∇ (Nabla) $\nabla = (\partial_x, \partial_y, \partial_z)$

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f) = (f_x, f_y, f_z)$$ Gradient

scalar function

$$\nabla f = \text{grad} f$$

vector field $\vec{u}(\vec{r}) = (P(\vec{r}), Q(\vec{r}), R(\vec{r}))$

$$\nabla \cdot \vec{u} = \partial_x P(\vec{r}) + \partial_y Q(\vec{r}) + \partial_z R(\vec{r})$$

Divergence

$$= \text{div}\, \vec{u}$$

notation for same thing

What is $\nabla \times \vec{u}$?

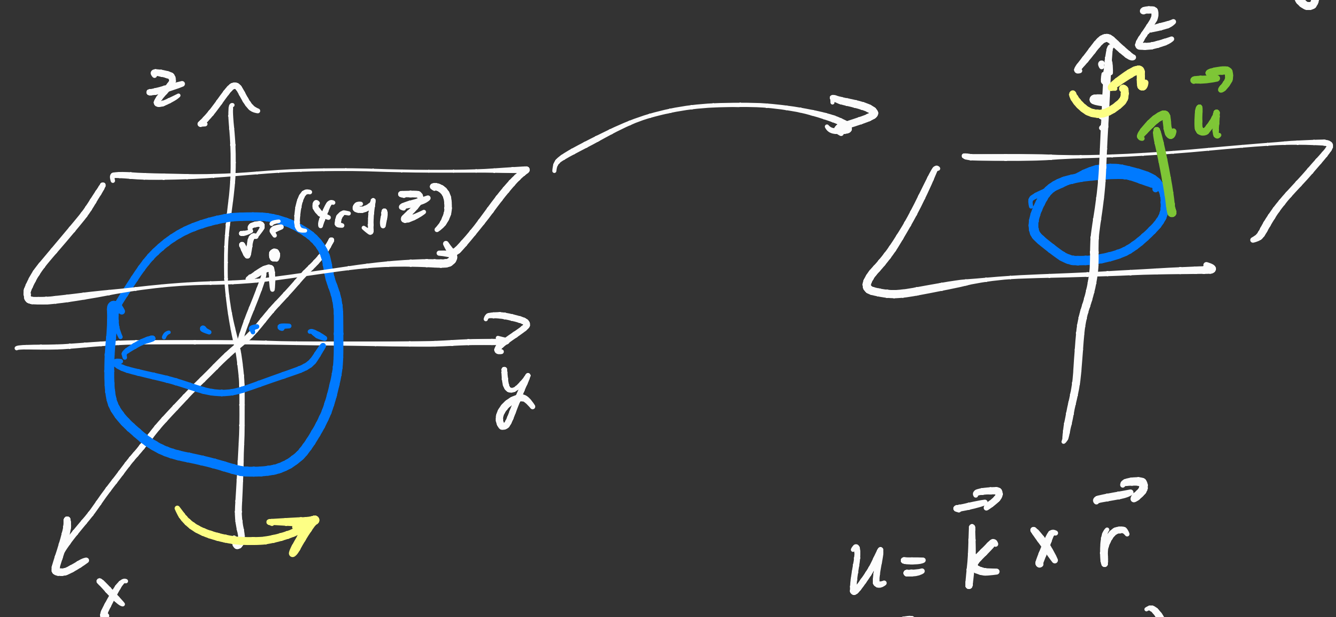
$$\nabla \times \vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= (\partial_y R - \partial_z Q)\vec{i} - (\partial_x R - \partial_z P)j + (\partial_x Q - \partial_y P)\vec{k}$$

$$= (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

$$\boxed{\nabla \times \vec{u} = \mathrm{curl}\,\vec{u}}$$

<u>Example:</u> Velocity field of rotation of a solid body



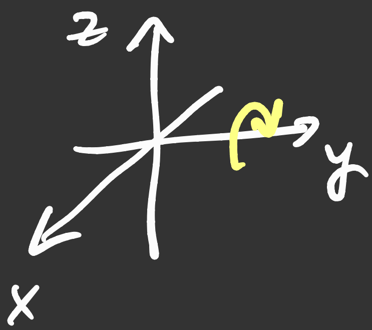
$$u = \vec{k} \times \vec{r}$$

$$\vec{k} = (0, 0, 1)$$

$$\vec{u} = \vec{k} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$$

$$\mathrm{curl}\,\vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0, 0, 2) = 2\vec{k}$$

If it rotates around a different axis



$$\vec{u}(\vec{r}) = \vec{j} \times \vec{r}$$

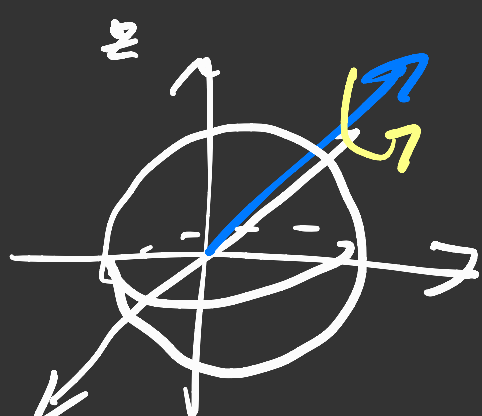
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z, 0, -y)$$

$$\operatorname{curl} \vec{u}(\vec{r}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & 0, & -x \end{vmatrix} = (0, 2, 0)$$
$$= 2\vec{j}$$

Around x-axis:

$$\vec{u}(\vec{r}) = \vec{i} \times \vec{r} \qquad \operatorname{curl} \vec{u} = 2\vec{i}.$$

Axis of rotation in direction $\omega = (\omega_x, \omega_x, \omega_z)$



with angular rotation speed

$\|\omega\|$

Then the velocity is

$$\vec{u}(\vec{r}) = \vec{\omega} \times \vec{r}$$

and

$$\operatorname{curl} \vec{u}(\vec{r}) = 2\vec{\omega}.$$

check yourselves!

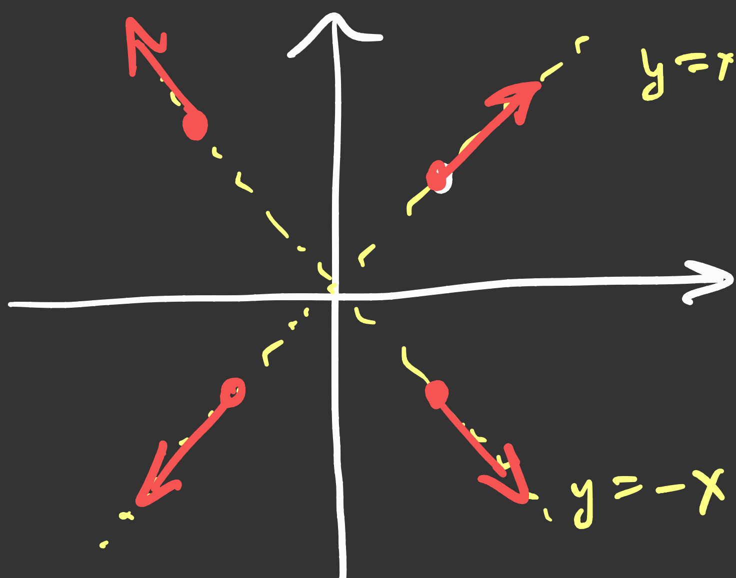
Consider now

$$\vec{u}(\vec{r}) = (y, x, 0)$$

$$\nabla \times \vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & x & 0 \end{vmatrix} = (0, 0, 1-1)$$

$$= (0, 0, 0)$$

This vector field has curl zero!



Consider a function $f(\vec{r}) = f(x,y,z)$

and its gradient $\nabla f(\vec{r}) = (f_x, f_y, f_z)$

$= \text{grad} f.$

Let's find it's curl.

$$\text{curl grad} f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix}$$

$$= (\partial_y \partial_z f - \partial_z \partial_y f, -(\partial_x \partial_z f - \partial_z \partial_x f), \partial_x \partial_y f - \partial_y \partial_x f)$$

$$= (0, 0, 0)$$

since partials commute. for functions with continuous second derivatives

Thus

$$\underset{\nabla \times \nabla f}{\overset{\text{curl grad} f}{\shortparallel}} = 0$$

for all functions f!

In particular, our previous example:

$$\vec{u} = (y, x, 0)$$

$$f = xy$$

$$\partial_x f = y, \quad \partial_y f = x, \quad \partial_z f = 0$$

Thus $\vec{u} = \nabla f$, and $\operatorname{curl} \vec{u} = 0$ by our general result.

Much more interesting and important is that the reverse is true:

> If $\vec{u}$ is a vector field with $\operatorname{curl} \vec{u} = 0$,
> then $\vec{u} = \operatorname{grad} f$ for some f.

We'll see this later...

A connection between div and curl:

$$\vec{u} = (P, Q, R)$$

$$\text{curl}\,\vec{u} = \nabla \times \vec{u}$$

What is $\text{div}\,\text{curl}\,\vec{u}$?

First

$$\text{curl}\,\vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= \left(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\right)$$

$$\text{div}\,\text{curl}\,\vec{u} = \partial_x(R_y - Q_z) + \partial_y(P_z - R_x) + \partial_z(Q_x - P_y)$$

$$= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy}$$

$$= 0!$$

Thus $\text{div}\,\text{curl}\,\vec{u}$ for any vector field $\vec{u}$.

Basic principles of hydrodynamics. There $\vec{u}$ is velocity field. If $\text{div}\,\vec{u} = 0$, fluid is incompressible.

$\omega = \text{curl}\,\vec{u}$ vorticity  $\Leftarrow$ ω directed along column