M&T Sections: 5.3, 6.2, 8.1

(1) Use Green's theorem with $\vec{F} = \frac{1}{2}(-y, x)$ to show that the area of an n–sided polygon D in the xy–plane with vertices $(x_1, y_1), \dots (x_n, y_n)$ is given by

$$\text{Area}(D) = \frac{1}{2}\sum_{i=1}^{n} x_i y_{i+1} - x_{i+1} y_i,$$

where, in the above formula, we use the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. This is how area is evaluated in computer graphics very efficiently, with no integration involved.

Note that $\text{curl}\vec{F} = 1$. Thus, by Green's theorem

$$\text{Area}(D) = \iint_D \text{curl}\vec{F} dA = \oint_C \vec{F} \cdot d\vec{r},$$

where C is the boundary of the polygon D. The line from (x_i, y_i) to (x_{i+1}, y_{i+1}) can be represented by the parametrization

$$\vec{r}(t) = (x_i, y_i) + t(x_{i+1} - x_i, y_{i+1} - y_i), \qquad t \in [0, 1].$$

Thus, along that piece of segment, a computation shows

$$\oint_{C_i} \vec{F} \cdot d\vec{r} = \oint_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \frac{1}{2}(x_i y_{i+1} - x_{i+1} y_i).$$

The result follows by summing up all contributions.

(2) Consider a torus, whose radius from the center of the hole to the center of the torus tube is a, and the radius of the tube is b. Consider therefore that $a > b$. Then the equation in Cartesian coordinates for a torus azimuthally symmetric about the z-axis is

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2.$$

To derive this, one can think of the torus as a surface of revolution generated by rotating the circle $(y - a)^2 + z^2 = b^2$ around the z-axis. Find its surface area in terms of a and b.

You can find the area of the torus using the fact that

$$z = f(x, y) = \pm\sqrt{b^2 - (\sqrt{x^2 + y^2} - a)^2}, \qquad r = \sqrt{x^2 + y^2}, \qquad f_x = -\frac{(r - a)x}{\sqrt{b^2 - (r - a)^2}r}$$

$$\begin{aligned}
A(S) &= 2\int\int_D \sqrt{1 + \frac{(r-a)^2}{b^2 - (r-a)^2}} dA \\
&= 2\int_0^{2\pi}\int_{a-b}^{a+b} \frac{b}{\sqrt{b^2 - (r-a)^2}} r dr d\theta \\
&= 4\pi b \int_{-b}^{b} \frac{s + a}{\sqrt{b^2 - s^2}} ds \qquad s = r - a \\
&= 4\pi b \int_{-b}^{b} \frac{a}{\sqrt{b^2 - s^2}} ds \qquad \text{by symmetry} \\
&= 4\pi ab \arcsin\frac{s}{b}\Big|_{-b}^{b} = 4\pi^2 ab.
\end{aligned}$$

(3) Find the flux of $\vec{F} = (yz, xz, xy)$ over the surface S which is the graph of $z = x^2 + y^2$ over the unit disk centered at the origin.

$$\begin{aligned}\int\int_{\Sigma} \vec{F} \cdot d\vec{S} &= \int\int_{D} (yz, xz, xy) \cdot (-2x, -2y, 1) dxdy \\ &= \int\int_{D} (-4xy(x^2+y^2) + xy) dxdy \\ &= \int_0^{2\pi} \int_0^1 -r^2 \frac{\sin 2\theta}{2}(1-4r^2) r dr d\theta = 0\end{aligned}$$

(4) Find the flux of $\vec{F}(x, y, z) = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

The surface is defined by the equation $\varphi(x, y, z) = x^2 + y^2 - z = 0$. And the constraint is $z \leq 1$¿. Since $\nabla\varphi = (2x, 2y, -1)$, the normal vector is

$$\vec{n} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}(2x, 2y, -1).$$

Thus $\vec{F} \cdot \vec{n} = \frac{8x^2+8y^2-2}{\sqrt{4x^2+4y^2+1}}$. Thus the integral to be computed is

$$\begin{aligned}\iint_S \vec{F} \cdot dS &= \iint_{R=\{x^2+y^2\leq 1\}} \frac{8x^2 + 8y^2 - 2}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1} dxdy \\ &= \int_0^{2\pi} \int_0^1 (8r^2 - 2) r dr d\theta = 2\pi.\end{aligned}$$

(5) Let S be the surface defined by the portion of the plane $x + y + z = -1$ satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and oriented so that the normal to the plane is pointing "up". Find the flux of the vector field $\vec{F} = (x, -2y, xz)$ across the surface S.

We can write this piece of the plane as a graph over the xy–plane. $z = f(x, y) := -1 - x - y$ for $(x, y) \in [0, 1] \times [0, 1]$. This parametrization orients the surface with an upward pointing normal. The normal is

$$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}} = \frac{(1, 1, 1)}{\sqrt{3}}$$

The flux is

$$\iint_S F \cdot dS = \iint_D (x, -2y, xf(x, y)) \cdot (1, 1, 1) dxdy = -\frac{19}{12}.$$