

- (1) Consider the vector field $F(x, y) = (2x/y, (1 - x^2)/y^2)$ for $y > 0$.

- Check that F is potential (conservative) and find a potential function

Note $\partial_y \left(\frac{2x}{y} \right) = -\frac{2x}{y^2}$, $\partial_x \left(\frac{1-x^2}{y^2} \right) = -\frac{2x}{y^2}$. Hence \vec{F} is conservative. We must now find a scalar function f which is a solution of

$$\partial_x f = \frac{2x}{y}, \quad \partial_y f = \frac{1-x^2}{y^2}.$$

From the first equation we find $f(x, y) = \frac{x^2}{y} + g(y)$ for some yet unknown free function g . Plugging into the second equation we find

$$-\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} = \frac{1}{y^2} - \frac{x^2}{y^2} \implies g'(y) = \frac{1}{y^2}.$$

Thus, $g(y) = -\frac{1}{y} + c$ where c is an arbitrary constant. Therefore, our potential is

$$f(x, y) = \frac{x^2 - 1}{y} + c.$$

- Let C be the curve $(x-3)^5 + y^2 = 3$, from $(2, -2)$ to $(2, 2)$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

Since F is conservative we have

$$\int_C \vec{F} \cdot d\vec{r} = f(2, 2) - f(2, -2) = 3.$$

- (2) Evaluate $\int_C F \cdot dr$ where

$$F(x, y, z) = \left(\frac{1}{y^2 + 1}, -\frac{2xy}{(y^2 + 1)^2} + ze^{yz}, ye^{yz} + 2z \right)$$

where C is part of the helix $r(t) = \langle \cos(t), \sin(t), t \rangle$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

We seek a potential for this vector field. Indeed it is

$$f(x, y, z) = \frac{x}{y^2 + 1} + e^{yz} + z^2$$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 0, 2\pi) - f(1, 0, 0) = 2 + 4\pi^2 - 2 = 4\pi^2.$$

- (3) Find the integral of $f(x, y) = x^2 + y^2$ on the domain

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 \leq y \leq x\}.$$

Sketch the region of integration.

D can be viewed as a type 1 domain with lower boundary $y = x^2$ and upper boundary $y = x$. Then

$$I = \iint_D f(x, y) dx dy = \int_0^1 \int_{x^2}^x (x^2 + y^2) dy dx = \int_0^1 (x^2(x - x^2) + \frac{1}{3}(x^3 - x^6)) dx = \frac{3}{35}.$$

- (4) Find the limits of integration of $\iint_D f(x, y) dx dy$ if

$$D := \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$$

when D is considered first as a type I and then as a type II domain.

As a type 1 domain, the lower boundary is $y = -2\sqrt{1 - \frac{x^2}{9}}$ and top boundary is $y = 2\sqrt{1 - \frac{x^2}{9}}$. As a type 2 domain, the left boundary is $x = -3\sqrt{1 - \frac{y^2}{4}}$ and right boundary is $x = 3\sqrt{1 - \frac{y^2}{4}}$.

- (5) Sketch the region bounded by the curves $y = \log(x)$, $y = 2\log(x)$ and $x = e$ in the first quadrant. Then express the region's area as an iterated double integral and evaluate.

To compute the area of this region we will write the integral using vertical cross-sections. The projection of the region D to the x -axis is the segment $(1, e)$ (because we are only interested in the region contained in the first quadrant). The vertical segment above x is the vertical segment connecting $(x, \log(x))$ and $(x, 2\log(x))$. The area of the region is

$$\int_1^e \int_{\log(x)}^{2\log(x)} dy dx = \int_1^e \log(x) dx.$$

A primitive for $\log(x) = x \log(x) - x$. Finally, the area enclosed is: $\int_1^e \int_{\log(x)}^{2\log(x)} dy dx = 1$.

- (6) Consider $\iint_D f dA = \int_0^3 \int_{-2\sqrt{1-(x/3)^2}}^{2(1-x/3)} f(x, y) dy dx$.

- Sketch the region of integration.

The limits in x are $0 \leq x \leq 3$. The limits in y are $-2\sqrt{1 - (x/3)^2} \leq y \leq 2(1 - x/3)$, so, a part of the ellipse $x^2/3^2 + y^2/2^2 = 1$.

- Switch the order of integration in the above integral.

If we are to integrate first in x and then in y , we must split the integral at $y = 0$. In the interval $-2 \leq y \leq 0$, the lower limit in x is 0 and the upper limit comes from $x^2/3^2 + y^2/2^2 = 1$, i.e. $x \leq 3\sqrt{1 - (y/2)^2}$. On the other hand, in the interval $0 \leq y \leq 2$, the lower limit in x is again 0 but the upper limit comes from $y = 2(1 - x/3)$, i.e. $x = 3 - 3y/2$. We conclude

$$\iint_D f dA = \int_{-2}^0 \int_0^{3\sqrt{1-(y/2)^2}} f(x, y) dx dy + \int_0^2 \int_0^{3-3y/2} f(x, y) dx dy.$$

- Compute the integral $\iint_D f dA$ if $f(x, y) = xy$.

We compute

$$\begin{aligned} \iint_D f dA &= \int_0^3 \int_{-2\sqrt{1-(x/3)^2}}^{2(1-x/3)} f(x, y) dy dx \\ &= \frac{1}{2} \int_0^3 4x [(1 - x/3)^2 - (1 - x^2/3^2)] dx = \frac{4}{3^2} \int_0^3 (x^3 - 3x^2) dx = -3. \end{aligned}$$

We can also compute

$$\iint_D f dA = \int_{-2}^0 \int_0^{3\sqrt{1-(y/2)^2}} xy dx dy + \int_0^2 \int_0^{3-3y/2} xy dx dy = -3.$$