Gradient

$\vec{r} = (x,y)$ is position vector

$$f(x,y) = f(\vec{r})$$

Def: $$\nabla f(\vec{r}) = \nabla f(x,y)$$
$$= \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$
$$= \frac{\partial f}{\partial x}(x,y)\, \vec{i} + \frac{\partial f}{\partial y}(x,y)\, \vec{j}$$

Ex: $$f(x,y) = x^2 - 2y^2$$
$$\nabla f(x,y) = (2x, -4y)$$

The symbol ∇ is called "nabla".
It is an ancient phoenician letter.
It was supposed to represent the head of a bull.

Also symbolizes a musical instrument.
But for us, it is the vector of partials.

Linear approximation: $\vec{r} = (x, y)$ $\vec{r}_0 = (x_0, y_0)$

$$f(\vec{r}) - f(\vec{r}_0) \approx f_x(\vec{r}_0)(x - x_0) + f_y(\vec{r}_0)(y - y_0)$$

$$= \nabla f(\vec{r}_0) \cdot (x - x_0, y - y_0)$$

since

$$\nabla f(\vec{r}_0) = (f_x(\vec{r}_0), f_y(\vec{r}_0))$$

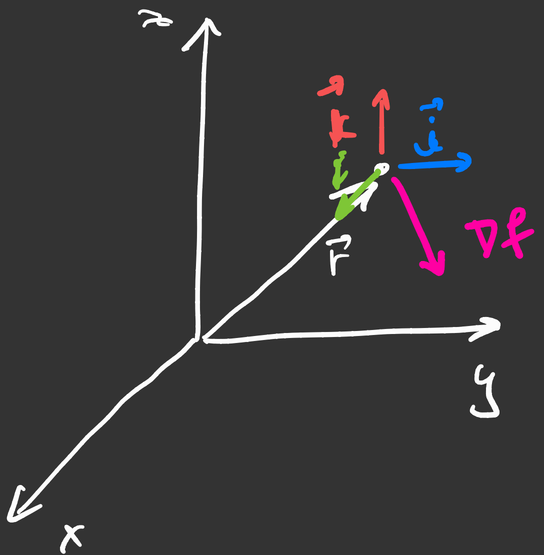
Thus

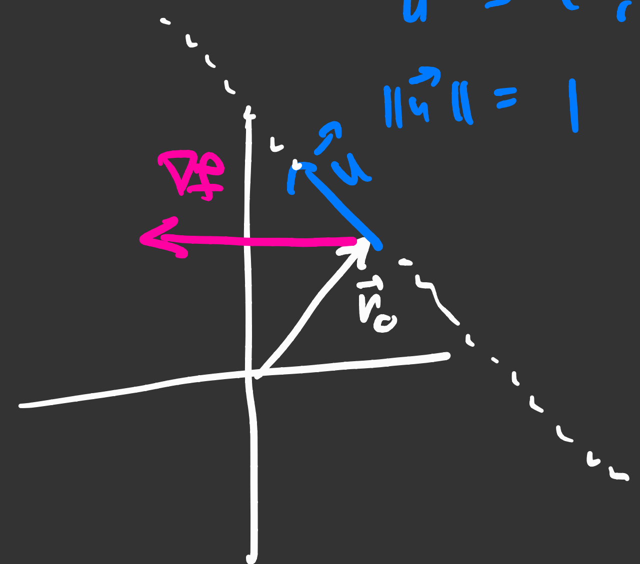
$$\boxed{f(\vec{r}) - f(\vec{r}_0) \approx \nabla f(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0)}$$

In particular

$$f_x = \nabla f \cdot \vec{i} \qquad f_y = \nabla f \cdot \vec{j}$$

Directional Derivative





$$l = \vec{r_0} + t\vec{u}$$

$$f(\vec{r_0} + t\vec{u}) = f(x_0 + ta, y_0 + tb)$$

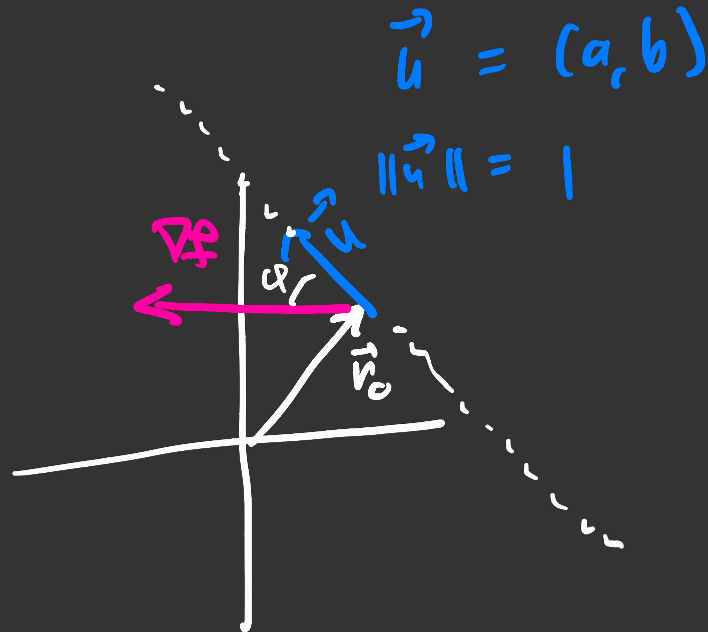
$$\frac{d}{dt} f(\vec{r_0} + t\vec{u}) = f_x(x_0 + ta, y_0 + tb)\, a + f_y(x_0 + ta, y_0 + tb)\, b$$

$$\frac{d}{dt} f(\vec{r_0} + t\vec{u}) = f_x a + f_y b$$

$$= \nabla f(x_0, y_0) \cdot \vec{u}$$

Derivative in direction of a unit vector

$$D_u f(x_0, y_0) = \vec{u} \cdot \nabla f(x_0, y_0)$$



$D_u f$ is rate of change of f as we move in direction $\vec{u}$.

$$\nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\varphi$$

$$-1 \le \cos\varphi \le 1$$

$\uparrow$ if $\vec{u}$ and ∇f are opposite directions

$\uparrow$ when $\vec{u}$ and ∇f are same direction

$\therefore$ $D_u f(\vec{r}_0)$ is maximal if $\vec{u} \parallel \nabla f(\vec{r}_0)$

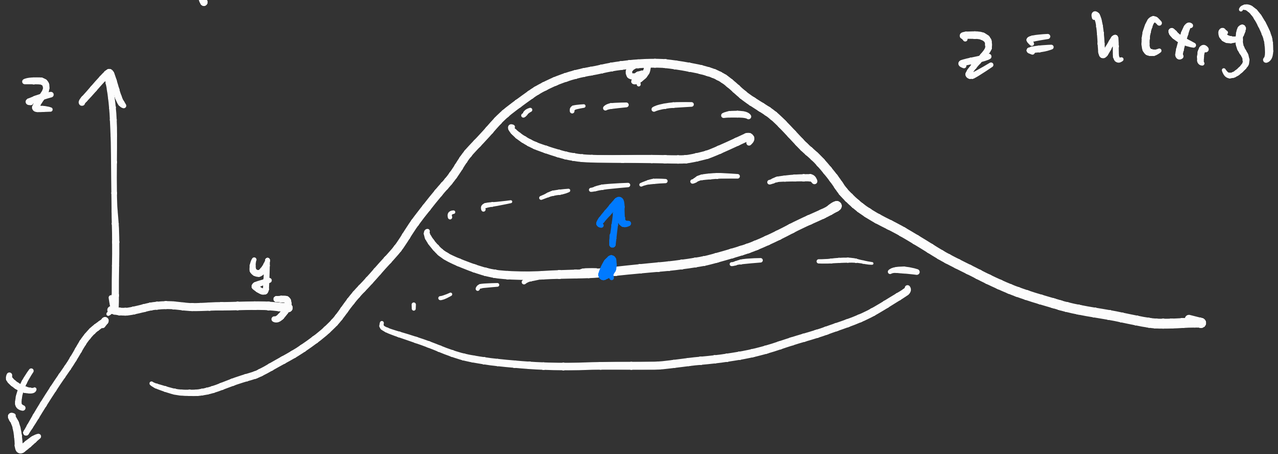
$D_u f(\vec{r}_0)$ is minimal if $-\vec{u} \parallel \nabla f(\vec{r}_0)$

i.e. $\vec{u}$ and $\nabla f(\vec{r}_0)$ antiparallel.

Maximal growth if you move along gradient.

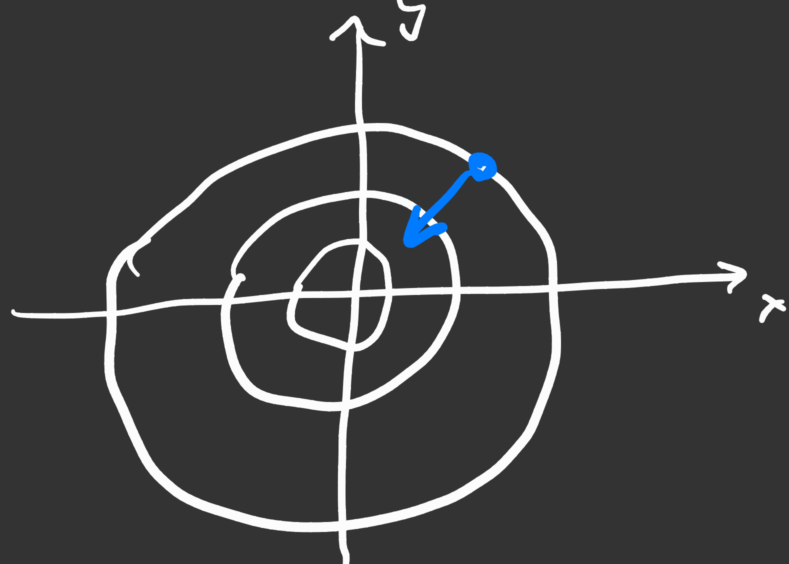
Maximal decay if you move opposite.

Example



To climb the fastest, you should go in the direction of $\nabla f(x_0, y_0)$.

To decend the fastest, go in direction of $-\nabla f(x_0, y_0)$.



How to find a point where a function is maximal?
choose initial point, and move up the gradient!
Basis of $\boxed{\text{Gradient method}}$ in optimization.