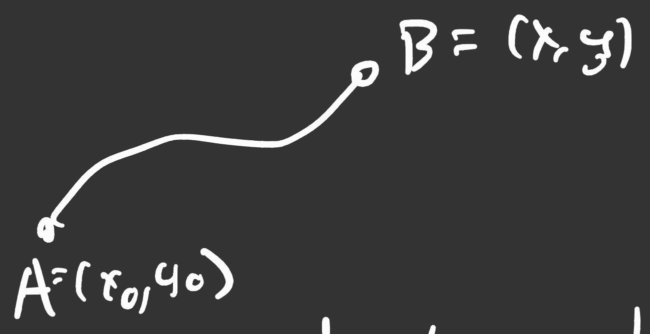
Define the function $f(\vec{r})$ as follows.

- fix $A = (x_0, y_0)$ and vary $B = (x, y)$

define

$$f(x,y) = f(B) = \int_C \vec{F} \cdot d\vec{r}$$

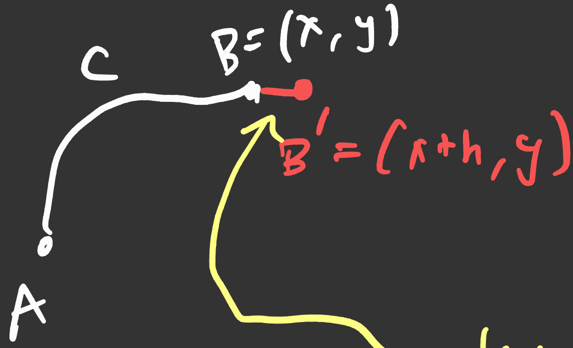
where $C$ is any curve from $A$ to $B$.




By (1), this integral does not depend on the curve, only on the point $B$. Thus it is well defined.

Suppose $\vec{F}(x,y) = (P(x,y), Q(x,y))$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left[ P(x(t),y(t))\, x'(t) + Q(x(t),y(t))\, y'(t) \right] dt$$




$$f(B') = \int_a^b \vec{F}\cdot d\vec{r} + \int_b^{b+h} \vec{F}\cdot d\vec{r}$$

where we move along the segment with unit speed.
$x'(t) = 1$

$$\int_b^{b+h} P(x(t),y(t))\,dt$$

Thus

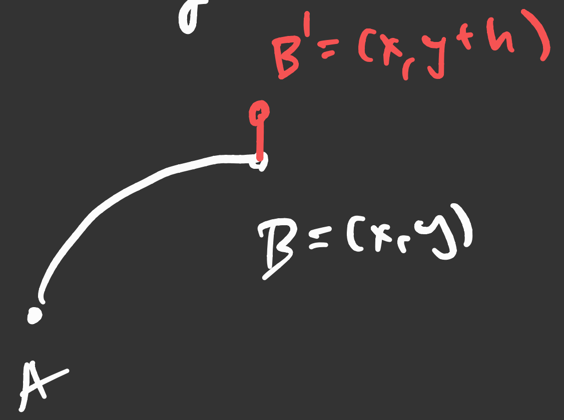
$$f(B') = f(B) + \int_b^{b+h} P(x(t),y(t))\,dt$$

$$\approx f(B) + P(B)\,h$$

But

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(B') - f(B)}{h} = P(B) = P(x,y)$$

Similarly




we find $\dfrac{\partial f}{\partial y} = Q(x,y)$

Thus $\nabla f = (P(x,y), Q(x,y))$

$= \vec{F}$

and we found our potential field by computing the integral of $\vec{F}$ along any path connecting $A$ and $B = (x,y)$.

Two questions

- How do you know if a field is potential.
- How to find the potential function.