(1) (M&T, §1.3: #2(a)) Find the determinant $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix}$.

$2 \times 3 + 1 \times 3(4 - 6) = 4.$

(2) (M&T, §1.3: #14) Compute $\vec{u} + \vec{v}$, $\vec{u} \cdot \vec{v}$, $\|\vec{u}\|$, $\|\vec{v}\|$ and $\vec{u} \times \vec{v}$ where $\vec{u} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{v} = -6\vec{i} - 2\vec{j} - 2\vec{k}$.

We have $\vec{u} = (3, 1, -1)$ and $\vec{v} = (-6, -2, -2)$. Thus, $\vec{u} + \vec{v} = (-3, -1, -3)$, $\vec{u} \cdot \vec{v} = -18 - 2 + 2 = -18$, $\|\vec{u}\| = \sqrt{11}$ and $\|\vec{v}\| = 2\sqrt{11}$. The cross product is $\vec{u} \times \vec{v} = (-4, 12, 0)$.

(3) (M&T, §1.3: #15(c)) Find an equation for the plane that is perpendicular to the line $\ell(t) = (5, 0, 2)t + (3, -1, 1)$ and passes through the point $(5, -1, 0)$.

A normal vector to the plane is $\vec{n} = (5, 0, 2)$. Given the point with position vector $\vec{r}_0 = (5, -1, 0)$ and the normal, we form the position vector for any point on the plane $\vec{r} = (x, y, z)$ and the equation for the plane is $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$. Note that $\vec{n} \cdot \vec{r} = 5x + 2z$ while $\vec{n} \cdot \vec{r}_0 = 25$.

(4) (M&T, §1.3: # 38) Give vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, do the equations $\vec{u} \times \vec{a} = \vec{b}$ and $\vec{u} \cdot \vec{a} = \|\vec{a}\|$ determine a unique vector $\vec{u}$? Argue *both* geometrically and analytically.

Note generally, since it might be that $\vec{b} \cdot \vec{a} \neq 0$, a contradiction. **If $\vec{a}$ is not orthogonal to $\vec{b}$, there is no solution**. Suppose that $\vec{a}$ and $\vec{b}$ are orthogonal. If $\vec{a} = 0$, then $\vec{b}$ must be zero to have a solution (otherwise there are none). If both are zero, any vector $\vec{u}$ is a solution, and thus it is not unique. **Suppose now that $\vec{a} \neq 0$ and $\vec{b} \neq 0$. We shall now determine $\vec{u}$ uniquely.** Note that since $\vec{a}$ and $\vec{b}$ are orthogonal, and $\vec{a} \times \vec{b}$ is orthogonal to both, the triple $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ form a basis of $\mathbb{R}^3$. As such, any vector $\vec{u}$ can be written uniquely as a linear combinations of them. We express

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{a}\|^2}(\vec{u} \cdot \vec{a})\vec{a} + \frac{1}{\|\vec{v}\|^2}(\vec{u} \cdot \vec{b})\vec{v} + \frac{1}{\|\vec{a} \times \vec{b}\|^2}(\vec{u} \cdot (\vec{a} \times \vec{b}))\vec{a} \times \vec{b} \\ &= \frac{1}{\|\vec{a}\|}\vec{a} + \frac{\|\vec{b}\|^2}{\|\vec{a} \times \vec{b}\|^2}\vec{a} \times \vec{b} = \frac{1}{\|\vec{a}\|}\vec{a} + \frac{1}{\|\vec{a}\|^2}\vec{a} \times \vec{b}, \end{aligned}$$

where we used that $\vec{b} \cdot \vec{u} = 0$ (which follows from dotting $\vec{u} \times \vec{a} = \vec{b}$ with $\vec{u}$), $\vec{b} \cdot (\vec{u} \times \vec{a}) = \vec{u} \cdot (\vec{a} \times \vec{b})$ (using the signed volume definition of this triple product), and on the other hand, $\vec{b} \cdot (\vec{u} \times \vec{a}) = \|b\|^2$. Finally, we noted that since $\vec{a}$ and $\vec{b}$ are orthogonal, $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|$ since the magnitude of the cross product is the area of the parallelogram made from $\vec{a}$ and $\vec{b}$, which in this case is a rectangle. Finally, this solution is also valid (namely $\vec{u} = \vec{a}/\|\vec{a}\|$) in the case of $\vec{b} = 0$ (the reader should check).

(5) Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors that are not co-planar, namely

$$\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0} \qquad \text{if and only if} \qquad \alpha = \beta = \gamma = 0.$$

(a) Show that $\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}$ are not co-planar.

Suppose

$$\alpha\vec{u} \times \vec{v} + \beta\vec{v} \times \vec{w} + \gamma\vec{w} \times \vec{u} = \vec{0}$$

Dotting it with $\vec{u}, \vec{v}, \vec{w}$ respectively, and recalling $V(\vec{u}, \vec{v}, \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \neq 0$ is the signed volume of the parallelepiped spanned the $\vec{u}, \vec{v}, \vec{w}$, we get that $\alpha = \beta = \gamma = 0$.

(b) Suppose $a, b, c \in \mathbb{R}$, find the point of intersections of the three planes

$$\vec{u} \cdot (x, y, z) = a, \qquad \vec{v} \cdot (x, y, z) = b, \qquad \vec{w} \cdot (x, y, z) = c$$

Express the solution (x, y, z) as

$$(x, y, z) = \alpha\vec{v} \times \vec{w} + \beta\vec{w} \times \vec{u} + \gamma\vec{u} \times \vec{v}.$$

Again, dotting with $\vec{u}, \vec{v}, \vec{w}$, we have $\alpha = \frac{a}{V(\vec{u},\vec{v},\vec{w})}$, $\beta = \frac{b}{V(\vec{u},\vec{v},\vec{w})}$, $\gamma = \frac{c}{V(\vec{u},\vec{v},\vec{w})}$.