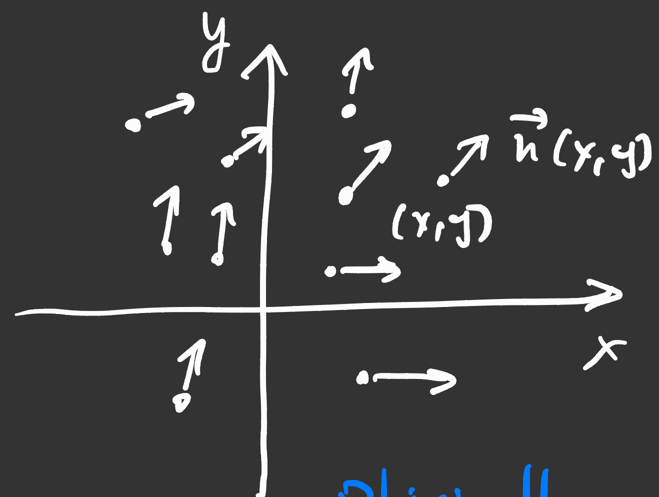
Vector fields



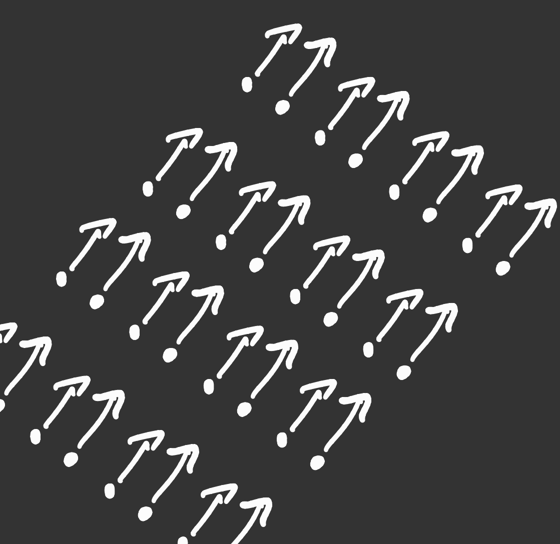
A field is a piece of the plane which at every point grows something (wheat, corn, etc). Here, vectors are grown.

Physically, think of a fluid: at every point there is a velocity.

$$\vec{u}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

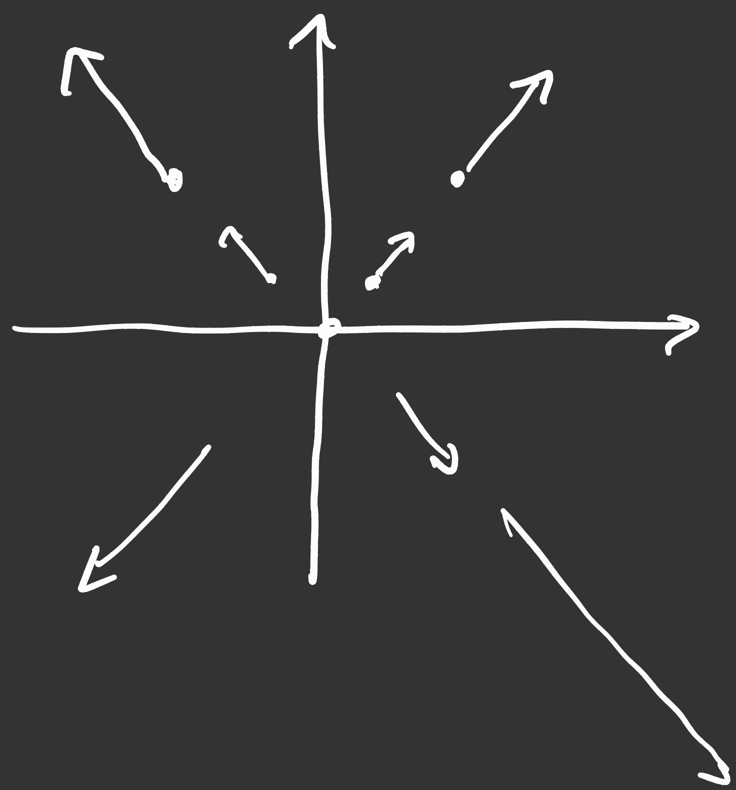
$$= P\,\vec{i} + Q\,\vec{j} \quad \text{(for short)}$$

Example: $\vec{u} = \text{const}$, e.g. $P = \text{const}$, $Q = \text{const}$

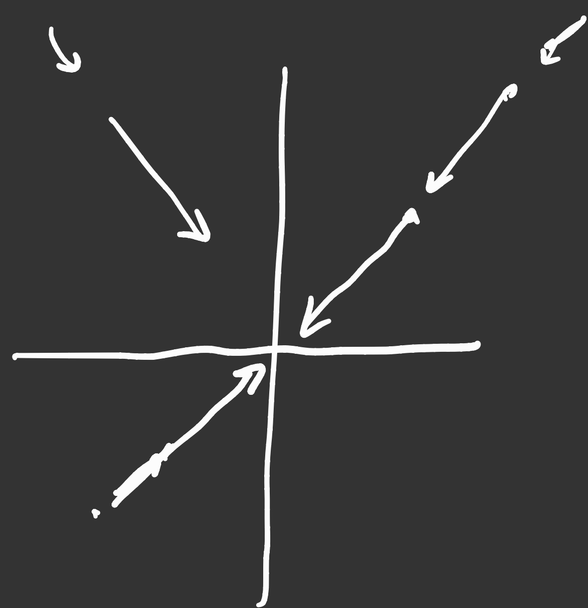


other physical examples
- electric field
- magnetic field
- gravitational force field

Example 2: $\quad u(x,y) = (x,y)$



Example 3 $\quad u(x,y) = u(\vec{r}) = -\dfrac{\vec{r}}{\|\vec{r}\|^3}$



Gravity field

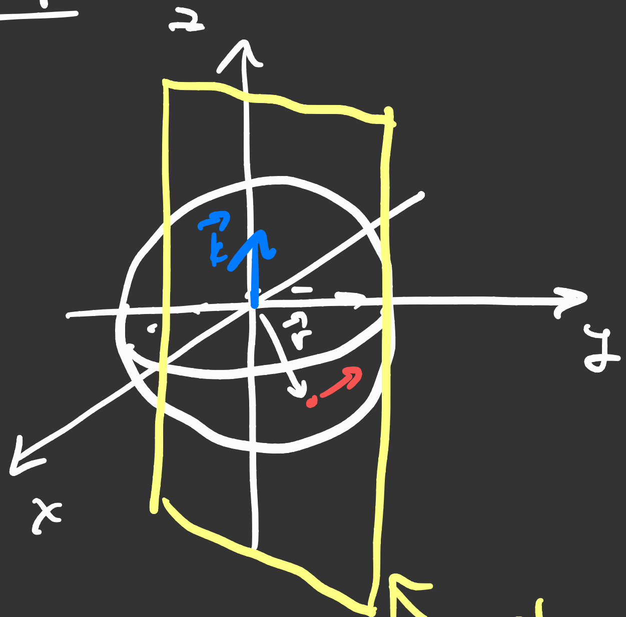
Closer you are to the origin (attracting mass), stronger the pull of gravity.

In 3D, same concept.

$$\vec{u}(\vec{r}) = \vec{u}(x,y,z)$$

$$= P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

Example:



rotating solid ball around z-axis $\hat{k}$

- rotates with unit angular velocity, e.g. in 1 second, it turns 1 radian.

plane containing z axis and $\vec{r}$.

The velocity of that point $\vec{r}$ is orthogonal to $\vec{r}$ and orthogonal to $\vec{k}$.

Thus, it is proportional to $\vec{k} \times \vec{r}$

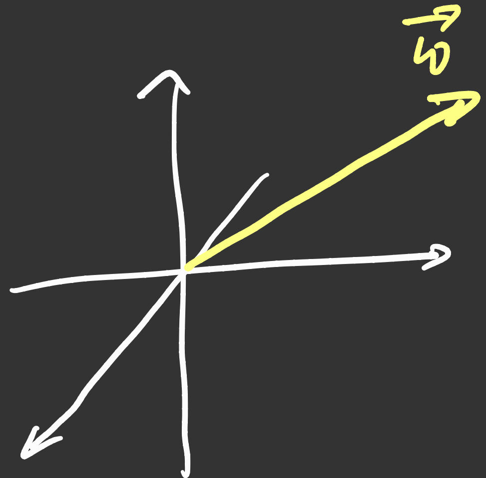
If the angular velocity is 1, then

$$\vec{u}(\vec{r}) = \vec{k} \times \vec{r}$$

We can imagine a rotation around any other axis.

Consider the vector

$\vec{w} = (w_1, w_2, w_2)$, a vector directed along axis of rotation, so if you look down tip, the rotation is ccw.



The length of $\vec{w}$ is equal to angular velocity of rotation.

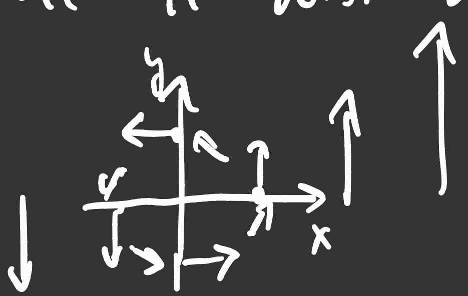
The velocity is then

$$\vec{u}(\vec{r}) = \vec{w} \times \vec{r}$$

In particular, if $\vec{w} = \vec{k} = (0,0,1)$, then

$$\vec{u}(\vec{r}) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$$

Note it doesn't depend on z (as it is rotating along z axis)



← rotating disk in the plane.

Operations on vector fields

Gradient ∇:

$$\nabla f(x,y,z) = (\partial_x f, \partial_y f, \partial_z f) \leftarrow \text{vector field.}$$

Divergence (div):

$$\vec{u}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$$

$$\operatorname{div}\vec{u} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Takes a vector field and results in a scalar.

Ex: $\vec{u}(\vec{r}) = \vec{r} = (x,y,z)$ $\qquad$ $P = x$, $Q = y$, $R = z$

$$\operatorname{div}\vec{u} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

Ex: (solid rotation) $\vec{u}(x,y,z) = (-y, x, 0)$ $\qquad$ $P = -y$, $Q = x$, $R = 0$

$$\operatorname{div}\vec{u} = -\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$$

we will understand why

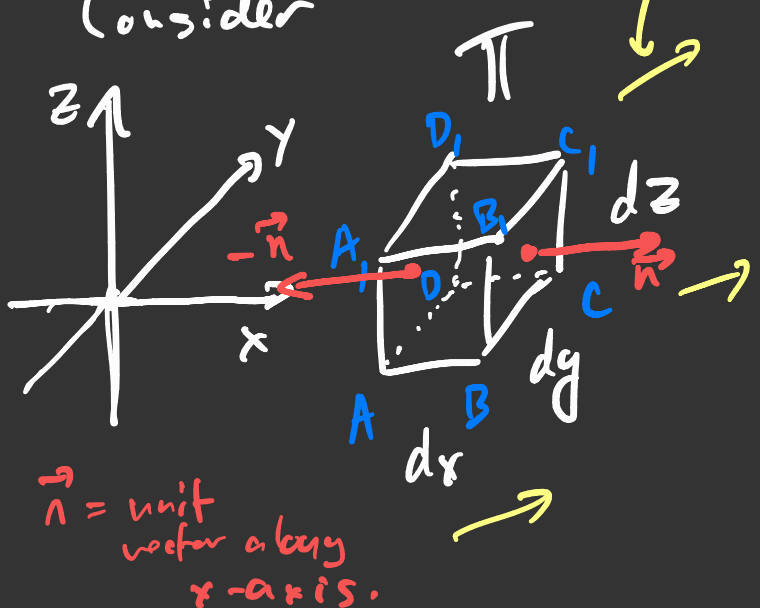
Divergence has many profound applications we will explore.

Physical meaning of divergence

$$\vec{u} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

Imagine $\vec{u}$ is the velocity field of a fluid.

Consider



How much fluid enters, and how much exits?

$$\text{Flux} := \begin{cases} \text{positive if fluid goes out} \\ \text{negative if fluid comes in} \end{cases}$$

Flux through faces ADD_1A_1 and BCC_1B_1

→ note second and third component ($\vec{j}$ and $\vec{k}$) are tangent to face, so no contribution to flux.

negative as $\vec{i}\cdot(-\vec{n}) = -1$ positive as $\vec{i}\cdot\vec{n} = 1$

$$\rightarrow \underbrace{dy\,dz}_{\text{area of face}}\left(-P(x,y,z) + P(x+dx,y,z)\right) \approx P_x(x,y,z)\,dx\,dy\,dz.$$

Similarly,

Flux through ABB_1A_1 and DCC_1D_1

$$\approx Q_y \, dx\,dy\,dz$$

Flux through $ABCD$ and $A_1B_1C_1D_1$

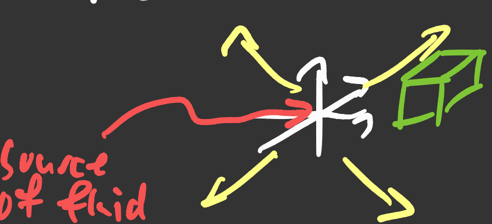
$$\approx R_z \, dx\,dy\,dz$$

In total, the flux through the surface Π is

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div} \vec{u}$$

If $\operatorname{div} \vec{u} > 0$, the fluid is somehow generated inside Π (e.g. from nothing, or sources)

Recall $\vec{u} = \vec{r}$



For any Π, the amount of fluid that leaves Π is more than amount that enters

Not in contradiction with cons law (if no source or sinks, as much enters, as much leaves)