

M&T Sections:

- (1) Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the $z = 0$ plane.

We parametrize the surface as $z = f(x, y)$, $f(x, y) = 9 - x^2 - y^2$ over the xy -plane. The region D over which this graph resides is bounded by the intersection of the surface $z = f(x, y)$ and $z = 0$. In other words, the set of (x, y) such that $9 - x^2 - y^2 = 0$, or a circle of radius 3. Then

$$\iint_S 1 dS = \iint_D \sqrt{1 + \partial_x f^2 + \partial_y f^2} dA = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2} dA = \iint_D \sqrt{1 + 4(x^2 + y^2)} dA.$$

To evaluate this integral, we use polar coordinates. It is

$$\int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta = 2\pi \int_0^3 \sqrt{1 + 4r^2} r dr = 2\pi \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_0^3 = \frac{\pi}{6} (37^{3/2} - 1).$$

- (2) (a) Let $f : [1, \infty) \rightarrow [0, \infty)$ be a continuously differentiable function. Let S be the surface of revolution obtained by revolving the graph of $y = f(x)$ around the x -axis. Recall that the volume enclosed and surface area are:

$$\text{Vol} = \pi \int_1^\infty f(x)^2 dx, \quad \text{Area} = 2\pi \int_1^\infty f(x) \sqrt{1 + f'(x)^2} dx.$$

Suppose that $f(x) \leq M$ for some finite $M > 0$. Show that, if the surface area is finite, then so is the volume enclosed.

Let A be the surface area of the graph.

$$\text{Vol} = \int_1^\infty f(x)(\pi f(x)) dx \leq \frac{M}{2} \int_1^\infty 2\pi f(x) dx \leq \frac{M}{2} \int_1^\infty 2\pi f(x) \sqrt{1 + f'(x)^2} dx = \frac{M}{2} \text{Area}.$$

- (b) (Torricelli's trumpet) Let $f(x) = 1/x$ on $[1, \infty)$, revolve the graph of $f(x)$ around the x -axis, we get a trumpet-shaped surface. Find the volume and surface area.

The volume is

$$\text{Vol} = \int_1^\infty f(x)(\pi f(x)) dx = \pi \int_1^\infty \frac{dx}{x^2} = \pi.$$

On the other hand, the surface area is

$$\text{Area} = 2\pi \int_1^\infty f(x) \sqrt{1 + f'(x)^2} dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx > 2\pi \int_1^\infty \frac{dx}{x} = 2\pi \log(x) \Big|_1^\infty = \infty.$$

Torricelli's trumpet is a surface of finite volume and infinite area! You can fill the trumpet with a finite amount of paint, but it requires infinite amount of paint to cover the surface!

- (3) Evaluate the integral $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = (x, y, 1)$ and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

The surface is a graph: $z = f(x, y)$, $f(x, y) = \sqrt{1 - x^2 - y^2}$ sitting above the unit disc D in the xy -plane. Note $f_x = -x(1 - x^2 - y^2)^{-1/2}$, $f_y = -y(1 - x^2 - y^2)^{-1/2}$. Thus

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D F(x, y, f(x, y)) \cdot (-f_x, -f_y, 1) dA = \iint_D \left(\frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}} + 1 \right) dA \\ &= \iint_D \frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}} dA + \pi\end{aligned}$$

since π is the area of the unit disc. We compute the other integral in polar coordinates.

$$\iint_D \frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}} dA = \int_0^{2\pi} \int_0^1 \frac{r^2}{\sqrt{1 - r^2}} r dt = 2\pi \left(\frac{1}{2} \int_0^1 \frac{1 - u}{\sqrt{u}} du \right) = \frac{4\pi}{3}.$$

Thus the flux is $\frac{4\pi}{3} + \pi = \frac{7\pi}{3}$.

- (4) Let B be the solid ball of radius 1 given by

$$x^2 + y^2 + z^2 \leq 1.$$

Evaluate the following integrals.

Hint: You can compute each integral independently and in any order. The principle of symmetry may play an important role in each part!

- (a) $\iiint_B (x^{2023} + y^{2023} + z^{2023}) dV$

Each term is an odd function on B , by symmetry,

$$\iiint_B (x^{2023} + y^{2023} + z^{2023}) dV = 0$$

- (b) $\iiint_B (x^2 + y^2 + z^2 - xy - yz - xz) dV$

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2 - xy - yz - xz) dV &= \iiint_B (x^2 + y^2 + z^2) dV - 2 \iiint_B (xy + yz + zx) dV \\ &= \iiint_B (x^2 + y^2 + z^2) dV = \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^4 \sin \phi d\rho d\phi d\theta = \frac{4}{5}\pi\end{aligned}$$

Note that by symmetry, the integral of the terms xy, yz, xz are all 0.

- (c) $\iiint_B (x^{2n} + y^{2n} + z^{2n}) dV$ where n is a positive integer.

Use symmetry of x, y, z :

$$\begin{aligned}\iiint_B (x^{2n} + y^{2n} + z^{2n}) dV &= 3 \iiint_B x^{2n} dV \quad \text{by symmetry} \\ &= 3 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} x^{2n} dz dy dx \\ &= 6 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} x^{2n} dy dx \\ &= 6 \int_{-1}^1 \frac{\pi(1-x^2)}{2} x^{2n} dx \\ &= 6\pi \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) = \frac{12\pi}{(2n+1)(2n+3)}.\end{aligned}$$

Alternatively,

$$\begin{aligned}
 \int \int \int_B (x^{2n} + y^{2n} + z^{2n}) dV &= 3 \int \int \int_B z^{2n} dV \\
 &= 3 \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^{2n+2} \cos^{2n} \phi \sin \phi d\rho d\phi d\theta \\
 &= \frac{6\pi}{(2n+3)(2n+1)} \cos^{2n+1} \phi \Big|_\pi^0 = \frac{12\pi}{(2n+1)(2n+3)}.
 \end{aligned}$$

- (5) Use Stoke's theorem to evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (-xy, -xz, -yz)$ and C is the triangle with vertices $(0, 1, 0)$, $(0, 1, 5)$ and $(3, 1, 0)$ oriented by taking the vertices in that order.

Let S be the solid triangle bounded by C . Since it is oriented by the order of the points, this gives a normal vector that points along the positive y -axis (draw figure, use right hand rule). Thus $\vec{n} = (0, 1, 0)$. Since \vec{F} is differentiable, Stokes theorem tells that the line integral in equation equals the $\iint_S \text{curl} \vec{F} \cdot \hat{n} dS$. The curl is computed to be $\text{curl} \vec{F} = (x - z, 0, z - x)$. Thus $\text{curl} \vec{F} \cdot \hat{n} = 0$ so $\oint_C \vec{F} \cdot d\vec{r} = 0$.

- (6) (Challenge) Find the probability that three random numbers chosen uniformly from $[0, 1]$ represent the side lengths of some triangle. (Yes, this is a Calc III question).

Hint: If the three numbers are x, y and z , they are sides of a triangle if and only if

$$x + y \geq z, \quad x + z \geq y, \quad \text{or} \quad y + z \geq x.$$

Find the probability that $0 \leq x \leq y \leq z \leq 1$ and $x + y \geq z$ first. Note that when $x \geq \frac{1}{2}$, $x + y \geq 1$; and when $x \leq \frac{1}{2}$, $x + y \leq 1$ if $y \leq 1 - x$ and $x + y \geq 1$ if $y \geq 1 - x$. Then consider the other 5 possibilities similarly: $x \leq z \leq y$, $y \leq x \leq z$, $y \leq z \leq x$, $z \leq x \leq y$, and $z \leq y \leq x$.

The probability that $0 \leq x \leq y \leq z \leq 1$ and $x + y \geq z$ is

$$\begin{aligned}
 \int_0^{1/2} \int_x^{1-x} \int_y^{x+y} 1 dz dy dx + \int_0^{1/2} \int_{1-x}^1 \int_y^1 1 dz dy dx + \int_{1/2}^1 \int_x^1 \int_y^1 1 dz dy dx \\
 = \frac{1}{24} + \frac{1}{48} + \frac{1}{48} = \frac{1}{12}
 \end{aligned}$$

Consider the other 5 possibilities similarly: $x \leq z \leq y$, $y \leq x \leq z$, $y \leq z \leq x$, $z \leq x \leq y$, and $z \leq y \leq x$, we have that the three numbers form the three sides of a triangle with probability $\frac{6}{12} = \frac{1}{2}$.