(1) Find the tangent plane to the the surface defined by $f(x,y) = x^3 - xy + y^2$ at $(2,1)$.

Note $f(2,1) = 7$, and $Df(x,y) = \begin{bmatrix} 3x^2 - y & -x + 2y \end{bmatrix}$, so $\nabla f(2,1) = (11,0)$. The tangent hyperplane has normal vector $(11,0,-1)$ and passes through the point $(2,1,7)$, so it has equation

$$(11,0,-1)\cdot(x-2,y-1,z-7) = 0, \qquad 11(x-2)-(z-7) = 0.$$

(2) Find the unit normal to the *graph* (namely, the two-dimensional surface sitting in three-dimensional space defined by $z = f(x,y)$) for $f(x,y) = e^x y$ at the point $(-1,1)$.

We compute $\nabla f = (e^x y, e^x)$. Let $F = z - f(x,y)$. Then $\nabla F = (e^x y, e^x, 1)$ and $\|\nabla F\| = \sqrt{e^{2x}(1+y^2)+1}$. Thus the normal vector is $\hat{n} = \nabla F/|\nabla F|$. At the point $(-1,1)$, we have $\hat{n}(-1,1) = (e^{-1}, e^{-1}, 1)/\sqrt{2e^{-2}+1}$.

(3) Captain Buzz is in trouble near the sunny side of Mars, at coordinate $(1,1,1)$. His ship's hull is melting. He measures the temperature in his vicinity to be $T(x,y,z) = e^{-x} + e^{-2y} + e^{3z}$. In what direction should he proceed in order to cool the fastest?
The temperature gradient is $\nabla T = (-e^{-x}, -2e^{-y}, 3e^{3z})$. At the point $(1,1,1)$, this is $\nabla T|_{(1,1,1)} = (-e^{-1}, -2e^{-1}, 3e^3)$. He should move in the direction $-\nabla T|_{(1,1,1)} = (e^{-1}, 2e^{-1}, -3e^3)$ to cool the fastest.

(4) Compute:

- f_{xyzxz} if $f(x,y,z) = x\exp(yz^2\sin(y+z))$,

  $f_{xyzxz} = \partial_y \partial_z^2 f_{xx} = 0$, as partial derivatives commute.

- $f_{tt} - f_{xx}$ if $f(x,t) = \sin(x)\cos(t)$,

  $f_{tt} = -\sin(x)\cos(t) = -f$ and likewise $f_{xx} = -f$. Thus $f_{tt} - f_{xx} = 0$, so f is a standing wave.

- all second order partial derivatives if $f(x,y) = x\log y$.

  $f_{xx} = 0$, $f_{xy} = 1/y = f_{yx}$ and $f_{yy} = -\frac{x}{y^2}$.

(5) Describe what $\mathrm{div}\vec{v}(\vec{r})$ and $\mathrm{curl}\vec{v}(\vec{r})$ means and compute them for the vector fields:

- $\vec{v}(\vec{r}) = \|\vec{r}\|\vec{r}$ where $\vec{r} = (x,y,z)$,
  Note that $\nabla\cdot\vec{r} = 3$. Also $\nabla\|\vec{r}\| = \frac{\vec{r}}{\|\vec{r}\|}$. Thus $\nabla\cdot\vec{v}(\vec{r}) = \vec{r}\cdot\nabla\|\vec{r}\| + 3\|\vec{r}\| = 4\|\vec{r}\|$. For the curl, we note that $\mathrm{curl}\vec{r} = 0$. Thus $\mathrm{curl}\vec{v} = \nabla\|\vec{r}\|\times\vec{r} = \frac{1}{\|\vec{r}\|}\vec{r}\times\vec{r} = 0$.

- $\vec{v}(r) = \frac{\vec{r}}{\|\vec{r}\|^3}$ where $\vec{r} = (x,y,z)$,
  Provided $\vec{r} \neq 0$, we compute $\nabla\cdot\frac{\vec{r}}{\|\vec{r}\|^3} = \frac{3}{\|\vec{r}\|^3} - \frac{3}{\|\vec{r}\|^4}\vec{r}\cdot\nabla\|\vec{r}\| = 0$. At $\vec{r} = 0$ we cannot tell what happens from this. We will find later that, in a suitable sense, the divergence is infinite at $\vec{r} = 0$. The curl, as in the first example, is zero.

- $\vec{v}(x,y,z) = (A\sin z + C\cos y, B\sin x + A\cos z, C\sin y + B\cos x)$, The divergence is zero since the e_1 component doesn't depend x variable, and so on. The curl is $\vec{v}(x,y,z)$ itself! That is, it is an *eigenfunction* of the "curl operator". This velocity is called ABC flow, so-named after Arnold, Beltrami and Childress (hence the symbols for the constants in the formula). It arises as a time-independent solution of a three-dimensional inviscid fluid equations.

(6) Let ∇^2 be the Laplacian operator defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

- If f is a twice-differentiable function, show that $\text{div}(f\nabla f) = \|\nabla f\|^2 + f\nabla^2 f$

$$\begin{aligned} \nabla \cdot (f\nabla f) &= \nabla \cdot (ff_x, ff_y, ff_z) \\ &= f_x^2 + ff_{xx} + f_y^2 + ff_{yy} + f_z^2 + ff_{zz} \\ &= \|\nabla f\|^2 + f\nabla^2 f \end{aligned}$$

Alternatively, use the fact that $div(f\vec{F}) = f div\vec{F} + \nabla f \cdot \vec{F}$,

$$\nabla \cdot (f\nabla f) = f\nabla \cdot \nabla f + \nabla f \cdot \nabla f$$

- Suppose $\vec{F}$ is a C^2 vector field, show that $\text{curl}(\text{curl}\vec{F}) = \nabla(\text{div}\vec{F}) - \nabla^2\vec{F}$

We may also write $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$. Let

$$\vec{F} = (P, Q, R),$$

$$\begin{aligned} \nabla \times (\nabla \times \vec{F}) &= \nabla \times (R_y - Q_z, P_z - R_x, Q_x - P_y) \\ &= (Q_{xy} - P_{yy} - P_{zz} + R_{xz}, R_{yz} - Q_{zz} - Q_{xx} + P_{xy}, P_{xz} - R_{xx} - R_{yy} + Q_{yz}) \end{aligned}$$

$$\begin{aligned} \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F} &= \nabla(P_x + Q_y + R_z) - (\nabla^2 P, \nabla^2 Q, \nabla^2 R) \\ &= (P_{xx} + Q_{xy} + R_{xz} - \nabla^2 P, P_{xy} + Q_{yy} + R_{yz} - \nabla^2 Q, P_{xz} + Q_{yz} + R_{zz} - \nabla^2 R) \\ &= \nabla \times (\nabla \times \vec{F}) \end{aligned}$$

Alternative, use the fact that $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\vec{c}$ to show that

$$\nabla \times (\nabla \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

(7) Consider the function $f(x, y) = \ln(\sqrt{x^2 + y^2} + y)$.

- Determine the domain of f and sketch it in the xy-plane.

We can only take the logarithm of nonnegative numbers, so

$$\sqrt{x^2 + y^2} + y > 0 \implies \sqrt{x^2 + y^2} > -y.$$

This is satisfied automatically if $y > 0$. If $y \leq 0$, both sides of the inequality are positive and we may safely square to get

$$x^2 + y^2 > y^2 \qquad \implies \qquad x^2 > 0 \qquad \implies \qquad x \neq 0.$$

Therefore the domain excludes all points along the negative y-axis (the origin is excluded as well). Note that there are no domain restrictions from the square root term $\sqrt{x^2 + y^2}$ since $x^2 + y^2$ is never negative.

- What is the linearization of f at $(3, -4)$?

We know that the linearization is

$$L(x, y) = f_x(3, -4)(x - 3) + f_y(3, -4)(y + 4) + f(3, -4).$$

First note $f(3, -4) = \ln(1) = 0$. Next, we compute

$$f_x(x, y) = \frac{1}{\sqrt{x^2 + y^2} + y} \cdot \left(\frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x \right) = \frac{x}{(\sqrt{x^2 + y^2} + y)\sqrt{x^2 + y^2}},$$

$$f_y(x,y) = \frac{1}{\sqrt{x^2+y^2}+y} \cdot \left(\frac{1}{2} \frac{1}{\sqrt{x^2+y^2}} \cdot 2x + 1 \right) = \frac{y}{(\sqrt{x^2+y^2}+y)\sqrt{x^2+y^2}} + 1.$$

Therefore

$$f_x(3,-4) = \frac{3}{(5-4)\cdot 5} = \frac{3}{5},$$

$$f_y(3,-4) = \frac{-4}{(5-4)\cdot 5} + 1 = \frac{1}{5}$$

and hence

$$L(x,y) = \frac{3}{5}(x-3) + \frac{1}{5}(y+4).$$