MAT 203, Calculus III with Applications, Fall 2023

M&T Sections: 5.3, 6.2, 8.1

(1) Find the volume of the region boundary by $x^2 + y^2 = 1$, x = z and z = 0. This region is known as the *hoof of Archimedes*. The floor of hoof is the half disk D with $x^2 + y^2 \leq 1$ and $x \geq 0$ and roof is z = x. Then

$$\iint_{D} x dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r^{2} \cos \theta dr d\theta = \frac{1}{3} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \frac{2}{3}$$

(2) Evaluate the following integrals using polar coordinates

•
$$I_1 = \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy,$$

• $I_2 = \int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dx dy,$
• $I_3 = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx.$

$$\begin{split} I_1 &= \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \frac{\pi}{8} \\ I_2 &= \int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^{y} dx dy = \int_{\pi/4}^{\pi/2} \int_2^{2 \csc \theta} r dr d\theta = \int_{\pi/4}^{\pi/2} (2 \csc^2 \theta - 2) d\theta = (-2 \cot \theta - \frac{1}{2}\theta) \Big|_0^{\pi/2} = 2 - \frac{\pi}{2} \\ I_3 &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx = 4 \int_0^{\pi/2} \int_0^1 \frac{2r}{(1+r^2)^2} dr d\theta = \pi. \end{split}$$

(3) Use Green's theorem to compute $\oint_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = (\arctan(y/x), \log(x^2 + y^2))$ along the curve C given by the boundary of the region defined by the polar inequalities $1 \le r \le 2$ and $0 \le \theta \le \pi/2$, oriented counterclockwise.

Note that
$$\partial_x \log(x^2 + y^2) = \frac{2x}{x^2 + y^2}$$
, and $\partial_y \arctan(y/x) = \frac{x}{x^2 + y^2}$ Thus

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2}\right) dxdy = \int_0^\pi \int_1^2 r \cos\theta drd\theta = \int_0^\pi \cos\theta d\theta = 0$$

(4) Find the work done by the force $\vec{F} = (4x - 2y, 2x - 4y)$ on a particle going counterclockwise around the circle C: $(x - 2)^2 + (y - 2)^2 = 4$.

Let
$$P(x, y) = 4x - 2y$$
 and $Q(x, y) = 2x - 4y$. Then $\partial_y P = -2$ and $\partial_x Q = 2$. Thus
 $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (2 - (-2)) dA = 4$ Area(circle of radius 2) = 16π .

(5) Show that the value of $\oint_C xy^2 dx + (x^2y + 2x)dy$ around any square C depends only on the area enclosed and not on its location in the plane.

Let D be the interior of any given square C. Let $P(x, y) = xy^2$ and $Q(x, y) = x^2y + 2x$. Then $\partial_y P = 2xy$ and $\partial_x Q = 2xy + 2$. Thus we find the integral is proportional to the area enclosed only:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (2xy + 2 - 2xy) dA = 4 \text{Area(square)}.$$