Celestial Mechanics and Kepler's laws

Theorem: (Newton) Suppose, for constants G, m, and M, that planetary motion about thus is governed by

$$m\vec{r}''(t) = -\frac{GMm}{\|\vec{r}\|^3}\vec{r}(t). \tag{1}$$

Then the planet's orbit, traced out by $\vec{r}(t)$, is a conic section.

(1) Show conservation energy. Namely, $E = \frac{m}{2}\|\vec{v}\|^2 - \frac{GMm}{\|\vec{r}\|}$ is a constant of motion.

(2) Deduce from conservation of energy that, provided $E_0 := \frac{m}{2}\|\vec{v}(0)\|^2 - \frac{GMm}{\|\vec{r}(0)\|} < 0$, the planet's orbit $\vec{r}(t)$ is bounded for all time. In fact, show that $\|\vec{r}(t)\| \leq \frac{GMm}{|E_0|}$.

(3) Show conservation angular momentum. Namely, $\vec{L} = \vec{r} \times \vec{v}$ is a constant of motion. Argue that the the motion is confined for all time to the plane $\Pi_{\vec{L}}$ that passes through the origin and is orthogonal to $\vec{L}$.

(4) The above is equivalent to Kepler's second law: "*The line segment from the sun to the planet sweeps out equal areas in equal times.*" That is, the vector $\vec{r}(t)$ sweeps out area $A(t)$ in the plane $\Pi_{\vec{L}}$ at a constant rate. Explain why by proving $A'(t) = \frac{1}{2}\|\vec{L}\|$.

(5) Prove conservation of the Laplace–Runge–Lenz vector $\vec{d} := \vec{v} \times \vec{L} - GM\frac{\vec{r}}{\|r\|}$.

(6) Argue that $\vec{d}$ is in the plane spanned by $\vec{r}$ and $\vec{v}$.

(7) Let θ be the angle between $\vec{d}$ and $\vec{r}/\|\vec{r}\|$. Let $L = \|\vec{L}\|$ and $d = \|\vec{d}\|$, then

$$\|r\| = \frac{p}{1 + e\cos\theta}, \qquad p = \frac{L^2}{GM}, \qquad e = \frac{d}{GM}.$$

(8) Rotating the plane containing $\vec{v}$ and $\vec{r}$ so that $\vec{d}$ coincides with the positive x-axis. Show the result of (c), in Cartesian coordinates $x = r\sin\theta$, $y = r\cos\theta$, is

$$(1 - e^2)x^2 + 2pex + y^2 = p^2.$$

Show that the curve $\{(x, y) \in \mathbb{R}^2 \mid (1 - e^2)x^2 + 2pex + y^2 = p^2\}$ is an
- ellipse if $|e| < 1$,
- parabola if $|e| = 1$,
- hyperbola if $|e| > 1$.

(9) Prove Kepler's third law for elliptical orbits: "*The square of the period is proportional to the cube of the major axis of the ellipse.*"