(1) (M&T, # 4.1.26) Let $\vec{r}(t)$ be the vector from the origin to the position of an object of mass $m > 0$, $\vec{v}(t) := \vec{r}'(t)$ be the velocity and $\vec{a}(t) := \vec{r}''(t)$ be the acceleration. Suppose that $\vec{F}(t)$ is the force acting at time t, so that $\vec{F} = m\vec{a}$.

(a) Prove that $\frac{d}{dt}(m\vec{r}\times\vec{v}) = \vec{r}\times\vec{F}$. What do you conclude if $\vec{F}$ is parallel to $\vec{r}$?
Note $\frac{d}{dt}(m\vec{r}(t)\times\vec{v}(t)) = m\vec{r}'(t)\times\vec{v}(t)+m\vec{r}(t)\times\vec{v}'(t) = m\vec{v}(t)\times\vec{v}(t)+\vec{r}(t)\times(m\vec{a})(t) = \vec{r}\times\vec{F}$, since $\vec{F} = m\vec{a}$ and $\vec{v}\times\vec{v} = 0$. If $\vec{F}$ is parallel to $\vec{r}$, then $\vec{r}\times\vec{F}$ and therefore the quantity $m\vec{r}\times\vec{v}$, the angular momentum, is preserved in time.

(b) Prove that a planet (say, of mass m) moving about the Sun (say, of mass M) does so in a fixed plane (one of Kepler's Laws). Recall, for planetary motion, $\vec{F} = -\frac{GmM}{\|\vec{r}\|^3}\vec{r}$.
Since angular momentum is conserved, $\vec{\ell} = \vec{r}(t)\times\vec{v}(t)$ for some fixed vector $\vec{\ell}\in\mathbb{R}^3$. Moreover, $\vec{r}(t)\cdot\vec{\ell} = 0$ since $\vec{r}(t)\cdot(\vec{r}(t)\times\vec{a}) = 0$ for any $\vec{a}\in\mathbb{R}^3$. This implies that the position vector $\vec{r}(t)$ is confined to a plane with normal vector $\vec{\ell}$.

(2) Let $\vec{r}(t) = (\cos(t), \sin(t), t)$ be the helix.

(a) Find the length of the helix for $0 \le t \le \pi$.
First we compute $\vec{r}'(t) = (\sin(t), \cos(t), 1)$ so that $\|\vec{r}'(t)\| = \sqrt{\sin^2(t)+\cos^2(t)+1} = \sqrt{2}$. Thus $s(t) = \sqrt{2}t$. The length from $0 \le t \le \pi$ is $s(\pi) = \sqrt{2}\pi$.

(b) Find the arc length parametrization for the helix.
The inverse function is $t(s) = s/\sqrt{2}$. The arc-length parametrization is therefore given by composing $\vec{r}(t)$ with $t(s)$, i.e. $\vec{r}(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$.

(3) Compute the curvature and principle normal vector of the helix $\vec{r}(t) = (\cos(t), \sin(t), t)$.
From the previous question, we have that $\vec{r}(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$. Unit tangent vector to the helix is therefore

$$\vec{T}(s) = \vec{r}'(s) = \frac{1}{\sqrt{2}}(-\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), 1).$$

Therefore, the acceleration vector is

$$\frac{d}{ds}\vec{T}(s) = \vec{r}''(s) = -\frac{1}{2}(\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), 0).$$

The curvature is

$$\kappa(s) = \left\|\frac{d}{ds}\vec{T}(s)\right\| = \frac{1}{2}\sqrt{\cos^2(s/\sqrt{2})+\sin^2(s/\sqrt{2})} = \frac{1}{2}.$$

The principle normal vector is

$$\vec{N}(s) = \frac{\frac{d}{ds}\vec{T}(s)}{\left\|\frac{d}{ds}\vec{T}(s)\right\|} = -(\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), 0).$$

(4) Compute the curvature of the exponential spiral $\vec{r}(t) = (e^t\cos(t), e^t\sin(t), 0)$. Draw a (rough) picture of this curve. What happens as $t\to\infty$?
Here we cannot easily compute the arc-length explicitly, so we instead use the formulae in terms of an arbitrary parametrization. We require

$$\vec{r}'(t) = e^t(\cos(t)-\sin(t), \cos(t)+\sin(t), 0), \qquad \vec{r}''(t) = 2e^t(-\sin(t), \cos(t), 0).$$

We require also

$$\begin{aligned}\|\vec{r}'(t)\| &= e^t\sqrt{(\cos(t)-\sin(t))^2+(\cos(t)+\sin(t))^2} \\ &= \sqrt{2}e^t\sqrt{\cos^2(t)+\sin^2(t)} = \sqrt{2}e^t\end{aligned}$$

so that $\|\vec{r}'(t)\|^3 = 2^{3/2}e^{3t}$. Finally, we need

$$\vec{r}'(t)\times\vec{r}''(t) = 2e^{2t}(0,0,1), \qquad \|\vec{r}'(t)\times\vec{r}''(t)\| = 2e^{2t}.$$

Then we assemble the curvature

$$\kappa(t) = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{e^{-t}}{\sqrt{2}}.$$

As time tends to infinity, the curvature tends to zero, in accord with it becoming flatter as it gets farther out. As time goes to minus infinity, the curve coils around the origin and the curvature diverges.

(5) Prove that if the curvature of a curve is identically zero, then the curve is a straight line.
This is intuitively obvious. To prove it, suppose that the curve is parametrized by arc-length. Then $\vec{v}(s) = \vec{T}(s)$ and $\vec{T}'(s) = 0$ since the curvature is zero, so $\vec{v}(s)$ is constant in s and therefore the same on the entire curve. Therefore the curve is a straight line, the only type of curve with equal tangent vector at all points.