

- (1) Let $\vec{F}(x, y, z) = (\cos x \sin y, \sin x \cos y, 1)$, and C is the line segment from $(1, 0, 0)$ to $(0, 0, 3)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{r}(t) = (1 - t, 0, 3t), \quad t \in [0, 1]$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^1 (\cos(1 - t) \sin 0, \sin(1 - t) \cos 0, 1) \cdot (-1, 0, 3) dt = \int_0^1 3 dt = 3$$

- (2) If $\vec{r}(t) = (a \cos t, a \sin t)$, $t \in [0, 2\pi]$, and $\vec{F}(x, y) = (-y, x)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-a \sin t, a \cos t) \cdot (-a \sin t, a \cos t) dt = 2\pi a^2.$$

- (3) If $\vec{r}(t) = (\cos \pi t, \sin \pi t)$, $t \in [0, 2]$, and $\vec{F}(x, y) = (x, y)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^2 (\cos \pi t, \sin \pi t) \cdot (-\pi \sin \pi t, \pi \cos \pi t) dt = 0.$$

- (4) Compute $\int_C x^2 dx - xy dy + dz$ where C is the parabola $z = x^2$, $y = 0$ from $(-1, 0, 1)$ to $(1, 0, 1)$.

We parametrize the curve as follows $(x, y, z) = (t, 0, t^2)$ for $-1 \leq t \leq 1$. Then

$$\int_C x^2 dx - xy dy + dz = \int_{-1}^1 (t^2 + 2t) dt = \left(\frac{t^3}{3} + t^2 \right) \Big|_{-1}^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}.$$

- (5) (M&T, # 7.2.6) Let $\vec{r}(t)$ be a parametrization of a curve C .

- Suppose that $\vec{F}$ is perpendicular to $\vec{r}'(t)$ at the point $\vec{r}(t)$. Show $\int_C \vec{F} \cdot d\vec{r} = 0$.

Being perpendicular $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 0$ for each t . Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = 0.$$

- Suppose that $\vec{F}$ is parallel to $\vec{r}'(t)$ at the point $\vec{r}(t)$ (e.g. $\vec{F}(\vec{r}(t)) = \lambda(t)\vec{r}'(t)$ for some $\lambda(t) > 0$). Show $\int_C \vec{F} \cdot d\vec{r} = \int_C \|\vec{F}\| ds$.

Noting that $\|\vec{F}(\vec{r}(t))\| = \lambda(t)\|\vec{r}'(t)\|$ since $\lambda(t) > 0$, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int \lambda(t)\|\vec{r}'(t)\|^2 dt = \int \|\vec{F}(\vec{r}(t))\|\|\vec{r}'(t)\| dt = \int_C \|\vec{F}\| ds.$$

- Suppose L is the length of C and $\|\vec{F}\| \leq M$. Prove $\left| \int_C \vec{F} \cdot d\vec{r} \right| \leq ML$.

From the above we have

$$\left| \oint_C \vec{F} \cdot d\vec{r} \right| = \left| \int \|\vec{F}(\vec{r}(t))\|\|\vec{r}'(t)\| dt \right| \leq M \int \|\vec{r}'(t)\| dt = ML.$$

(6) Let

$$\vec{F}(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right),$$

and C be a parametrized curve defined, for $t \in [0, 1]$, by

$$\vec{r}(t) = \left(a \cos(2k\pi t), b \sin(2k\pi t) \right),$$

where k is a positive integer and $0 < b \leq a$.

- Note that in polar coordinates, $\tan \theta = \frac{y}{x}$, so $\theta = \arctan \frac{y}{x}$. Show that

$$\frac{d\theta}{dt} = \vec{F} \cdot (x'(t), y'(t)),$$

and hence conclude that

$$\int_0^1 \frac{d\theta}{dt} dt = \oint_C \vec{F} \cdot d\vec{r} = 2k\pi.$$

In fact, for any smooth closed curve C in $\mathbb{R}^2$ that intersects itself finitely many times and does not pass through the origin, the line integral $\frac{1}{2\pi} \oint_C \frac{-ydx + xdy}{x^2 + y^2}$ is always an integer (known as the winding number of C around the origin).

$\nabla \arctan \frac{y}{x} = \vec{F}$. By Chain Rule, $\frac{d\theta}{dt} = \nabla \arctan \frac{y}{x} \cdot (x'(t), y'(t))$.

$$\int_0^1 \frac{d\theta}{dt} dt = \theta(1) - \theta(0)$$

by Fundamental Theorem of Calculus, which is $2k\pi$.

- Using the above, evaluate the integral

$$\int_0^1 \frac{dt}{a^2 \cos^2(2k\pi t) + b^2 \sin^2(2k\pi t)}.$$

Hint: In computing $\oint_C \vec{F} \cdot d\vec{r}$, write $x'(t)$ in terms of $y(t)$, and $y'(t)$ in terms of $x(t)$.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F} \cdot (x'(t), y'(t)) dt = \theta(1) - \theta(0) = 2k\pi. \\ 2k\pi &= \oint_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \cdot \left(-\frac{2k\pi a}{b} y, \frac{2k\pi b}{a} x \right) dt, \\ &= 2k\pi \int_0^1 \frac{\frac{a}{b} y^2 + \frac{b}{a} x^2}{x^2 + y^2} dt = 2k\pi \int_0^1 ab \frac{\cos^2(2k\pi t) + \sin^2(2k\pi t)}{a^2 \cos^2(2k\pi t) + b^2 \sin^2(2k\pi t)} dt. \end{aligned}$$

Hence

$$\int_0^1 \frac{dt}{a^2 \cos^2(2k\pi t) + b^2 \sin^2(2k\pi t)} = \frac{1}{ab}.$$