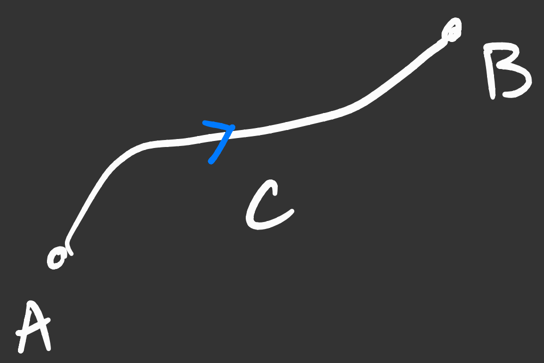
Line Integrals.



C is an oriented curve, going from A to B.

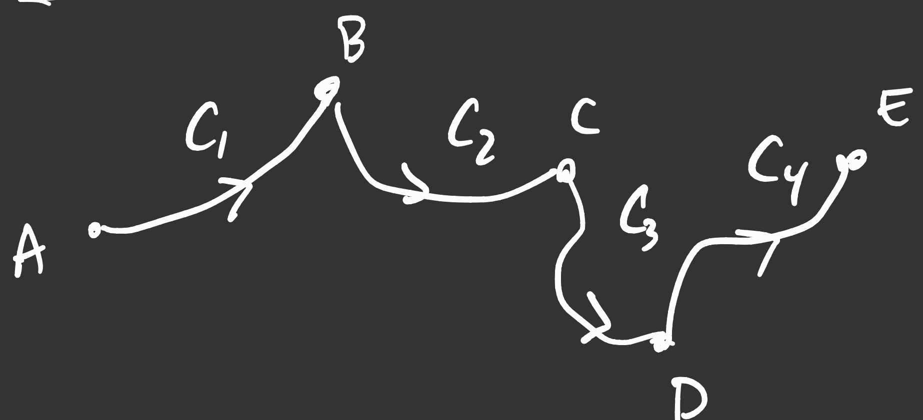
$$C: \quad x = x(t), \quad y = y(t), \quad z = z(t)$$

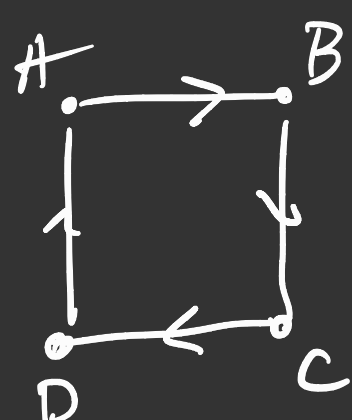
x, y, z differentiable and

moving along curve with positive speed. $\rightarrow (x'(t))^2 + (y'(t))^2 + (z'(t))^2 > 0$

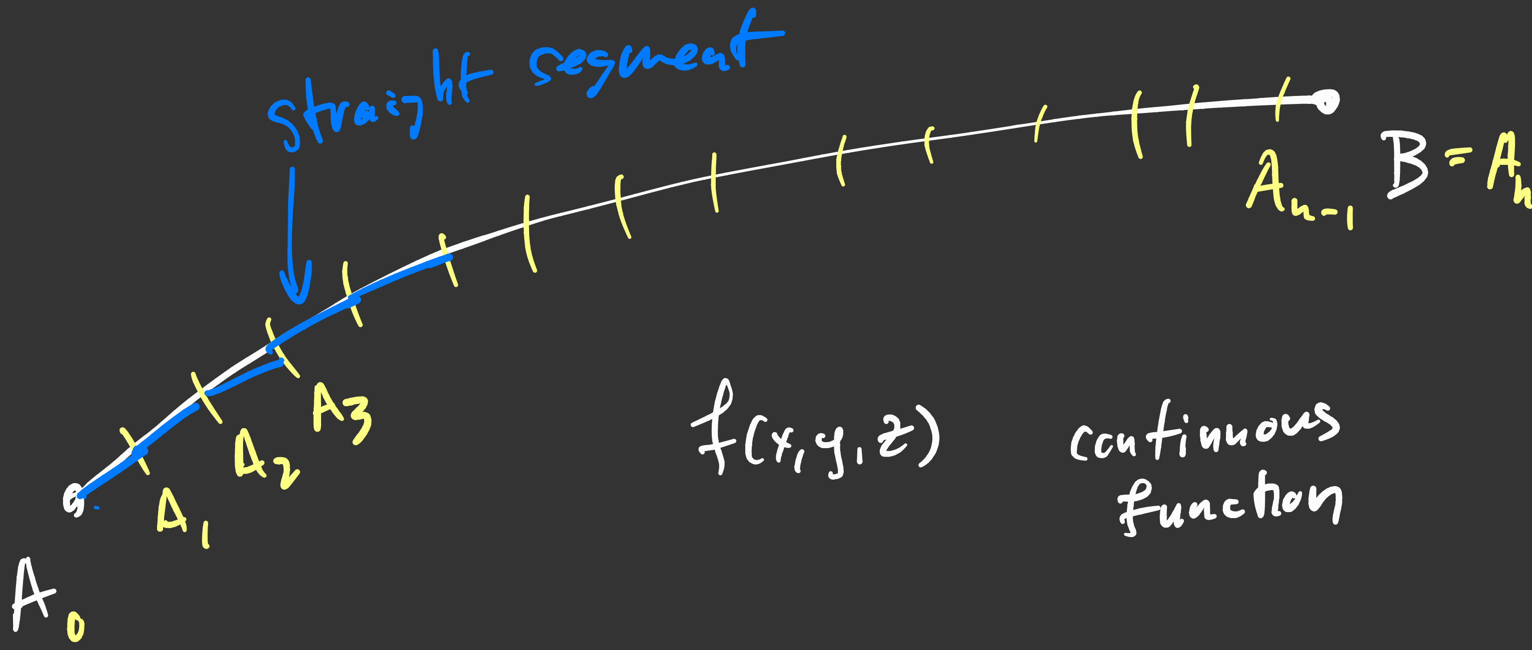
At every point, there is a tangent line.

Piecewise regular curve





We now define the line integral



$f(x,y,z)$ continuous function

$$\int_C f(x,y,z)\,ds$$

length of straight segment b/w A_0 and A_1

$$S_N = f(A_0)\,|A_0A_1| + f(A_1)\,|A_1A_2| + \dots + f(A_{n-1})\,|A_{n-1}A_n|.$$

If number of points $n \to \infty$, then

$$\max_{k \le N} |A_kA_{k+1}| \to 0 \quad \text{and} \quad S_N \to \int_C f\,ds.$$

To find it analytically, we introduce a parametrization:

$$C: \vec{r} = \vec{r}(t), \quad a \le t \le b.$$

$$A_k = \vec{r}(t_k) \quad \text{for some } t_k$$

Then

$$S_N = f(\vec{r}(t_0)) \, |\vec{r}(t_1) - \vec{r}(t_0)| + f(\vec{r}(t_1)) \, |\vec{r}(t_2) - \vec{r}(t_1)| + \dots$$

When $n \to \infty$,

$$\|\vec{r}(t_{k+1}) - \vec{r}(t_k)\| \approx \|\vec{r}\,'(t_k)\| \, (t_{k+1} - t_k)$$

so

$$\boxed{S_N \to \int_a^b f(\vec{r}(t)) \, \|\vec{r}(t)\| \, dt}$$

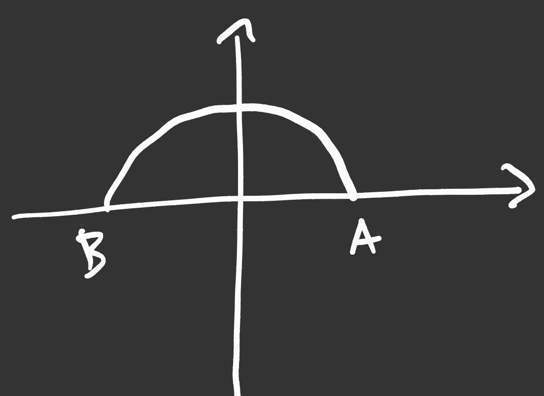
$$= \int_C f \, ds$$

Thus

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt.$$

Rare that integral can be found explicitly.

Example. $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$

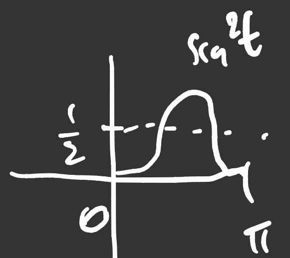


$f(x,y) = y^2$

$$\int_C y^2 \, ds = \int_0^{\pi} \sin^2(t) \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt$$

$$= \int_0^{\pi} \sin^2(t) \, dt = \int_0^{\pi} \frac{1}{2}(1 - \cos(2t)) \, dt$$

$$= \frac{\pi}{2}.$$

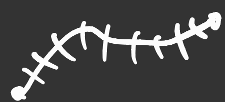


$$\sin^2 t = \frac{1}{2}(\sin^2 + \cos^2 t - \cos^2 t + \sin^2 t)$$

$$= \frac{1}{2}(1 - \cos(2t))$$

This integral itself is not very useful, that is not many problems require it. In fact

$\int_C f \, ds$ does not require parametrization of the curve C, or orientation.

The construction  does not require it. However, to analytically compute it, we use it.

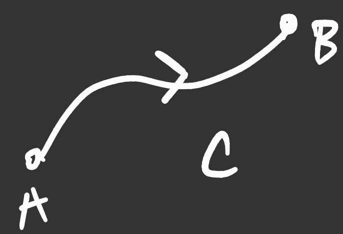
Line integral is invariant under change of parametrization, and change of orientation.

A much more important integral is...

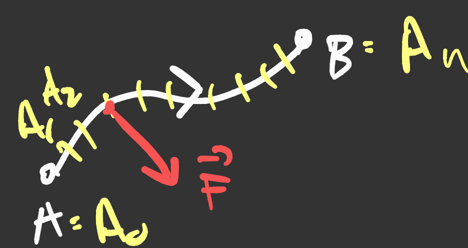
Integral of a vector field along a curve

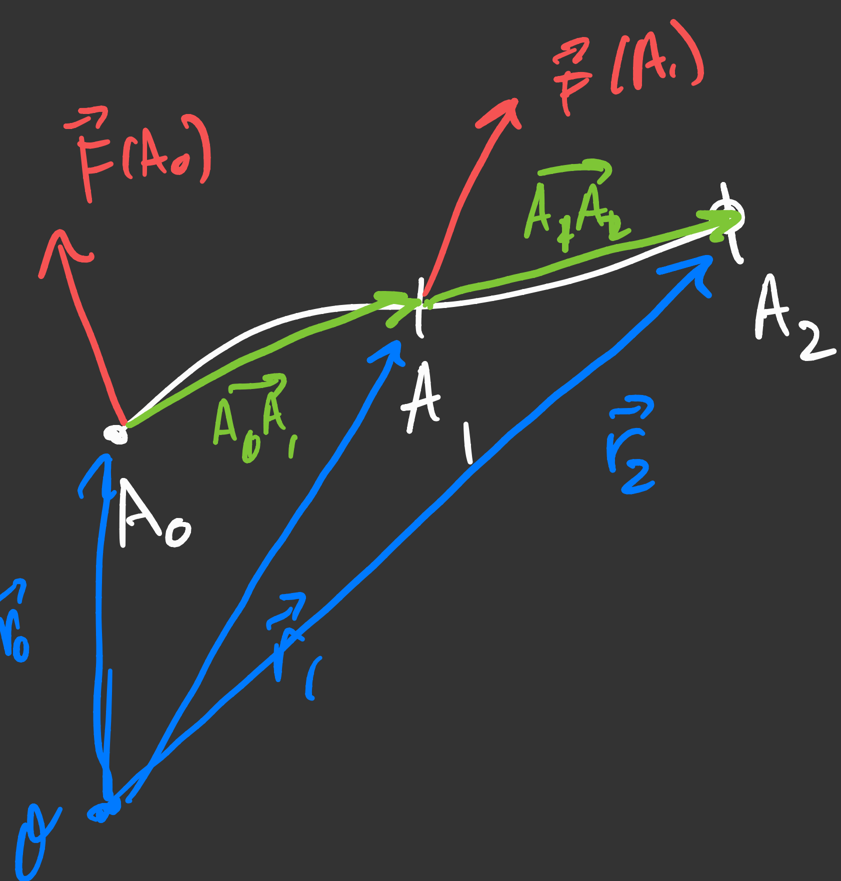
Given a vector field $\vec{F} = \vec{F}(x,y,z)$
$= (P(x,y,z), Q(\dots), R(\dots))$.

and a oriented curve C.



Now consider the following construction.
At each point, there is a vector





Consider the sum:

$$S_N = \vec{F}(\vec{r}_0)\cdot(\vec{r}_1-\vec{r}_0) + \vec{F}(\vec{r}_1)\cdot(\vec{r}_2-\vec{r}_1) + \dots + \vec{F}(\vec{r}_{N-1})\cdot(\vec{r}_N-\vec{r}_{N-1}).$$

Consider $N \to \infty$, so distances b/w points go to zero,

$$\max_{k<N} \|\vec{r}_{k+1}-\vec{r}_k\| \to 0$$

If the curve and vector field are regular, the limit exists and is denoted

$$\lim_{N\to\infty} S_N = \int \vec{F}\cdot dr.$$

To find a formula, introduce a parametrization

$$C: \quad \vec{r} = \vec{r}(t) \qquad a \leq t \leq b$$

$$S_N = \vec{F}(\vec{r}(t_0))\,(\vec{r}(t_1)-\vec{r}(t_0)) + \dots + \vec{F}(\vec{r}(t_{N-1}))\,(\vec{r}(t_N)-\vec{r}(t_{N-1}))$$

$$\approx \vec{F}(\vec{r}(t_0))\cdot\vec{r}\,'(t_0)\,(t_1-t_0) + \dots + \vec{F}(\vec{r}(t_{N-1}))\cdot\vec{r}\,'(t_{N-1})\,(t_N-t_{N-1}).$$

Thus

$$\lim_{N\to\infty} S_N = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Thus

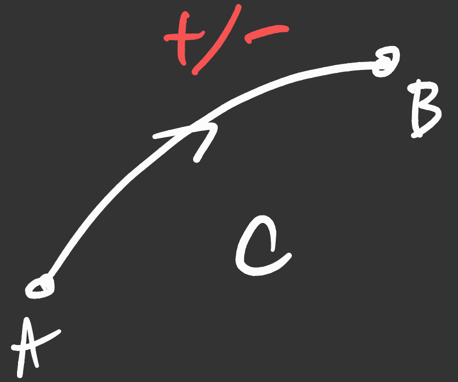
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

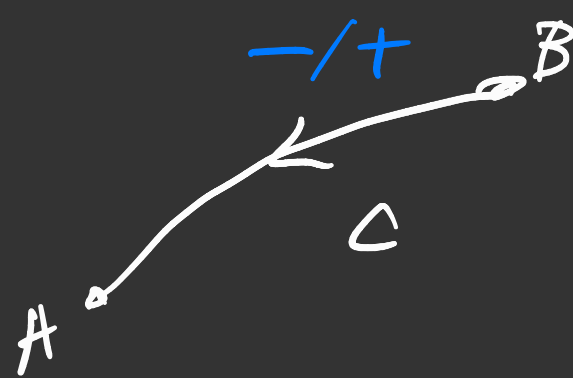
Note, again, that since the sums S_N were defined without any parametrization, the limit also is <u>independent of parametrization</u>

However, <u>orientation does enter.</u>

If you go from B to A instead of A → B, the factors $\vec{r}\,'(t_{N-1})\,(t_N - t_{N-1})$ change direction, so the integral changes sign.

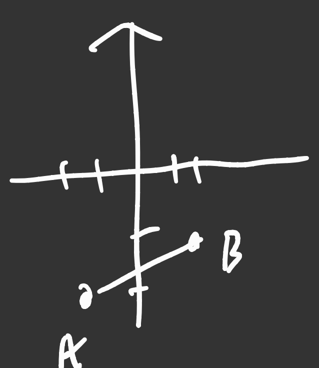
$\int_C \vec{F} \cdot dr$ changes sign with the change of orientation of C.





But independent of parametrization, e.g. $\vec{r}(t^2)$ instead of $\vec{r}(t)$.

<u>Example:</u> $A = (-1,-2)$, $B = (2,-1)$, $\vec{F}(x,y) = (x,y)$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (x(t),y(t)) \cdot (3,1)\,dt$$
$$= \int_0^1 (-1+3t, -2+t)(3,1)\,dt$$
$$= \int_0^1 [(-1+3t)3 + (-2+t)1]\,dt$$
$$= \int_0^1 [-5+10t]\,dt = 5-5 = 0$$

$\vec{r}(t) = \vec{r}_0 + t\,\vec{AB} = (-1,-2) + t(3,1)$

$\vec{F}(0) = A$, $\vec{r}(1) = B$ $\qquad \vec{AB} = (2-(-1), -1-(-2)) = (3,1)$

What is the meaning of this integral?

$$\int_C \vec{F} \cdot d\vec{r} = $$ Work done by the Force, $\vec{F}$, upon a body moving along curve C in direction corresponding to orientation.

For example $\vec{F}$ could be gravitational force. In general $\vec{F}(\vec{r})$ is the force applied to the body if it is positioned at $\vec{r}$.

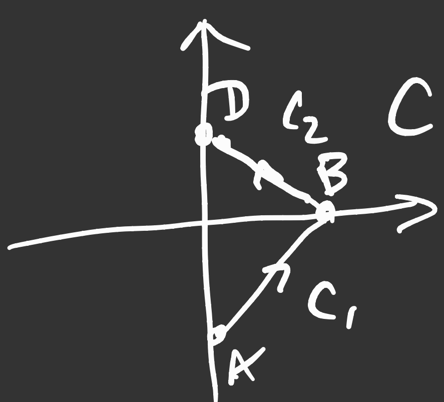
<u>Orientation:</u> If you lift a body in a gravitational field, you perform work opposite in sign to the work done by the field if the body falls down.

If $\vec{F}(\vec{r}) = (P, Q, R)$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (P, Q, R) \cdot \underbrace{d\vec{r}}_{(x'(t),\, y'(t),\, z'(t))\, dt}$$

$$= \int_C \left(P(\vec{r})\, dx + Q(\vec{r})\, dy + R(\vec{r})\, dz \right)$$

<u>Example:</u> $A = (0, -1)$ $B = (1, 0)$ $D = (0, 1)$



$$\int_C \left(y\, dx + x^2\, dy \right)$$

$$= \left(\int_{C_1} + \int_{C_2} \right) (\cdots)$$

Introduce parametrization:

C_1: $\begin{array}{l} x(t) = t \\ y(t) = t-1 \end{array}$ $\quad 0 \le t \le 1$

C_2: $\begin{array}{l} x(t) = 1-t \\ y(t) = t \end{array}$ $\quad 0 \le t \le 1$

$$\int_{C_1} (y\,dx + x^2\,dy) = \int_0^1 \left((t-1)\,dt + t^2\,dt\right)$$

$$= \frac{1}{2} - 1 + \frac{1}{3} = -\frac{1}{6}.$$

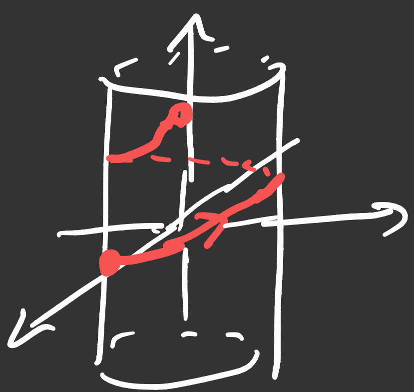
$$\int_{C_2} (y\,dx + x^2\,dy) = \int_0^1 t(-1)\,dt + (1-t)^2\,dt$$

$$= -\frac{1}{2} + 1 - 1 + \frac{1}{3} = -\frac{1}{6}.$$

$$\int_{C} (y\,dx + x^2\,dy) = -\frac{1}{6} - \frac{1}{6} = -\frac{1}{3}.$$

Ex: $C:\ \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \qquad 0 \le t \le 2\pi.$

Helix



$$\vec{F}(x,y,z) = (z, x, y)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{(t, \cos t, \sin t)}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{(-\sin t, \cos t, 1)}_{dr}\, dt$$

$$= \int_0^{2\pi} \left(-t\sin t + \cos^2 t + \sin t\right) dt$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

$$\mathrm{I} = \int_0^{2\pi} -t\sin t\, dt = \int_0^{2\pi} t \frac{d}{dt}\cos t\, dt$$

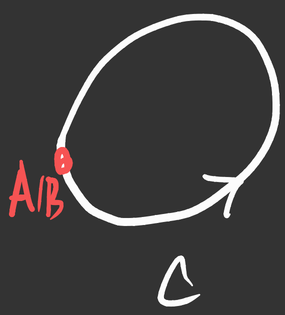
$$= -\int_0^{2\pi} \cos t\, dt + t\cos t\Big|_0^{2\pi} = 2\pi.$$

$$\mathrm{II} = \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt = \pi.$$

$$\mathrm{III} = \int_0^{2\pi} \sin t\, dt = 0.$$

Thus $\boxed{\int_C F\cdot dr = 3\pi}$

Integral of a vector field along a closed curve



no objective first or last points
Pick point A/B arbitrarily and call it the first and last.

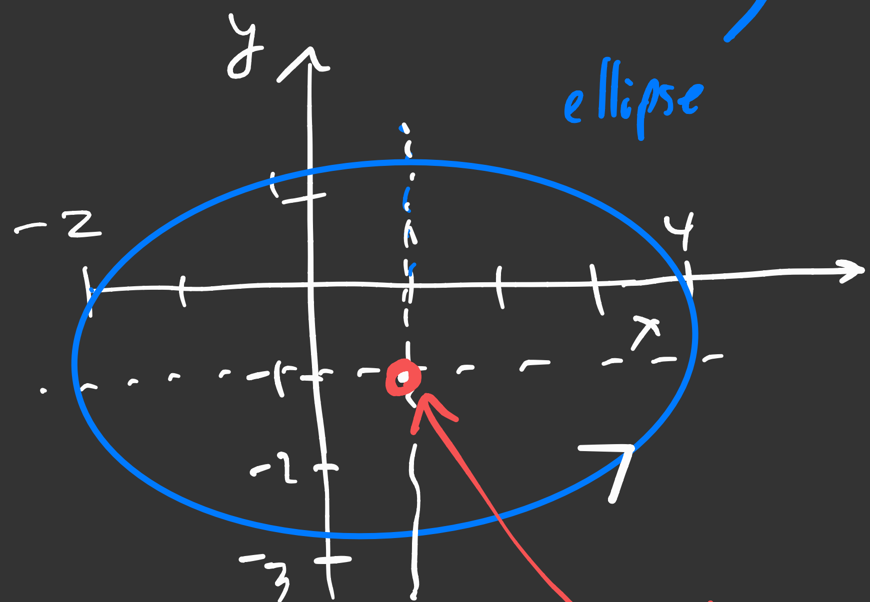
We may define

$$\int_C \vec{F}\cdot d\vec{r} \equiv \oint_C \vec{F}\cdot d\vec{r}$$

notation for integral over closed curve / contour

Again, this definition does not depend on any parametrization, but it depends on the direction of orientation (must choose proper parametrization coherent with the given orientation of the curve).

Example: $C: \frac{(x-1)^2}{9} + \frac{(y+1)^2}{4} = 1$ with counterclockwise parametrization



center at $(1,-1)$

parametrization $\vec{r}(t) = (x(t), y(t))$ where

$x(t) = 1 + 3\cos t$ $\quad 0 \le t \le 2\pi$

$y(t) = -1 + 2\sin t$

$$\vec{F} = (x+y, 2x-y)$$

$$\vec{F}(\vec{r}(t)) = (1+3\cos t + (-1+2\sin t), 2(1+3\cos t) + 1 - 2\sin t)$$

$$= (2\sin t + 3\cos t, 3 + 6\cos t - 2\sin t)$$

$$\vec{r}'(t) = (-3\sin t, 2\cos t)$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$$

$$= -6\sin^2 t - 9\sin t\cos t + 6\cos t$$

$$+ 12\cos^2 t - 4\sin t\cos t$$

$$= 12\cos^2 t - 6\sin^2 t - 13\sin t\cos t + 6\cos t$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(12\cos^2 t - 6\sin^2 t - 13\sin t\cos t + 6\cos t\right) dt$$

Note

$$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \sin^2 t\, dt = \pi.$$

$$\int_0^{2\pi} \cos t\, dt = 0 \qquad \int_0^{2\pi} \sin t\cos t\, dt = \int_0^{2\pi} \frac{1}{2}\sin 2t\, dt = 0$$

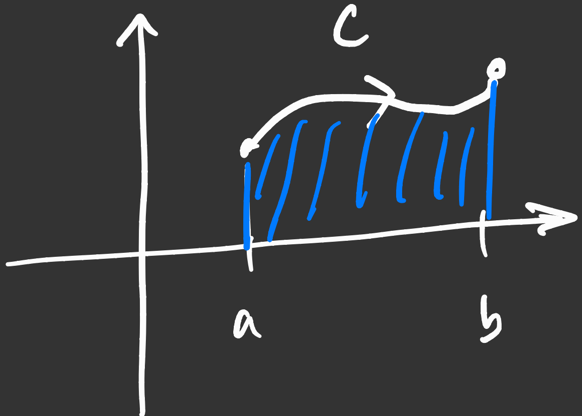
So

$$\oint_C \vec{F}\cdot d\vec{r} = (12-6)\pi = 6\pi.$$

Area inside a closed curve

$C: \quad y = f(x)$

$x = t, \quad y = f(t)$

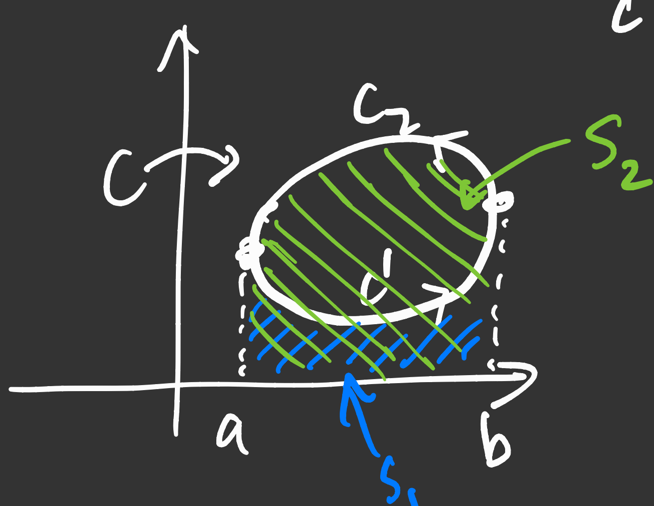


area below the graph

$$\int_C y\,dx = \int_a^b f(t)\,dt$$

Now consider

$$\oint_C y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx$$



$$= S_1 - S_2$$

area under C_1

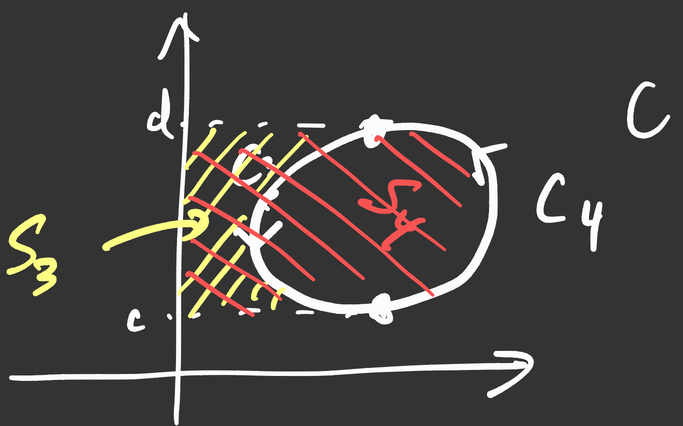
parametrization switched direction

area under C_2

Thus

$$\oint_C y\,dx = -(\text{Area inside } C)$$

↑ if orientation is c.c.w.



$$\oint_C x\,dy = \int_{C_3} x\,dy + \int_{C_4} x\,dy$$

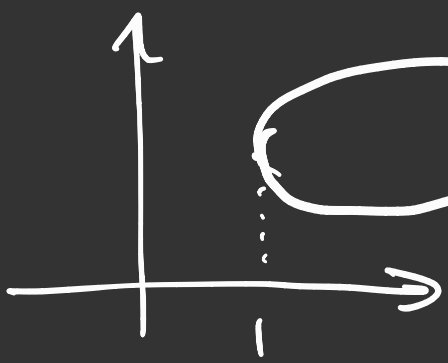
orientation

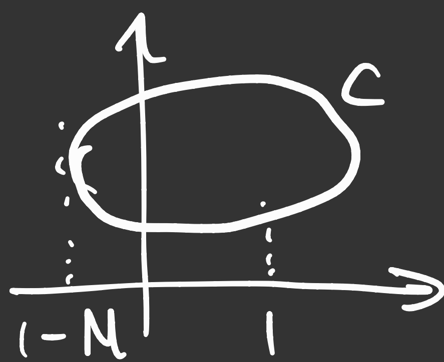
$$= -S_3 + S_4$$

$$= \text{Area enclosed by } C.$$

↑ if orientation is c.c.w

Note that the area does not change if the curve is shifted. Suppose





e.g. shift C over by M in x.

Then

$$\int_C x\,dy = \int_C (x-M)\,dy + \underbrace{\cancel{\int_C M\,dy}}_{0}$$

since curve is closed so no net change in y.

Ex: Area inside an ellipse

$$x = 1 + 3\cos t \qquad 0 \le t \le 2\pi$$
$$y = -1 + 2\sin t$$

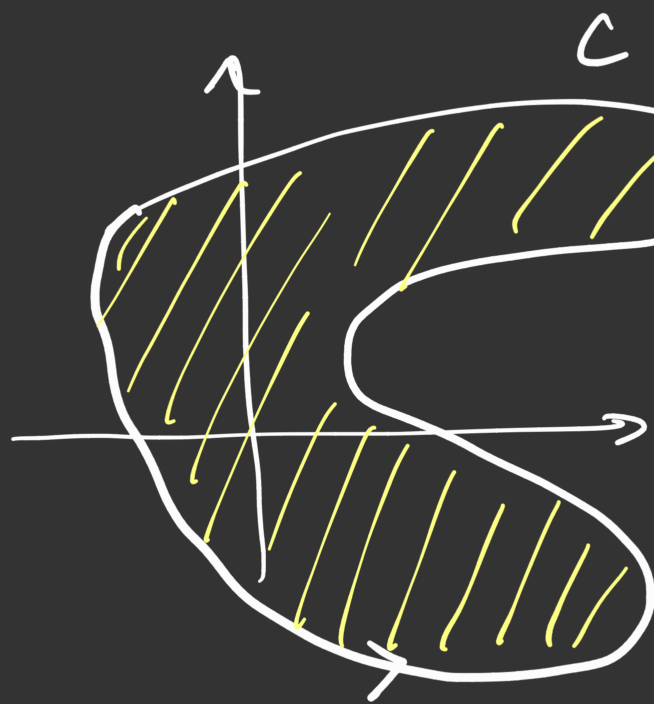
$$S = \oint_C x\,dy = \int_0^{2\pi} (1+3\cos t)(2\cos t)\,dt$$
$$= \int_0^{2\pi} (2\cos t + 6\cos^2 t)\,dt$$
$$= 6\pi.$$

This is correct, since the ellipse is a unit circle expanded in the horizontal by 3 and then in the vertical by 2.

Thus

$$(\text{Area of ellipse}) = \left(\begin{array}{c}\text{Area of circle}\\ \text{of radius } 1\end{array}\right)(2)(3)$$
$$= \pi \cdot 2 \cdot 3 = 6\pi.$$

We could consider more interesting domains

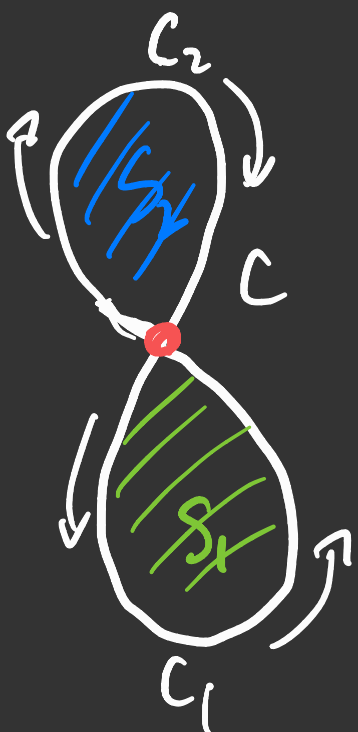


Claim:

$$\int_C x\,dy = \text{Area inside } C.$$

Not so obvious, since now we must break up curve in different regions to find pieces of the area, but it works.

$$\oint_C x\,dy = \text{Area inside curve}$$



How to find area?

$$\oint_C x\,dy = \oint_{C_1} + \oint_{C_2}$$

$$= S_1 - S_2$$

opposite orientation for C_2.

If curve is symmetric, $S_1 = S_2$, then this vanishes. In general $\oint_C x\,dx$ gives signed area!